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Derivation of Ohm's Law Using Quantum Mechanics Principles

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Objective

The purpose of this paper is to derive Ohm's Law using Quantum Mechanics Principles.

Approach

Ohm's Law will be derived in three different ways: a) using Classic Physics b) using Quantum Mechanics Schroedinger's Equation in an equivalent way to the Classic Physics solution and c) using Quantum Principles of Scattering Theory.

Note: Equation numbers, when given in square brackets [], correspond to the equations used in the 2 textbooks used as reference. The corresponding pages are included at the end of this document for convenience. Equations numbered (1.x) are derived as part of this paper.

a) Classic Physics Approach

The average velocity of the electrons in direction *x* (current flowing in the –*x* direction) is called the drift velocity, v_{dx}. At a given time, the current density J_x depends on the drift velocity and the applied electric field. This relation is expressed as:

$$J_x(t) = env_{dx}(t) \qquad [2.2]$$

The drift mobility, μ_d, is defined as

$$\mu_d = \frac{e\tau}{m_e} \qquad [2.5]$$

where e is the electron charge, m_e is the mass of the electron and τ is the mean scattering time. Relating the drift velocity in terms of the drift mobility, we have:

$$v_{dx} = \frac{e\tau}{m_e} E_x \qquad [2.3]$$

which is equivalent to:

$$v_{dx} = \mu_d E_x \qquad [2.4]$$

where E_x is the electric field in the x direction. Replacing [2.4] in [2.2] we obtain:

$$J_x = en\mu_d E_x \qquad [2.6]$$

Therefore, the current density is proportional to the electric field and the conductivity σ is the term multiplying E_x, that is,

$$\sigma = en\mu_d \tag{2.7}$$

Replacing [2.7] in [2.6] we have:

$$J_x = \sigma E_x$$

Since resistivity ρ is the inverse of the conductivity σ, that is, $\rho = 1/\sigma$, we obtain

$$J_x = \frac{1}{\rho} E_x$$

If we have a conductor (or resistor) of cross-section area, *A*, and length *L*, we multiply the current density *J* by the cross-section area *A* to obtain the current *I*, multiply the resistivity ρ by the length *L* to obtain the resistance *R*, and multiply the electric field *E* by the length *L* to obtain the voltage *V*, as follows:

$$J_x A = \frac{1}{\rho L} E_x L$$

$$I = \frac{1}{R} V$$

or simply $V = IR$ which is the Ohm's Law.

b) Quantum Mechanics Approach: Schroedinger's Equation

In this approach we model the electron as a wave, not as a particle. More exactly, we model the electron as a wavefunction using Schroedinger's Equation. Since the solutions to Schroedinger's Equation behave like traveling waves, we can obtain the equivalent translation velocity of the wave in the *x* direction, assuming that the electron is moving in that direction under the influence of an electric field in the same direction *x*.

Since the electrons are expected to move in only one direction in average (scattering actually makes electrons move in all directions), we can use the time-independent one dimension Schroedinger's Equation:

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V)\psi = 0 \tag{3.11a}$$

where E is the free electron potential energy and V is the potential energy due to the electric field, in this particular case.

For an electron moving under the effect of an externally applied electric field, the free electron potential E can be neglected, since most of the energy of the electron is due to the electric field. Therefore,

$$\frac{d^2\psi}{dx^2} - \frac{2m}{\hbar^2}V\psi = 0$$

or

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0$$

where $k^2 = -\frac{2m}{\hbar^2}V$. Solving the differential equation we obtain:

$$\psi(x) = Ae^{jkx} \qquad \text{or} \qquad \psi(x) = Be^{-jkx}$$

The total wavefunction is obtained by multiplying $\psi(x)$ by $e^{-jEt/\hbar}$. We can define a fictitious frequency for the electron as $\omega = E/\hbar$ and multiply $\psi(x)$ by $e^{-j\omega t}$:

$$\psi(x) = Ae^{j(kx-\omega t)} \qquad \text{or} \qquad \psi(x) = Be^{-j(kx+\omega t)}$$

Each of these is a traveling wave. The first solution is a traveling wave in the +*x* direction and the second one is in the –*x* direction. Thus, the electron has a traveling wave solution with a wavenumber $k = 2\pi/\lambda$. The energy of the electron is its kinetic energy:

$$KE = \frac{(\hbar k)^2}{2m}$$

but the kinetic energy of the electron as a particle is:

$$\frac{1}{2}mv_{dx}^2$$

therefore,

$$\frac{(\hbar k)^2}{2m} = \frac{1}{2}mv_{dx}^2$$

from which we can derive the equivalent *quantum drift velocity*:

$$v_{dx} = \frac{\hbar k}{m} \qquad \textbf{(1.0)}$$

recalling that

$$J_x = en\mu_d E_x = en\frac{e\tau}{m}E_x \qquad \text{[2.5] and [2.6]}$$

and

$$J_x = env_{dx} = en\frac{\hbar k}{m}\beta E_x \qquad \text{[2.2] and (1.0)}$$

where β is a constant that depends on the crystal structure, impurities and defects. We conclude that the *quantum mean scattering time* is

$$\tau_q = \frac{\hbar k}{e}\beta \qquad \textbf{(1.1)}$$

and similarly, the *quantum drift mobility* is

$$\mu_{qd} = \frac{e\tau_q}{m_e} = \frac{e\frac{\hbar k\beta}{e}}{m_e} = \frac{\hbar k\beta}{m_e} \qquad \text{[2.2] and (1.1)}$$

and similarly, as in the Classic Physics approach, the current density is proportional to the electric field and the conductivity σ is the term multiplying E_x, that is,

$$\sigma = en\mu_{qd} \qquad \text{[2.7]}$$

Replacing [2.7] in [2.6] we have:

$$J_x = \sigma E_x$$

Since resistivity ρ is the inverse of the conductivity σ, that is, $\rho = 1/\sigma$, we obtain

$$J_x = \frac{1}{\rho}E_x$$

If we have a conductor (or resistor) of cross-section area, *A*, and length *L*, we multiply the current density *J* by the cross-section area *A* to obtain the current *I*, multiply the resistivity ρ by the length L to obtain the resistance R, and multiply the electric field E by the length L to obtain the voltage V, as follows:

$$J_x A = \frac{1}{\rho L} E_x L$$

$$I = \frac{1}{R} V$$

or simply $V = IR$ which is the Ohm's Law.

c) Quantum Mechanics Approach: Scattering Theory

Another way to derive Ohm's Law is by assuming that the electrons are scattered by a Coulomb potential. Typically, a beam of incident particles is directed towards a target, and the scattered particles are collected (detected) and counted at various angular locations. Quantum mechanically, the particle trajectories are replaced, at best, by probabilistic wavepacket motion, but variations of the probability of scattering at different angles and energies still gives information on the nature of the scattering potential.

In a three-dimensional scattering experiment, a given intensity of incident articles,

$$J_{inc}^{(3)}(r,t) \equiv \frac{dN_{inc}}{dtdA} \qquad [12.1]$$

that is, the number of particles incident on a target per unit time per unit area, can be directly associated with the probability flux, defined in three dimensions via:

$$J(r,t) = \frac{\hbar}{2mi}\left[\psi^*(r,t)\nabla\psi(r,t) - \nabla\psi^*(r,t)\psi(r,t)\right] \qquad [12.2]$$

In this particular project, J corresponds to the electrical current density due to a potential V, and the particles are electrons being scattered by the crystal structures of the conductor (or resistor).

For a right or left moving plane wave of the form

$$\psi(x) = Ae^{j(\pm kx - \hbar k^2 t/2m)} \qquad [12.8]$$

the probability flux is given by

$$J(x,t) = \pm\frac{\hbar k}{m}|A|^2 \qquad [12.9]$$

The simplest one-dimensional potential that can be investigated analytically corresponds to a step potential, V, defined as

$$V(x) = \begin{cases} 0 & \text{for } x < 0 \\ V_0 & \text{for } x > 0 \end{cases} \qquad [12.10]$$

Considering a particle with energy $E > V_0 > 0$, the wave function is,

$$\psi(x) = \begin{cases} Ie^{ikx} + \mathrm{Re}^{-ikx} & \text{for } \mathrm{x} < 0 \\ Te^{iqx} + Se^{-iqx} & \text{for } x > 0 \end{cases} \qquad [12.11]$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}} \qquad \text{and} \qquad q = \sqrt{\frac{2m(E - V_0)}{\hbar^2}} \qquad [12.12]$$

For this particular problem, we are only interested in the wavepacket solution for x>0 and a traveling wave in only one direction. Therefore, from [12.11] we need that S=0 (the reflected wave) and only T (the transmitted wave) will be part of the solution.

This gives,

$$\psi(x) = Te^{iqx} = Te^{i\sqrt{\frac{2m(E-V_0)}{\hbar^2}}x} \qquad \mathbf{(1.2)}$$

Going back to equation [12.9] that gives us the probability flux

$$J(x,t) = \pm\frac{\hbar k}{m}|A|^2 \qquad [12.9]$$

we can see that by multiplying J by the electric charge e, we obtain the electric current I, that is

$$J(x,t)e = \pm\frac{\hbar k}{m}e|A|^2$$

$$I = \pm \frac{\hbar k}{m} e|A|^2 \quad \textbf{(1.3)}$$

The $\pm$ can be eliminated since we are only interested in electrons moving in one direction. Also, we can see that the term $\hbar k / m$ is the wave velocity, which in this case corresponds to the drift velocity. The term $|A|^2$ is directly proportional to the magnitude of the electric field and for that matter to the potential V. Equation 1.3 can therefore be rewritten as

$$I = \frac{\hbar k}{m} e\beta V \quad \textbf{(1.4)}$$

where β is a constant that relates the potential V to the maximum probability of $\psi(x)$ (its amplitude T in Equation 1.2). From here we can conclude that the conductance is

$$G = \frac{\hbar k}{m} e\beta \quad \textbf{(1.5)}$$

or that the resistance is

$$R = \frac{m}{\hbar k e\beta} \quad \textbf{(1.6)}$$

and replacing in equation 1.4 we have

$I = \frac{1}{R}V$ or $V = IR$ which is the Ohm's Law

CONCLUSION

The Ohm's Law was derived using both Classic and Quantum Physics principles, which proves the usefulness of modeling an electron both as a particle and as a wave.

REFERENCES

[1] Principles of Electronic Materials and Devices, Second Edition, Textbook by S.O.Kasap, McGraw-Hill, 2002. ISBN 0-07-239342-4

[2] Quantum Mechanics, Texbook by Richard W Robinett, 1997. Oxford University Press. ISBN 0-09-509202-3