TRANSFORM Your Geometry Students

into Active Learners

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by

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The study of transformations on the coordinate plane is a geometry concept that is explicitly presented in the *NCTM Principles and Standards for School Mathematics*, Geometry Standard for Grades 9-12 (NCTM 2000, p. 308):

Apply transformations and use symmetry to analyze mathematical situations. *In grades 9-12 all students should--*

- *understand and represent translations, reflections, rotations, and dilations of objects in the plane by using sketches, coordinates, vectors, function notation, and matrices;*
- *use various representations to help understand the effects of simple transformations and their compositions.*

In addition, the concept of transformations is a building block for the study of functions in more advanced math courses.

We were not satisfied in our search for materials and a method for teaching transformations to our high school informal geometry students, so we prepared the unit described in this article. The lessons actively engage students in the investigative process and give them a hands-on model for physical representations of the mathematical concepts of transformations.

By the time students have completed the unit, they understand simple reflections, translations, rotations, and dilations and are familiar with the related vocabulary and symbolism. The students have worked through a hands-on investigative procedure and have moved through the various levels of organization, analysis, and mathematical communication outlined in the *NCTM Standards* (NCTM 2000, p. 348). The lessons are good ones for practice in communicating mathematically because the patterns are simple enough for students at a variety of learning levels to understand and express in writing.

With advanced or highly motivated students much of the discovery could be independent. The format of the unit also works well with teacher-led discovery where that method better fits the needs of the students.

The unit of study requires several days for completion. Previous learning and the ability of the students will dictate the number of days required. In our experience students have needed about four days to work through the development of the concepts, followed by an appropriate number of days for practice and assessment. Eight to ten 50-minute periods are usually sufficient to complete the unit.

Materials

The materials required for the investigations are inexpensive and easy to work with. Students work through a set of worksheets during the discovery portion of the lessons. Several sheets of graph paper accompany these worksheets. Each student also needs a piece of foam board cut to the approximate size of a sheet of paper, several sheets of patty paper, at least two pushpins, and at least three or four colored pencils.

Foam board can be purchased in a variety of sizes at craft or discount stores. Pieces of corrugated cardboard could be substituted for the foam board. Patty paper can be ordered through various teacher supply companies. It is similar to wax paper and works well for tracing and marking. Patty paper comes in sheets measuring approximately 5-inches square.

During each stage of the lesson students use graph paper to draw a specified triangle or rectangle. The patty paper is used for tracing the figure and performing the transformations. Students use pushpins through the patty paper to mark the positions of the new coordinates on the graph paper. The foam board is placed underneath the graph paper so the pushpins will go through and mark locations on the paper without causing any damage to the desks. Students are also provided with colored pencils for drawing the transformed figures. The teacher can prepare transparencies and go through the steps of the transformations at the overhead projector while the students are working at their desks.

We used an additional teaching tool this year with the transformation unit. We connected a Play Station to a classroom TV and provided a Tetris game. The students had the opportunity to play the game for the first few minutes of class each day, as they were entering the room and getting ready to begin the day's lesson. The game provided an opportunity to show the use of transformations in a setting that was familiar and fun. The students were able to see translations and rotations in action as they enjoyed competing with each other and even with the teacher.

Lessons and Procedures

Reflections – The first type of transformation we study in this unit is reflections. We start with reflections across the x-axis and the y-axis.

We define reflections and also refer to them as “flips,” a term that is particularly meaningful in the physical act of making the reflections.

We instruct the students that throughout the lesson, the “prime” markings (’, ’’, ’’’) indicate whether a figure is transformed from the original or from a previous step in the chart. For instance, if a point is labeled *A’*, its pre-image point is the original point *A*. If it is labeled *A’’*, it is a transformation from the previous point *A’.*

Students draw ∆ABC on a sheet of graph paper and trace the figure on patty paper, also labeling the coordinates. It is helpful to trace a cross on the patty paper over the graph’s origin as a point of reference. Students then pick up the paper and flip it over the x-axis, physically modeling the concepts and the vocabulary already defined. Once the paper has been flipped and the cross on the patty paper aligned with the origin on the graph paper, the students use a pushpin to mark the coordinates of the reflected triangle. They then remove the patty paper and connect the small holes they have made on the graph paper to draw the reflected figure. The coordinates on the new figure are labeled A’, B’ and C’.

Table 1 shows the coordinates for the vertices of the original triangle and provides spaces for the students to record their results after making the assigned reflections. Students follow the procedure outlined above to make each reflection described in the table. After they have completed the chart and compared answers with neighbors, students are asked to find and describe in writing patterns in the coordinates of the transformed figures. With group discussion, these patterns are refined into the following rules:

reflection across x-axis: $(x,y) \rightarrow (x,-y)$

reflection across y-axis: $(x,y) \rightarrow (-x,y)$

We have discovered in working with our students that we must be consistent in expressing the reading and meaning of "-x" as "the opposite of x" and not "negative x." This is an area where confusion is very common among students.

We continue the study of reflections by reflecting figures across the line $y = x$. Before drawing ΔABC students are instructed to draw the line $y = x$ on a new sheet of graph paper. Students follow the same procedure described previously for making the reflections. On this step, we have found it beneficial for the students to mark the cross at the origin and also to trace part of the $y = x$ line on the patty paper. Table 2 is used with these reflections. The rule that the students discover for reflections across $y = x$ is: $(x,y) \rightarrow (y,x)$.

Translations – The next type of transformation that we study is translations. We define translations as "slides" and physically slide the patty paper across the graph paper, using the cross marked at the origin as a guide. Here we introduce the students to the concept of the translation vector, written in the form (T_x, T_y). We discuss the fact that a vector gives direction and distance, and those are shown by the sign and the value of each coordinate.

Table 3 gives the coordinates for the original triangle and its translations. (Notice that each translation is made from the original, as indicated by the single prime marking.)

The rule discovered at this point is: $(x, y) \rightarrow (x + T_x, y + T_y)$.

Rotations – The third kind of transformation in the unit is rotations about the origin. Again we define rotations in simple terms, as "turns." Students learn that rotations must be described in terms of direction and degrees. We begin with clockwise rotations and make rotations only in 90-degree increments. When doing the investigative activity, students place one pushpin through their patty paper at the origin, or center of rotation. They then physically turn the paper the required number of degrees and mark the new coordinates with the second pushpin. Table 4 provides working space for the clockwise rotations.

The rules discovered for clockwise rotations around the origin are:

90° rotation: $(x, y) \rightarrow (y, -x)$

180° rotation: $(x, y) \rightarrow (-x, -y)$

270° rotation: $(x, y) \rightarrow (-y, x)$

360° rotation: $(x, y) \rightarrow (x, y)$

We next work with counterclockwise rotations. Students soon discover that it is not necessary to actually draw all the figures indicated in Table 4 because they see the relationship between the counterclockwise and clockwise rotations. They find that rotating 90° counterclockwise is the same as 270° clockwise, and 270° counterclockwise is the same as 90° clockwise. They see that the results of 180° and 360° rotations are the same in either direction.

The rules discovered for counterclockwise rotations are:

90° rotation: $(x, y) \rightarrow (-y, x)$

180° rotation: $(x, y) \rightarrow (-x, -y)$

270° rotation: $(x, y) \rightarrow (y, -x)$

360° rotation: $(x, y) \rightarrow (x, y)$

Dilations – The last kind of transformation in this unit is a dilation, or size change. The investigation of dilations is handled a little differently from the other transformations. In Table 6 we give the students the coordinates for three rectangles, two of which are dilations of the first.

We ask students to compare the lengths of the sides of the rectangles they have drawn and the relationships between the coordinates of the vertices. We then discuss the scale factor, or magnitude (*k*), for each of the dilations.

The rule the students discover for dilations is: $(x,y) \to (kx,ky)$.

Practice and Assessment

After working through the discovery process and summarizing the rules developed, students then are given assignments in which they practice the different types of transformations. At the end of the unit they are given a written test over the transformations. We have also used an alternate assessment problem that allows a more open-ended assessment and creativity on the part of the students. That problem is shown in Figure 7.

Summary

This transformation unit has been a successful unit of study in our informal geometry classes. Because of the hands-on, investigative approach, students have a stake in the lessons and seem to internalize the concepts well. Students practice the important skill of communication in mathematical language. During the practice that follows the discovery lessons, students progress from using patty paper to physically model the transformations to the more abstract process of using the rules they have helped discover.

Reference

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics. Reston, Va.: NCTM, 2000.*

Table 1 – Reflections across the x-axis and the y-axis

Coordinates of ΔABC	Reflect across x-axis Coordinates of ΔA’B’C’	Reflect across y-axis Coordinates of ΔA’’B’’C’’	Reflect across x-axis Coordinates of ΔA’’’B’’’C’’’
A (5, 3)	A’ (,)	A’’ (,)	A’’’ (,)
B (9, 16)	B’ (,)	B“ (,)	B’’’ (,)
C (14, 8)	C’ (,)	C’’ (,)	C’’’ (,)

1. What pattern relating coordinates of pre-image points to image points do you observe on a reflection across the x-axis? the y-axis?

2. Write a rule which tells you how to take any point (x, y) and find its reflection image across the x-axis and across the y-axis. State your rules in words and in symbols.

Table 2 – Reflections across the line y = x

Coordinates of ΔABC	Coordinates of ΔA’B’C’		Coordinates of ΔDEF	Coordinates of ΔD’E’F’
A (-5, 5)	A’ (,)		D (3, 7)	D’ (,)
B (-1, 18)	B’ (,)		E (5, 12)	E’ (,)
C (4, 10)	C’ (,)		F (10, 3)	F’ (,)

3. What pattern relating coordinates of pre-image points to image points do you observe on a reflection across the line y = x?

4. Write a rule which tells you how to take any point (x, y) and find its reflection image across the line y = x. State your rule in words and in symbols.

Table 3 – Translations on the coordinate plane

Coordinates of ΔABC	Translation vector (9, 0) Coordinates of ΔA'B'C'	Translation vector (0, -12) Coordinates of ΔA'B'C'	Translation vector (3, -8) Coordinates of ΔA'B'C'	Translation vector (-7, 9) Coordinates of ΔA'B'C'
A (-7, 9)	A' (,)	A' (,)	A' (,)	A' (,)
B (7, 2)	B' (,)	B' (,)	B' (,)	B' (,)
C (-2, -6)	C' (,)	C' (,)	C' (,)	C' (,)

5. What pattern relating coordinates of pre-image points to image points do you observe on a translation with the vector (T_x, T_y)?

6. Write a rule which tells you how to take any point (x, y) and find its translation with the vector (T_x, T_y). State your rule in words and in symbols.

Table 4 – Clockwise rotations around the origin

Coordinates of ΔABC	Coordinates of ΔA'B'C' (Rotation of 90°)	Coordinates of ΔA'B'C' (Rotation of 180°)	Coordinates of ΔA'B'C' (Rotation of 270°)	Coordinates of ΔA'B'C' (Rotation of 360°)
A (3, 6)	A' (,)	A' (,)	A' (,)	A' (,)
B (8, 2)	B' (,)	B' (,)	B' (,)	B' (,)
C (12, 10)	C' (,)	C' (,)	C' (,)	C' (,)

7. What pattern relating coordinates of pre-image points to image points do you observe on a clockwise rotation of 90°, 180°, 270°, and 360°?

8. Write a rule which tells you how to take any point (x, y) and find its clockwise rotation of 90°, 180°, 270°, and 360°. State your rules in words and in symbols.

Table 5 – Counterclockwise rotations around the origin

Coordinates of ΔABC	Coordinates of ΔA'B'C' (Rotation of 90°)	Coordinates of ΔA'B'C' (Rotation of 180°)	Coordinates of ΔA'B'C' (Rotation of 270°)	Coordinates of ΔA'B'C' (Rotation of 360°)
A (3, 6)	A' (,)	A' (,)	A' (,)	A' (,)
B (8, 2)	B' (,)	B' (,)	B' (,)	B' (,)
C (12, 10)	C' (,)	C' (,)	C' (,)	C' (,)

9. What pattern relating coordinates of pre-image points to image points do you observe on a counterclockwise rotation of 90°, 180°, 270°, and 360°?

10. Write a rule which tells you how to take any point (x, y) and find its counterclockwise rotation of 90°, 180°, 270°, and 360°. State your rules in words and in symbols.

Table 6 – Dilation (size transformation) with magnitude k

Coordinates of rectangle ABCD	Coordinates of rectangle A'B'C'D'	Coordinates of rectangle A'B'C'D'
A (2, 6)	A' (4, 12)	A' (1, 3)
B (6, 6)	B' (12, 12)	B' (3, 3)
C (6, 12)	C' (12, 24)	C' (3, 6)
D (2, 12)	D' (4, 24)	D' (1, 6)

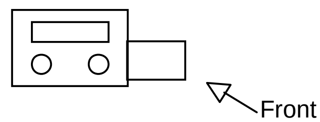
11. a. In each transformation above, what pattern relating pre-image points to image points do you observe?
    b. What comparison do you see between the lengths of the sides of ABCD and A'B'C'D'?

12. Write a rule which tells you how to take any point (x, y) and find its dilation with magnitude *k*. State your rule in words and in symbols.

Figure 7—Alternate Assessment for Transformations

On the attached graph paper is a drawing of a car and a garage. You are to “move” the car by a combination of transformations on the six outer vertices so that it will go into the garage. Each transformation must be explained and drawn on the graph paper. The following conditions must be met.

a) The car must “fit” in the garage, and the last transformation must put the car into the garage with the front of the car going in first.



b) You may use any of the transformations we have studied.

b) You may use as many transformations as you want, but more skill is required when using a smaller number of transformations.

c) You must describe each transformation and include terms such as the reflection line, the translation vector, and the magnitude of the dilation.

d) You must draw the location of the car after each transformation.

f) Only the six outer vertices of the car must be transformed. The inner details can be freely sketched.