Some Notes on Subsets Enumerability

Paola Cattabriga
CIRFID - Università di Bologna
Via Galliera, 3 I-40121 Bologna, ITALY.
paola@cirfid.unibo.it

Abstract

In this paper we develop a natural technique for enumerating and generating all the parts of a given set. The method is in some sense similar to Cantor's method for enumerating rational numbers. Unlike Cantor's our method disallows diagonalization both on the set of the parts and on the set of reals.

1

It is well known that an *enumerable* set is one whose members can be enumerated, i.e. it is an arrangement in which each member of the set is associated with one of the natural numbers. An infinite set which is enumerable is said to be *enumerably infinite* or *denumerable*. For instance, as shown in Figure 1, both the set of all the even numbers (the higher succession) and the set of all the rational numbers (the lower succession) are enumerable. The method shown in Figure 1 is that used by Cantor; fractions are not listed in a natural order but in the order determined by the sum of the numerator and the denumerator.

2,	4,	6,	8,	10,	12,	14,	16,	18,	...
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
1,	2,	3,	4,	5,	6,	7,	8,	9,	...
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
1/1,	1/2,	2/1,	3/1,	2/2,	1/3,	1/4,	2/3,	3/2,	...

Figure 1: Even and Rational Numbers Enumeration.

An infinite list, or denumerable sequence, determines a function f whose domain is the set of natural numbers and whose codomain is the list. For example, the infinite list which enumerates the set of the even numbers determines the function f for which we have $f(1) = 2,\ f(2) = 4,\ f(3) = 6,\ f(4) = 8,\ \dots$. For a detailed treatment see [He61, BJ74].

Let us now recall two basic notions.

Definition 1.1 *The set A is a* **subset** *of the set B if every element of A is an element of B. When this is the case, we say that A is* **included** *in B, written $A \subseteq B$, and B* **includes** *A, written $B \supseteq A$. The following properties hold:*

i) For all sets A, $\emptyset \subseteq A$ and $A \subseteq A$.

ii) For all sets A and B, $A = B$ if and only if $A \subseteq B$ and $B \supseteq A$.

iii) For all sets A, B and C if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

Definition 1.2 *Let A be a subset of B. Then A is a* **proper subset** *of B if A is not equal to B itself. Thus there is some element of B not in A. The subset A is then said to be* **poperly** *(or* **strictly***)* **included** *in B and this is written $A \subset B$.*

Given a finite set {a,b,c,d}, we produce its subsets as in Figure 2. The disposition in Figure 2 is natural and intuitive, namely the position of each subset mirrors the order a human would follow to make sure to write all of them. Let us describe informally the generation process in Figure 2.

$$\begin{array}{llll}
\emptyset & & & \\
\{a\} & \{ab\} & \{abc\} & \{abcd\} \\
\{b\} & \{ac\} & \{abd\} & \\
\{c\} & \{ad\} & \{acd\} & \\
\{d\} & \{bc\} & \{bcd\} & \\
 & \{bd\} & & \\
 & \{cd\} & &
\end{array}$$

Figure 2: Parts of {a,b,c,d}.

The process is started generating all the singletons, then we find the way to generate all the pairs, then all the terns and so on. Singletons are written following the order of occurrence of each element into the set.

1. {a},
2. {b},
3. {c},

4. {d}.

   Then, to make sure all couples are generated, we follow a plain order of combination.

   All the couples starting from the first element.

5. {ab},

6. {ac},

7. {ad}.

   All the couples starting from the second element *(leaving out the preceding combination and the repetitions of element).*

8. {bc},

9. {bd} *({ba} already exists, hence it is left out).*

   All the couples starting from the third element leaving out the preceding combination.

10. {cd} *({ca} and {cb} already exist, hence they are left out).*

    All the couples starting from the fourth element leaving out the preceding combination *({da}, {db}, {dc} already exist, hence they are left out).*

    Now, to make sure that all the terns are generated we follow a plain order of combination.

    All the terns starting from the first couple.

11. {abc},

12. {abd}.

    All the terns starting from the second couple *(leaving out the preceding combination).*

13. {acd} *({acb} already exists, hence it is left out).*

    All the terns starting from the third couple *({adb} and {adc} already exist, hence they are left out).*

    All the terns starting from the fourth couple *(leaving out the preceding combination).*

14. {bcd}.

Finally, as a consequence of the Definition 1.1, the empty set and the set itself belong to the set of parts. Since all possible combinations have been considered, we achieved all the subset of the starting set, even though – for simplicity – nothing was said about the *repetitions of the elements.*

The subsets esclusion process is not simply formed by the elimination of the preceding combinations, but also by a super-rule on the order of the combinations which automatically becomes exclusion. In other words, when a couple of elements is combined, they are combined with the elements which immediately follows the last element of the couple into the starting set. This is done until the last couple of the first series which do not generate anything is reached. Clearly, the same is done for terns, and so on.

Let also notice that the process was already implicit at the first level of combination, i.e. into the passage from the singleton to the couples. That procedure of non repetition arises naturally and later will be shown to be efficient.

In conclusion, it can be argued that the exclusion process, already implicit into the order displayed in Figure 1, consists of excluding combinations of elements preceding the last element of the singleton, of the couple, of the terna.

2

Let us suppose we have an infinite alphabet of letters, which is always possible even with a finite alphabet, for example we can suppose $\{a, b, c, \ldots, a^*, b^*, c^*, \ldots, a^{**}, b^{**}, c^{**}, \ldots\}$. Let us consider now any infinite set. We denote its elements with the letters of an infinite alphabet: $\{a, b, c, d, \ldots\}$. We ask if, for such a set, we are able to reproduce, in some way, the previous process of generation of its parts.

$$
\begin{array}{ccccccccc}
\frac{1}{1} & & \frac{2}{1} & \longrightarrow & \frac{3}{1} & & \frac{4}{1} & \longrightarrow & \frac{5}{1} \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & \swarrow & \\
\frac{1}{2} & & \frac{2}{2} & & \frac{3}{2} & & \frac{4}{2} & & \\
 & \swarrow & & \nearrow & & \swarrow & & & \\
\frac{1}{3} & & \frac{2}{3} & & \frac{3}{3} & & & & \\
\downarrow & \nearrow & & \swarrow & & & & & \\
\frac{1}{4} & & \frac{2}{4} & & & & & & \\
 & \swarrow & & & & & & & \\
\frac{1}{5} & & & & & & & & \\
\vdots & & & & & & & &
\end{array}
$$

Figure 3: Rational Numbers Enumeration.

Cantor's method for enumerating rational numbers, see Figure 3, is a way of finding an arrangement over the twodimensional space. Then we conjecture there can be a similar way of disposing all the subsets of an infinite set by reproducing the previous process of generation. In order to do this let us recall a very basic theorem, [MB88, AA65].

Theorem 1 *Any infinite set contains a countable subset.*

Proof. Let S be the infinite set; choose for s_1 any element in it. From $S - \{s_1\}$, then choose a second element s_2; from $S - \{s_1, s_2\}$ a third element s_3, and so on. Since S is infinite, $S - \{s_1, s_2, \ldots, s_n\}$ can never be empty; hence we can always choose an s_{n+1} in it, and the process can never stop untill we have constructed an infinite sequence of different elements of S.

Let us turn our attention to the process of generate the parts of a set, even when the set is infinite. We know from Definition 1.1 *i)* that the empty set and the set itself are subsets of a given set. We still know that it is very simple to generate all the singletons. Let us try to generate, starting from $\{a, b, c, d, \ldots\}$ all the couples, all the terns, all the 4-lists, ... all the n-lists. We proceed by analogy, i. e. we are drawing each time an element from the set (this can be done as asserted by Theorem 1), and generating for every new element all the parts not previously generated, by applying the previous procedure of combination and exclusion. Each new element generates new parts. The steps of the generated list correspond therefore to choosing a new element and generating the relative parts.

A representation of the generation of the parts of $\{a, b, c, d, \ldots\}$ is given below. Each new generated part is associated with a natural number, emphasized in boldface, and written for simplicity without commas and brackets. For example the part **abcdf** is associated to 59 and must be meant as

$$59 \leftrightarrow \{a, b, c, d, f\}.$$

Sections (the section of the singletons, the section of the couples, the section of the ternas, ...) are separated by a space.

The last section of any **Step n** is always an **n**-lists, one part composed by the same elements of the singletons at the first section of the step.

Generation of the parts of $\{a, b, c, d, \ldots\}$

Step 0	
1	$\emptyset$
Step 1	
2	**a**
Step 2	
3	a,**b**
4	**ab**
Step 3	
5	a,b,**c**
6	ab,**ac**
7	**bc**
8	**abc**
Step 4	
9	a,b,c,**d**
10	ab,ac,**ad**
11	bc,**bd**
12	**cd**
13	abc,**abd**
14	**acd**
15	**bcd**
16	**abcd**
Step 5	
17	a,b,c,d,**e**
18	ab,ac,ad,**ae**
19	bc,bd,**be**
20	cd,**ce**
21	**de**
22	abc,abd,**abe**
23	acd,**ace**
24	**ade**
25	bcd, **bce**
26	**bde**
27	**cde**
28	abcd,**abce**
29	**abde**
30	**acde**
31	**bcde**
32	**abcde**
Step 6	
33	a,b,c,d,e,**f**
34	ab,ac,ad,ae,**af**
35	bc,bd,be,**bf**
36	cd,ce,**cf**
37	de,**df**
38	**ef**
39	abc,abd,abe,**abf**
40	acd,ace,**acf**
41	ade,**adf**
42	**aef**
43	bcd,bce, **bcf**
44	bde,**bdf**
45	**bef**
46	cde,**cdf**
47	**cef**
48	**def**
49	abcd,abce,**abcf**
50	abde,**abdf**
51	**abef**
52	acde,**acdf**
53	**acef**
54	**adef**
55	bcde,**bcdf**
56	**bcef**
57	**bdef**
58	**cdef**
59	abcde,**abcdf**
60	**abcef**
61	**abdef**
62	**acdef**
63	**bcdef**
64	**abcdef**
Step 7	
$\Downarrow$	
o	
o	
o	

This representation shows a method for enumerating the parts of a set S for its elements $\{s_1, s_2, s_3 \ldots\}$. Let us denote the generated parts of S with the set $\{c_1, c_2, c_3, \ldots\}$ and call it C. The the representation of the parts of $\{a, b, c, d, \ldots\}$ shows a one-one mapping of N onto C (N, the set of the natural numbers) as follows:

$$\mathcal{F}(1) = \emptyset,\ \mathcal{F}(2) = a,\ \mathcal{F}(3) = b,\ \mathcal{F}(4) = ab,$$

$$\mathcal{F}(5) = c,\ \mathcal{F}(6) = ac,\ \mathcal{F}(7) = bc,\ \ldots$$

We are going to show that if the process of generating parts gives us $\mathcal{F}(1) = c_1$ and $\mathcal{F}(n) = c_n$ then we are always able to produce $\mathcal{F}(n+1) = c_{n+1}$. We are now able to assert the following theorem.

Theorem 2 *If the process of generating parts of a set S by choosing its elements $\{s_1, s_2, s_3, \ldots\}$ has produced $\mathcal{F}(n) = c_n$ (and $\mathcal{F}(1) = \emptyset$), then we are able to generate $\mathcal{F}(n+1) = c_{n+1}$.*

Proof. There are three possible cases.

i) Let us suppose $\mathcal{F}(1) = \emptyset$ and $\mathcal{F}(n) = c_n$, the n-lists. Let us write the elements of c_n as $\xi_1 \ldots \xi_j$. Then ξ_j tell us the number of the step of generation we are, **Step** s_j. $\xi_1 \dots \xi_{j-1}$ tell us the last combined $n-1$-list. This means that now we are able to obtain c_{n+1}: we go back to the previous section of $n-1$-lists (generated at the **Step** s_j), we search for the position of $\xi_1 \dots \xi_{j-1}$ and we choose exactly the first $n-1$-list on the right of $\xi_1 \dots \xi_{j-1}$. Let us denote it with $\zeta_1 \dots \zeta_i$. $c_{n+1} = \zeta_1 \dots \zeta_i, \xi_j$. If the $n-1$-list on the right of $\xi_1 \dots \xi_{j-1}$ is in boldface we must choose exactly the first $n-1$-list placed in the row below.

ii) If this last one is still in boldface this means that we finished the section of n-lists and c_{n+1} will be given combining ξ_j and the first n-list.

iii) When the elements $\xi_1 \dots \xi_j$ are as many as s_j then $c_{n+1} = s_{j+1}$, namely c_{n+1} will be the next chosen element of S.

Example i). Let us suppose that $\mathcal{F}(1) = \emptyset$ and $\mathcal{F}(50) = abdf$. Then f corresponds to the element s_6 of S and the step of generation is **Step 6**; abd tell us what's the last combined tern. Now we look at the previous sections of terns and we choose the first tern on the right, abe. $c_{n+1} = abef$.

Example ii). Let us now suppose that $n = 63$ and $\mathcal{F}(63) = bcdef$. If we look the section of the quadernas we see there are nor no boldface quadernas on the right nor in the row below. This means that $\mathcal{F}(64)$ start a new section, the section of 6-lists, combining f with $abcde$, the first (no boldface) 5-lists. $c_{n+1} = abcdef$.

Example iii). If we take $\mathcal{F}(64) = abcdef$ then $c_{n+1} = g$.

Theorem 2 inductively demonstrates that it is possible to generate all the parts of a set S. Here the crucial role is played by Theorem 1 and Definitions 1.1, 1.2. By Theorem 1 $\{s_1, s_2, s_3, \dots\} \subseteq S$ and not necessarely just $\{s_1, s_2, s_3, \dots\} \subset S$, so nothing prevents us from supposing the process of choosing always a new s_i will produce $S = \{s_1, s_2, s_3, ...\}$. For even if $S - \{s_1, s_2, \dots, s_n\}$ will never be empty, not necessarily $\{s_1, s_2, s_3, \dots\} \subset S$, i. e. we do not know whether there exists a $s_j \in S$ *and* $s_j \notin \{s_1, s_2, s_3, \dots\}$. This *unknown* do not prevent us from thinking $S = \{s_1, s_2, s_3, \dots\}$. What is equally unknown is whether $S \subseteq \{s_1, s_2, s_3, \dots\}$. To summarize, we only know that $\{s_1, s_2, s_3, \dots\} \subseteq S$, namely

$$\{s_1, s_2, s_3, \dots\} = S \text{ or } \{s_1, s_2, s_3, \dots\} \subset S$$

and nothing allows us to prefer the first instead of the second and vice versa. Theorem 2 says that we can generate all the parts of $\{s_1, s_2, s_3, \dots\}$, hence we can generate all the parts $\{c_1, c_2, c_3, \dots\}$ of S or all the parts $\{c_1, c_2, c_3, \dots\}$ of a proper subset of S. At this point there are two main objections to the assertion that the previous representation shows a method for enumerating [1] all the subsets of a given set S (finite or infinite):

- *a)* $S \not\sim C$
- *b)* the generated parts $c_i \in C$ are always finite, namely they are of the kind $\{\xi_1, \xi_2, \dots, \xi_m\}$ and not of the kind $\{\xi_1, \xi_2, \xi_3, \dots\}$.

where $X \not\sim Y$ means "X and Y are not equinumerous", namely there is not a one-to-one mapping of X onto Y. The next two sections will discuss rispectively *a)* and *b)* .

[1] In the Theory of the Computability a set A is *generable* if and only if A is *enumerable*.

3

Objection *a)* is the Cantor diagonalization, a method for demonstrating that a set A and the set of all its subsets, $\mathcal{P}(A)$ have not the same cardinality, namely it is not possible to construct a one - one correspondence between A and $\mathcal{P}(A)$, once achieved $A = \mathcal{P}'$ and $\mathcal{P}' \subset \mathcal{P}(A)$, assuming for example that $\mathcal{P}'$ is the set of the singletons (and therefore in a one-one correspondence with A).

We recall briefly the whole method and we show that our generating function $\mathcal{F}$ prevent its use.

Diagonalization on $\mathcal{P}(A)$. Let us argue by reductio ad absurdum. We suppose that between A and $\mathcal{P}(A)$ there exists a mapping f which associates with each element $x \in A$ an element $p_x \in \mathcal{P}(A)$ and vice versa:

$$f : x \in A \leftrightarrow p_x \in \mathcal{P}(A).$$

Now, given an element $a \in A$, it is known that among the elements of $\mathcal{P}(A)$ some contain a and some do not; two cases are possible: either the p_a which f makes correspond to a contains a or it does not contain a. This happens for every element of A. We call X the set formed by all and only the elements $x \in A$ which the associated p_x into the mapping f does not contain x:

$$X = \{x \in A \mid x \notin p_x\}$$

X is a subset of A (its elements are among the elements of A) and, for this reason X is an element among the elments of $\mathcal{P}(A)$. Let be b the element of A which corresponds to X into f:

$$b \leftrightarrow p_b \text{ and } p_b = X.$$

One can gives two cases:

1. p_b *contains* b, namely $b \in X \in \mathcal{P}(A)$: then its corresponding $b \in A$ can not be one of that $x \in A$ which f associates to a p_x which does not contain x; i. e. $b \notin X \subset A$. Hence, if $b \in X$ then $b \notin X$.

2. p_b *does not contain* b, namely $b \notin X \in \mathcal{P}(A)$: then its corresponding $b \in A$ is one of that $x \in A$ which f associates to a p_x which does not contain x; i. e. $b \in X$. Hence, if $b \notin X$ then $b \in X$.

These two contradictions demonstrate that the supposed one-one correspondence f does not exist.

We now show how the previous generating function $\mathcal{F}$ blocks the diagonalization. Let us apply it to the set of natural numbers N and let S=N, $C = \mathcal{P}(N)$. We will obtain:

$$\begin{array}{lcl}
1 & \leftrightarrow & \emptyset \\
2 & \leftrightarrow & 1 \\
3 & \leftrightarrow & 2 \\
4 & \leftrightarrow & \{1,2\} \\
5 & \leftrightarrow & 3 \\
 & & \\
6 & \leftrightarrow & \{1,3\} \\
7 & \leftrightarrow & \{2,3\} \\
 & & \\
8 & \leftrightarrow & \{1,2,3\} \\
9 & \leftrightarrow & 4
\end{array}$$

10	$\leftrightarrow$	$\{1,4\}$
11	$\leftrightarrow$	$\{2,4\}$
12	$\leftrightarrow$	$\{3,4\}$
13	$\leftrightarrow$	$\{1,2,4\}$
14	$\leftrightarrow$	$\{1,3,4\}$
15	$\leftrightarrow$	$\{2,3,4\}$
16	$\leftrightarrow$	$\{1,2,3,4\}$
17	$\leftrightarrow$	5
18	$\leftrightarrow$	$\{1,5\}$
19	$\leftrightarrow$	$\{2,5\}$
20	$\leftrightarrow$	$\{3,5\}$
21	$\leftrightarrow$	$\{4,5\}$
22	$\leftrightarrow$	$\{1,2,5\}$
23	$\leftrightarrow$	$\{1,3,5\}$
24	$\leftrightarrow$	$\{1,4,5\}$
25	$\leftrightarrow$	$\{2,3,5\}$
26	$\leftrightarrow$	$\{2,4,5\}$
27	$\leftrightarrow$	$\{3,4,5\}$
28	$\leftrightarrow$	$\{1,2,3,5\}$
29	$\leftrightarrow$	$\{1,2,4,5\}$
30	$\leftrightarrow$	$\{1,3,4,5\}$
31	$\leftrightarrow$	$\{2,3,4,5\}$
32	$\leftrightarrow$	$\{1,2,3,4,5\}$

$\Downarrow$
o
o
o

It is easy to see that it will never happen

$$f : n \in N \leftrightarrow p_n \in \mathcal{P}(N) \textit{ and } n \in p_n$$

since we can demonstrate that for each association $s_n \leftrightarrow c_n$ $(c_n = \{\xi_1, \xi_2, \ldots\})$ we have for all $\xi_i \in c_n$

$$s_n > \xi_i.$$

Hence we can say

$$\mathcal{F} : n \in N \leftrightarrow c_n \in C \textit{ and } n \notin c_n.$$

This means that $X = \{c_1, c_2, c_3, \ldots\}$, i. e.

$$C = \{n \in N | n \notin p_n\}.$$

Then *i)* C is a subset of S (its elements are among the elements of S), but *ii)* C is not an element of $\mathcal{P}(S)$, i. e. of itself. We recall the attention about that if *i)* is true $(C \subseteq S)$, it is even true that $\mathcal{F}(S)$ construct an one-to-one mapping of S onto C. Therefore $C \not\subseteq S$ and $C \sim S$, namely C and S are equinumerous. For this reason *ii)* is true. This contradicts the objection *a)*.

4

Let us consider now objection *b)*. We recall here what it is commonly meant with the notion of "finite".

Definition 4.1 *A set A is called* **finite** *if there exists a natural number n such that A is equinumerous with the set $\{0, 1, 2, \ldots, n-1\}$ of all natural numbers which are smaller than n, i. e. the member of A can be put into a one-one correspondence with the natural numbers less than n.*

If we look at the representation *Generation of the parts of* $\{a, b, c, d, \ldots\}$ in section 2, Definition 4.1 seems definitely confirm the validity of b. The c_i placed on the final row of any **Step n** is always $\{\xi_1, \xi_2, \ldots \xi_n\}$), i. e. it has n elements; each c_j for $j < i$ has always less elements than n. But this is misleading, if the relation to have less elements is considered a part from total $\mathcal{F}$. The representation of $\mathcal{F}$ n section 2 shows that there exists no kind of one-one correspondence among c_i and each c_j generated at the **Step n**. For example let it be $i = 64$ and $c_i = \{a, b, c, d, e, f\}$. If we take out f from c_i, we have $c_{i-1} = \{a, b, c, d, e\}$. It easy to see that there is no one-one correspondence with each c_j, $32 < j < 64$, generated at the **Step 6**. The same holds for $c_{i-2}, c_{i-3}, \ldots$. If we apply $\mathcal{F}$ to N, see section 3, there is no one-one correspondance of c_{i-x} with each c_j $(x \in \{1, \ldots, n-1\})$. Since all c_j are lists of *all* the naturals (for the construction and definition of $\mathcal{F}$), there is no one-one correspondance with the natural numbers less than n. Thus definition 4.1 contradicts objection *b)*.

5

We consider now the method of the Cantor diagonalization applyed to the set of the real numbers, R.

Diagonalization on R. The set of real numbers can be pictured by the continuum of the points on a line. Diagonalization states that R is not countable, the method is displayed in Figure 4. Supposing there were an enumeration of all the real numbers there would be a one-one correspondance with the naturals, as in Figure 4. The lenght of the list does not matter; it is always possible to construct a new decimal which defines a real number not included in the list. From the digits along the diagonal of this array we construct a new real number between 0 and 1 as follows. We write as the first decimal a 9 if the first digit is 1 otherwise we write 1. We do the same for the second digit and so on. Thus the written real number differs from all the real numbers included in the list in at least one decimal place. This contradicts our supposition that the enumeration included *all* real numbers.

1	$\longleftrightarrow$	$\boxed{0{,}1}$	1	1	1	1	$\cdots$
2	$\longleftrightarrow$	0,3	$\boxed{0}$	1	0	2	$\cdots$
3	$\longleftrightarrow$	0,4	7	$\boxed{7}$	1	2	$\cdots$
4	$\longleftrightarrow$	0,6	0	2	$\boxed{0}$	5	$\cdots$
5	$\longleftrightarrow$	0,6	9	8	9	$\boxed{7}$	$\cdots$
$\vdots$							$\ddots$
		0,9	1	1	1	1	$\cdots$

Figure 4: Diagonalization on R.

Let us applay now $\mathcal{F}$ to the set {0,1,2,3,4,5,6,7,8,9,0,1,2, 3,4,5,6,7,8,9,0,1,...}, let call it D. Following we give a representation of the generated subsets.

Step 0	
1	$\emptyset$
Step 1	
2	**0**
Step 2	
3	0,**1**
4	**01**
Step 3	
5	0,1,**2**
6	01,**02**
7	**12**
8	**012**
Step 4	
9	0,1,2,**3**
10	01,02,**03**
11	12,**13**
12	**23**
13	012,**013**
14	**023**
15	**123**
16	**0123**
Step 5	
17	0,1,2,3,**4**
18	01,02,03,**04**
19	12,13,**14**
20	23,**24**
21	**34**
22	012,013,**014**
23	023,**024**
24	**034**
25	123, **124**
26	**134**
27	**234**
28	0123,**0124**
29	**0134**
30	**0234**
31	**1234**
32	**01234**
Step 6	
33	0,1,2,3,4,**5**
34	01,02,03,04,**05**
35	12,13,14,**15**
36	23,24,**25**
37	34,**35**
38	**45**
39	012,013,014,**015**
40	023,024,**025**
41	034,**035**
42	**045**
43	123,124, **125**
44	134,**135**
45	**145**
46	234,**235**
47	**245**
48	**345**
49	0123,0124,**0125**
50	0134,**0135**
51	**0145**
52	0234,**0235**
53	**0245**
54	**0345**
55	1234,**1235**
56	**1245**
57	**1345**
58	**2345**
59	01234,**01235**
60	**01245**
61	**01345**
62	**02345**
63	**12345**
64	**012345**
Step 7	
$\Downarrow$	
∘	
∘	
∘	

Let us suppose that the lists $c_n = \{\xi_1, \xi_2, \xi_3, \ldots\}$ are lists of decimals and each one of them is an abbreviation of $0, \xi_1, \xi_2, \xi_3, \ldots$. Since each new section expands of a digit the lists of decimals of the preceding section, $\mathcal{F}$ is a construction of all the combinations of decimals to infinity. It is easy to see that $\mathcal{F}$ bloks the diagonalization. It is not possible to construct a real number that differs from all $c_i \in C$. If we try to construct a new list of decimal writing 9 in the place of 1, then C will certainly include a c_i with the digit 9 in the same position. For example for the position ξ_1, 9 will appear at the **Step 11** (not represented for reason of space). The same holds for 1 otherwise, and for all the positions of the decimals.

6

It is not easy for our intuition to think that $\mathcal{F}$ will generate for example the infinite list of all the even number, or an infinite list of all the primes, but this is exactly what $\mathcal{F}$ does, if we think it like a machine, a sequencial alghoritm which generates all the subset of any given set, which will never stop.

As we showed in this paper it is theoretically possible to construct such an algorithm and nothing apparently prevents us from establishing an enumeration of the generated subsets and to fix a one-one correspondence with naturals.

Obviously, our purpose was not to state a one-one correspondence with the elements of *any* set. In fact it is well known that such a correspondence does not hold on the elements of finite sets. Because of the theorem "The cardinal of the set of the subsets of a set of cardinality n is 2^n". We want to call in discussion the possibility of stating the same for infinite sets too. Namely we discuss the fundamental statement of the Cantor theory of transfinite: "the set of all the subsets of a denumerable set has cardinality greater than the enumerable". In other terms, we agree with Cantor when he says that the set (C) of the parts of a denumerable set (S) has cardinality greater than the one of the set itself

(S). But even if $C > S$, both C and S can be enumerated. Thus we disagree with Cantor when he says that C has cardinality greater than the enumerable.

7

To build diagonal functions, defined by general classes of sets, is usual in classical Mathematic. Such fuctions are basic also for the Theory of Computability and for the Theory of Demonstration [Tü36,37, TMR53, Gö31]. In connection with these procedures many recent researches in Artificial Intelligence and Computer Science are devoted to systems with reflective capabilities and to meta-level architectures.

We think that our method could have consequeces for Logic and Mathematic, since we clearly claim for a new Set Theory, but we especially hope that it will improve the inquiries about the use of diagonalizating and self-referring procedures.

References

[AA65] A. Abian. *The Theory of Sets and Transfinite Arithmetic.* W. B. Saunders Company, London, 1965.

[BJ74] G. Boolos, R. Jeffrey *Computability and Logic.* Cambridge University Press, London, 1974.

[Ca95,97] G. Cantor. *Contributions to the Founding of the Theory of Transfinite Numbers.* Dover Publications, New York, 1955.

[Co66] P. J. Cohen. *Set theory and the continuum hypothesis.* W. A. Benjamin, New York, 1966.

[Da58] M. Davis. *Computability and Unsolvability.* McGraw Hill, New York, 1958.

[Gö31] K. Gödel. Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. in *Monatsh. Math. Phys.*, 38, pp 173-98, 1931. English translation in [vH67].

[Gö40] K. Gödel. *The Consistency of the Axiom of Choice and of the generalized Continuum Hypothesis with the Axioms of Set Theory.* Annals of Mathematics Studies, 3, Princeton University Press, Princeton, 1940.

[Gö64] K. Gödel. What is Cantor's continuum problem? in *Philosophy of mathematics*, P. Benacerraf and H. Putnam. Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

[He61] H. Hermes. *Enumerability, Decidability, Computability.* Springer, Berlin, 1969.

[MB88] S. MacLane., G. Birkhoff. *Algebra.* Chelsea Pub. Co., New York, 1988.

[Me64] E. Mendelson. *Introduction to Mathematical Logic.* Wadsworth & Brooks Cole Advanced Books & Software, Pacific Grove, California, 1987.

[Sk22] T. Skolem. Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre. in *Wissenschaftliche Vorträge gehalten auf dem Fünften Kongress der Skandinavishen Mathematiker in Helsingfors vom 4. bis 7. Juli 1922.* Helsingfors, pp. 217-232, 1923. English translation in [vH67].

[TMR53] A. Tarski, A. Mostowski, R. M. Robinson. *Undecidable theories.* Noth-Holland, Amsterdam, 1968.

[Tü36,37] A. Türing. On Computable Numbers, with an Application to the Entscheidungsproblem. *Proc. Lond. math. Soc.*, 42, pp. 230-265, 43, pp. 544-546.

[vH67] J. van Heijenoort. *From Frege to Gödel: a source book in Mathematical Logic, 1879-1931.* Harvard University Press, Cambridge, Mass, 1967.