Research

International Investment

Arnut Paothong
ID 4104611910

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Department of Economics
Thammasat University

**** The origin paper is Thai

Contents

Preface

Because of limited capital of investor, he can't invest freely into each country. This paper will answer the question "How much do investor separate his capital into each country?" To separate it, he should understand "What are the factors for his decision?"

This paper will show the direction of changing capital when some factors changed. I will use the tools of Mathematics, called comparative static analysis.

I parted this paper into five parts

Chapter 1 Introduction, Objective, Restriction and Assumption

Chapter 2 The optimal ratio of capital: case of two countries and two factor of production

Chapter 3 The change of the ratio of investment

Chapter 4 The optimal ratio of capital: case of n countries and k factors of production

Chapter 5 Conclusion

Variable Definition

TC Total capital money of investment

TC_i Capital money of investment in country i

γ_i Ratio of investment in the i^{th} country for $\gamma_i = \frac{TC_i}{TC}$ and $0 < \gamma_i < 1$

X_i^P Quantity of goods that the i^{th} country will supply

X_i^S Quantity of goods that the i^{th} country demanded

X_{ij}^E Quantity of goods, exported from the i^{th} country to the j^{th} country

ε_{ij} Ratio between X_{ij}^E and X_i^P by $\varepsilon_{ij} = \frac{X_{ij}^E}{X_i^P}$ and $0 < \varepsilon_{ij} < 1$

TrC_{ij} Transaction cost for exporting from the i^{th} country to the j^{th} country

t_{ij} Transaction cost per unit by $t_{ij} = \frac{TrC_{ij}}{X_{ij}^E}$

P_i^X Price of goods in the i^{th} country

TR_i Total revenue from the i^{th} country by $TR_i = P_i^X X_i^S - \sum_{\substack{j \neq i \\ j=1}}^{n} TrC_{ij}$

B_i Marginal revenue by $B_i = \frac{\partial TR}{\partial X_i^P}$

Z_i^k The k^{th} of the factor in the i^{th} country

P_i^{Zk} Price of Z_i^k

$X_i^P = f_i(Z_i^1, Z_i^2, \ldots, Z_i^k)$ Production function of the i^{th} country by

$$f_i(Z_i^1, Z_i^2, \ldots, Z_i^k) = A_i Z_i^{1\alpha 1_i} \cdot Z_i^{2\alpha 2_i} \cdot \ldots \cdot Z_i^{k\alpha k_i} = A_i \prod_{k=1}^{k} Z_i^{k\alpha k_i}$$

A_i Coefficient of production in the i^{th} country

αk_i Coefficient of Z_i^k of production in the i^{th} country

Chapter 1 Introduction

In the past, the frequent questions of investors were "What do we invest?" and "How much could we invest?" in order to get maximum profit. In the fact that, there was no the question "Which country should we invest?" because they didn't know the factors in other country and couldn't move their money into some countries.

Unlike in the present, the technology is developed rapidly. Because of that, the capital can move in anywhere they want.

This paper will show the factors those they should concern and consider the changing of ratio of investment when some factors changed.

1.1 Objective

1. To know the factors those determine the ratio of investment.
2. To know how the ratio of investment change.

1.2 Restriction

1. There are only one goods for investment.
2. Use Cobb-Douglas function in production function.

1.3 Assumption

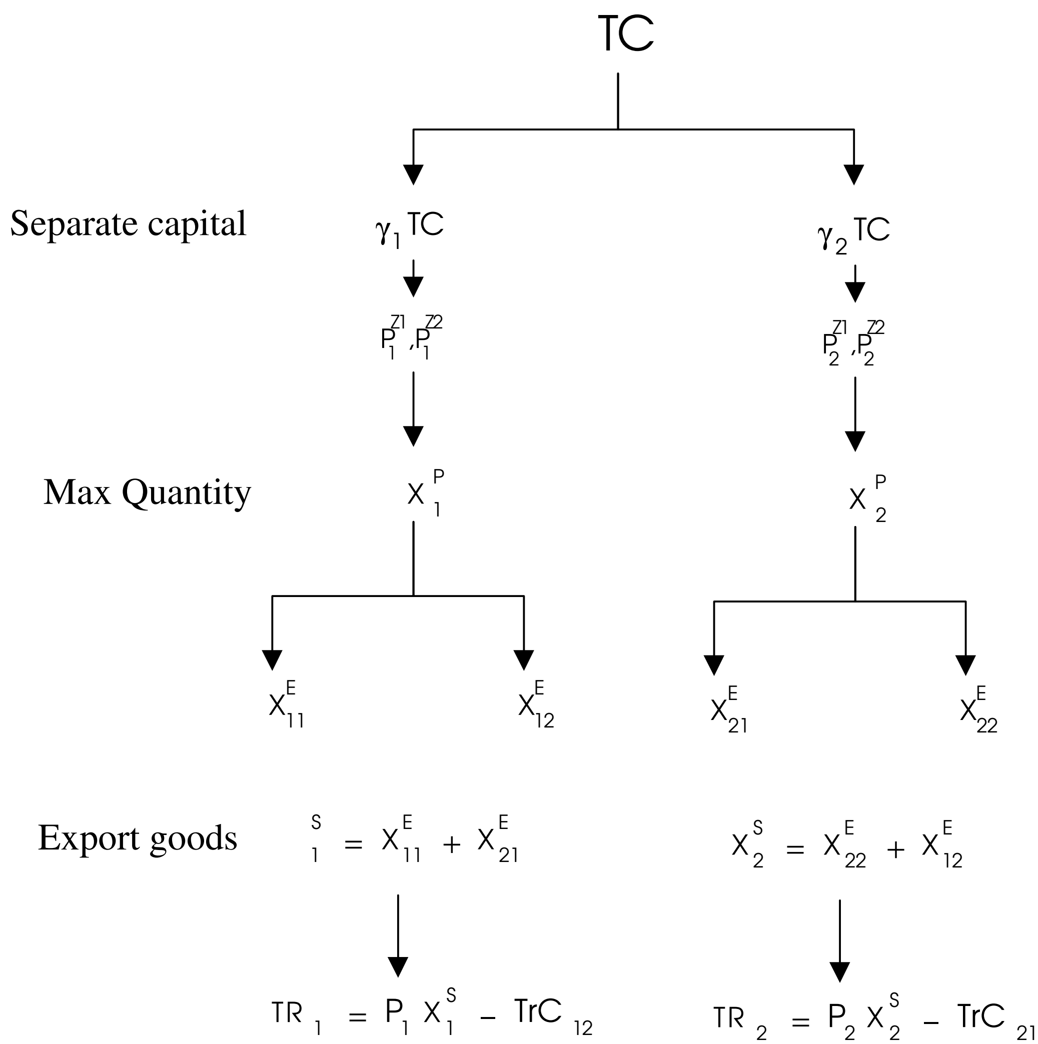
1. Perfect capital mobility in every country
2. Constant technology
3. Can mobile the factors freely in the country but can't mobile to other countries
4. The investor is a price taker.
5. The transaction cost depends on the amount of goods.

Chapter 2 2x2 Model

In this chapter, I will create the model simply by 2x2 Model. The investor has only two countries and two factors of production. I parted this chapter into five sections. Those are

Section 1 Separating the capital money
Section 2 The quantity of produced goods in two countries
Section 3 Exporting the goods form the first country to the second country
Section 4 Net total revenue in two countries
Section 5 The optimal ratio of investment for two countries

Figure 2.1 Investing chart of 2x2 model



Section 1 Separating the capital money

From the figure 2.1, the investor has a limited capital, TC , and want to produce goods in each country. He will separate his money, TC_1 and TC_2 by $TC_i = \gamma_i TC$, $i \in \{1,2\}$ and $\gamma_1 + \gamma_2 = 1$.

Section 2 The quantity of produced goods in two countries

Producing the goods in each country, He faces the limited capital, called $TC_i = \gamma_i TC$ and the two factors of production, called Z_i^1, Z_i^2. And the price of them was P_i^{Z1}, P_i^{Z2} respectively. He wants to maximize quantity of production. Thus, for $i \in \{1,2\}$

Capital	$TC_i = \gamma_i TC$
Production function	$X_i^P = f_i(Z_i^1, Z_i^2) = A_i Z_i^{1\alpha 1_i} Z_i^{2\alpha 2_i}$
Price of factors	P_i^{Z1}, P_i^{Z2}

From the above, We've got a problem of optimization of one constrain.

Max $X_i^P = f_i(Z_i^1, Z_i^2) = A_i Z_i^{1\alpha 1_i} Z_i^{2\alpha 2_i}$ s.t. $\gamma_i TC = P_i^{Z1} Z_i^1 + P_i^{Z2} Z_i^2$

From Appendix A.1 in equation (A.1.5), (A.1.6) and (A.1.7) we've got.

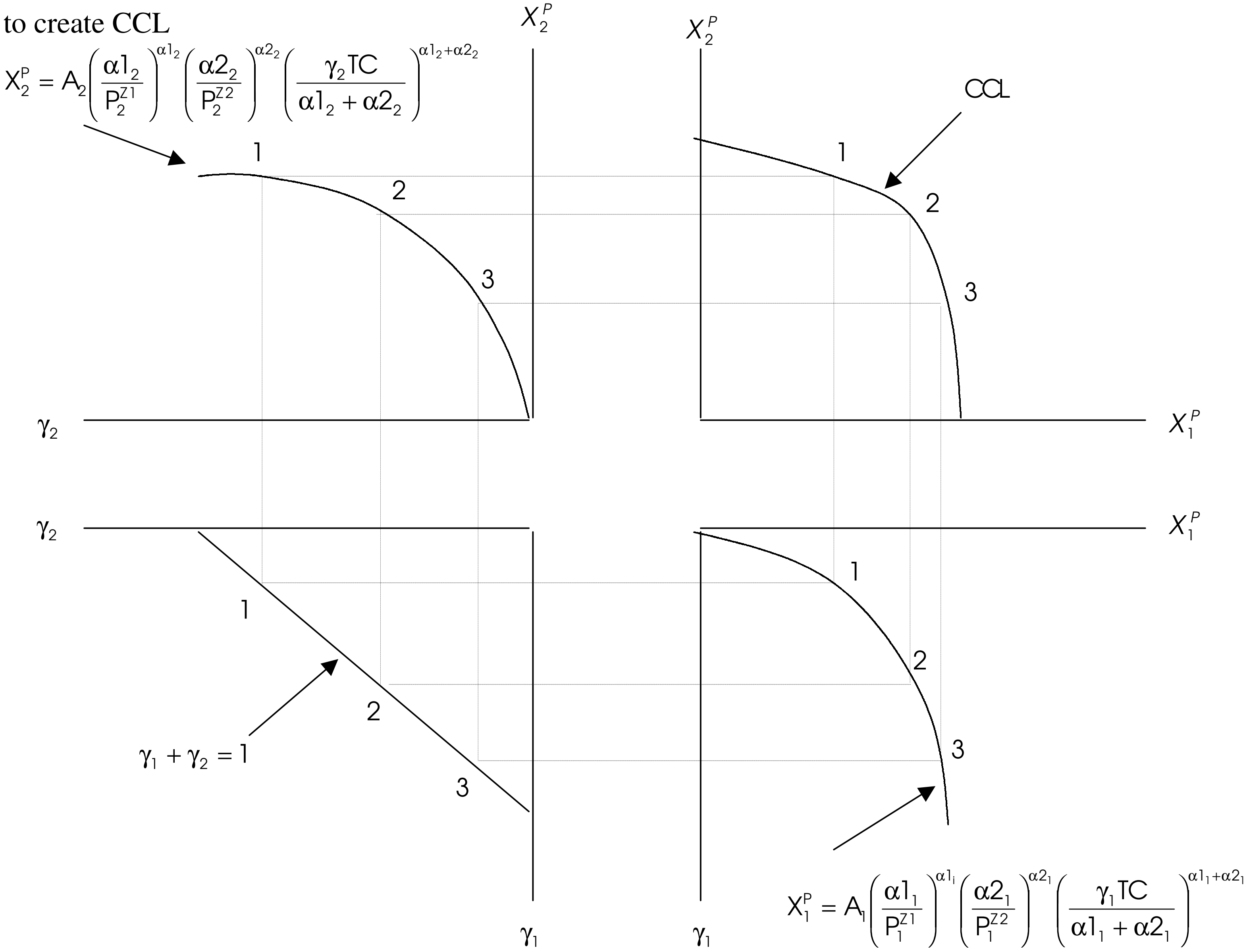
$$Z_i^1 = \left(\frac{\alpha 1_i}{P_i^{Z1}}\right)\left(\frac{\gamma_i TC}{\alpha 1_i + \alpha 2_i}\right)$$

$$Z_i^2 = \left(\frac{\alpha 2_i}{P_i^{Z2}}\right)\left(\frac{\gamma_i TC}{\alpha 1_i + \alpha 2_i}\right)$$

$$X_i^P = A_i \left(\frac{\alpha 1_i}{P_i^{Z1}}\right)^{\alpha 1_i} \left(\frac{\alpha 2_i}{P_i^{Z2}}\right)^{\alpha 2_i} \left(\frac{\gamma_i TC}{\alpha 1_i + \alpha 2_i}\right)^{\alpha 1_i + \alpha 2_i}$$

From this result, we can create the curve, supposed Constant Cost Line (CCL). It represents the relation of amount of goods in two countries (X_1^P, X_2^P) when the capital is constant. From Appendix A.2 in equation (A.2.6), it has negative slope and concave to origin. See the figure 2.2

Figure 2.2 How to create CCL



Section 3 Exporting the goods from the first country to the second country

From the section 2, through, we've got the optimum quantity of each country (X_i^P), that amount isn't sold in owned country. The investor has to export/import the goods to/from the others because the demand of each country may be different from the supply of it.

Suppose the amount of export from the i th country to the j th country by X_{ij}^E and $X_{ij}^E = \varepsilon_{ij} X_i^P$. In fact that, the exporter must expend transaction cost such as shipping, tariff, etc. I denote the transaction cost per unit, t_{ij}. Thus $TrC_{ij} = t_{ij} X_{ij}^E$

The quantity of demand of each country is parted two parts, the former is from owned country, and the latter is from the others. The same idea, the quantity of supply of each country is parted two parts, the former is from owned country, and the latter is from the others. That is

$$X_i^P = X_{ii}^E + X_{ij}^E = (1-\varepsilon_{ij})X_i^P + \varepsilon_{ij} X_i^P$$

$$X_i^S = X_{ii}^E + X_{ji}^E = (1-\varepsilon_{ij})X_i^P + \varepsilon_{ji} X_j^P$$

Section 4 Net total revenue in two countries

This section will calculate about total revenue when the investor faces price of goods, P_i^X. Then the total revenue from the i^{th} country is

$$TR_i = P_i^X X_i^S - TrC_{ij} = P_i\{(1-\varepsilon_{ij})X_i^P + \varepsilon_{ji} X_j^P\} - t_{ij}\varepsilon_{ij} X_i^P$$

From $TR = TR_1 + TR_2$

$$TR = P_1\{(1-\varepsilon_{12})X_1^P + \varepsilon_{21} X_2^P\} - t_{12}\varepsilon_{12} X_1^P + P_2\{(1-\varepsilon_{21})X_2^P + \varepsilon_{12} X_1^P\} - t_{21}\varepsilon_{21} X_2^P$$

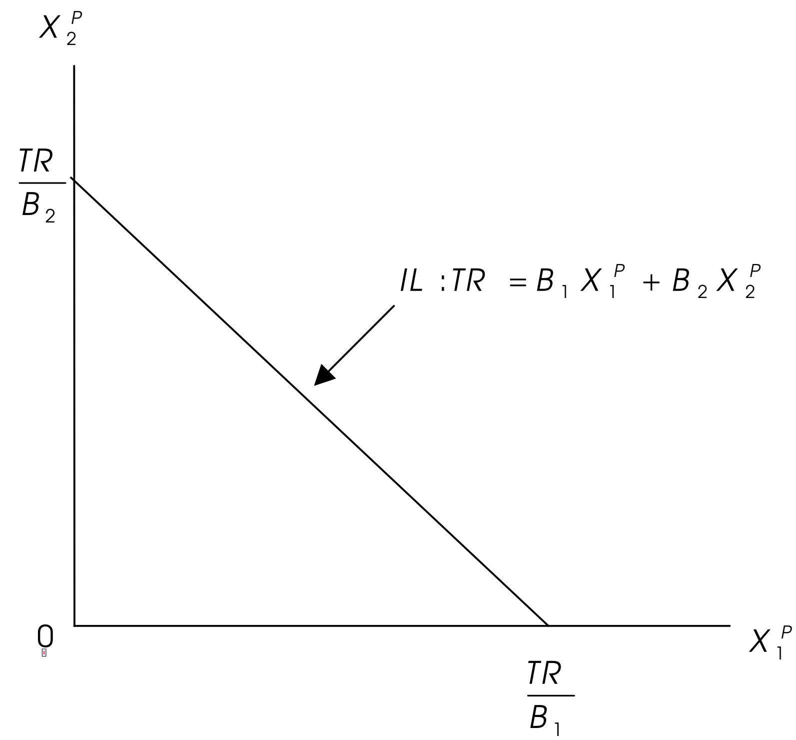
$$TR = \{P_1(1-\varepsilon_{12}) + \varepsilon_{12}(P_2 - t_{12})\}X_1^P + \{P_2(1-\varepsilon_{21}) + \varepsilon_{21}(P_1 - t_{21})\}X_2^P$$

From $B_i = \dfrac{\partial TR}{\partial X_i^P} = P_i(1-\varepsilon_{ij}) + \varepsilon_{ij}(P_j - t_{ij})$

Then $TR = B_1 X_1^P + B_2 X_2^P$

Like the CCL, we can create the curve, supposed Iso Revenue Line (IL). It represents the relation of amount of goods in two countries (X_1^P, X_2^P) what the investor receives the equal revenue. It is the line that has the intercept, $\dfrac{TR}{B_1}, \dfrac{TR}{B_2}$, see the figure 2.3.

<u>Figure 2.3</u> Iso Revenue Line (IL)



<u>Section 5</u> The optimal ratio of investment for two countries

From the section 4, we found that the difference of ratio makes the difference of revenue. It means that the investor has to decide the optimum ratio what makes maximal profit.

From $\pi = B_1 X_1^P + B_2 X_2^P - TC$

Max π s.t. $\gamma_1 + \gamma_2 = 1$

From Appendix A.3 in equation (A.3.5)

$$\gamma_i = \left(1 + \frac{B_j}{B_i} \frac{\alpha 1_j + \alpha 2_j}{\alpha 1_i + \alpha 2_i} \cdot \frac{X_j^P}{X_i^P}\right)^{-1}$$

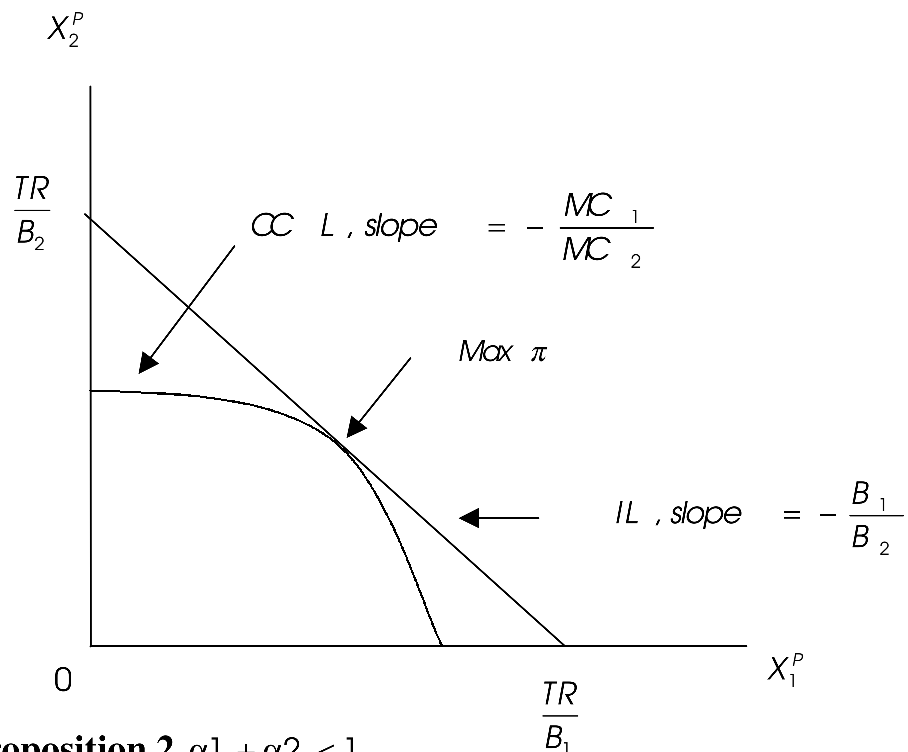
Then, we've got the function of the optimum ratio of investment in the term of many factors. Because of the maximization method, we've got two interested propositions. From Appendix A.4

<u>Proposition 1</u> $\frac{MC_1}{B_1} = \frac{MC_2}{B_2}$

Because of the first order condition of maximization in Appendix A.4 in equation (A.4.2), we've got the well-known condition, $\frac{MC_1}{B_1} = \frac{MC_2}{B_2}$. It's the necessary condition for maximal profit. To get the maximal profit, the investor should separate his money like this condition.

According to this condition, in figure 2.4 we can illustrate you with the curve of CCL and IL. The slopes of CCL and IL are $-\frac{\frac{\partial X_2^P}{\partial TC_2}}{\frac{\partial X_1^P}{\partial TC_1}} = -\frac{MC_1}{MC_2}$ and $-\frac{B_1}{B_2}$, respectively. Then, the point that their slopes are equal can solve the optimal produced goods.

Figure 2.4 The optimal produced goods in each country that makes the maximal profit



Proposition 2 $\alpha 1_i + \alpha 2_i < 1$

Because of the second order condition in Appendix A.5 in equation (A.5.4), we've got the other condition, $\alpha 1_i + \alpha 2_i < 1$. It's the sufficient condition for maximal profit. To hold this condition, the production function of each country must be strict character, called decreasing return to scale. If the characteristic of production function of any country isn't hold, the investor will move all of his capital into that country.

If this condition is true, the CCL will be stern character, call strictly concave to origin. Unless it is hold, the CCL will not be strictly concave to origin. The equilibrium may be multi equilibrium, corner solution or both.

Chapter 3 The Change of Investment

In this chapter, we will look over the effect of changing of some factors. We consider only four factors. Those are

1. The price of the first factor of production of the first country
2. The total capital
3. The transaction cost per unit from the first country to the second one
4. The price of goods of the first country

From Appendix A.6 in equation (A.6.1) and (A.6.2)

$$d\gamma_1 = -\left(1 + \frac{B_2}{B_1}\frac{\alpha 1_2 + \alpha 2_2}{\alpha 1_1 + \alpha 2_1} \cdot \frac{X_2^P}{X_1^P}\right)^{-2} dW_1$$

3.1 The price of the first factor of production of the first country

From Appendix A.6 in equation (A.6.4)

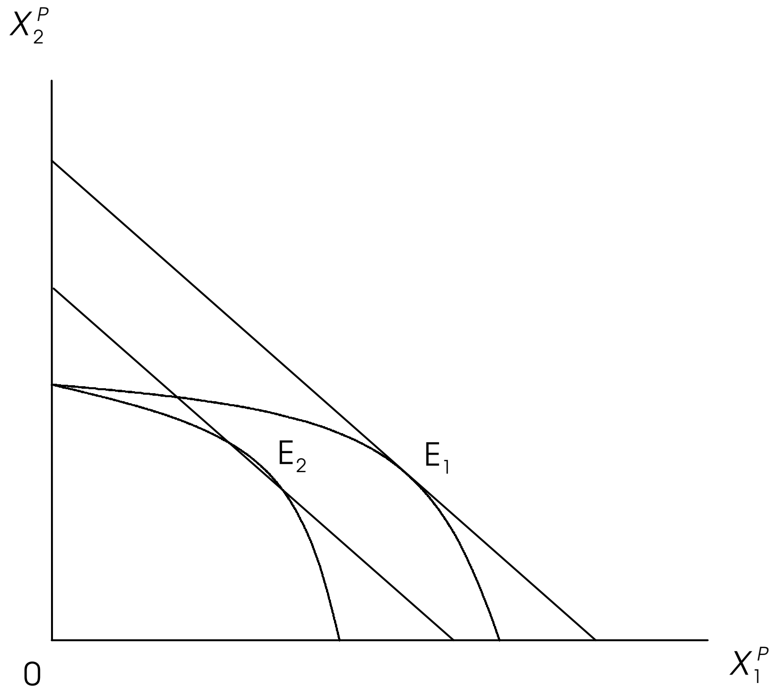
$$\frac{\partial \gamma_1}{\partial P_1^{Z1}} = -\left(1 + \frac{B_2}{B_1}\frac{\alpha 1_2 + \alpha 2_2}{\alpha 1_1 + \alpha 2_1} \cdot \frac{X_2^P}{X_1^P}\right)^{-2} \frac{\partial W_1}{\partial P_1^{Z1}}$$

$$\frac{\partial \gamma_1}{\partial P_1^{Z1}} = -\alpha 1_1 \frac{W_1}{P_1^{Z1}}\left(1 + \frac{B_2}{B_1}\frac{\alpha 1_2 + \alpha 2_2}{\alpha 1_1 + \alpha 2_1} \cdot \frac{X_2^P}{X_1^P}\right)^{-2} < 0$$

Because of above equation, If the price of the first factor of production of the first country, P_1^{Z1}, increases then its marginal cost, MC_1, will increase. Then, the investor will decrease the level of production by moving his capital from the first country to the second country. Moving his capital, the investor will increase the level of production in the second country, and then, the marginal cost of the second country increase, MC_2, too. From this changing, he can hold the first order condition.

Consider the figure 3.1, increasing of P_1^{Z1} affect the CCL, rotated clockwise because level of production in the first country decrease. From that, we've got a new equilibrium, E_2.

Figure 3.1 Effect of the increasing of P_1^{Z1}



3.2 The total capital

From Appendix A.6 in equation (A.6.6)

$$\frac{\partial \gamma_1}{\partial TC} = -\left(1 + \frac{B_2}{B_1}\frac{\alpha 1_2 + \alpha 2_2}{\alpha 1_1 + \alpha 2_1} \cdot \frac{X_2^P}{X_1^P}\right)^{-2} \frac{\partial W_1}{\partial TC}$$

$$\frac{\partial \gamma_1}{\partial TC} = -(\alpha 1_2 + \alpha 2_2 - \alpha 1_1 - \alpha 2_1)\frac{W_1}{TC}\left(1 + \frac{B_2}{B_1}\frac{\alpha 1_2 + \alpha 2_2}{\alpha 1_1 + \alpha 2_1} \cdot \frac{X_2^P}{X_1^P}\right)^{-2}$$

From the equation above, adding his capital, TC, the investor should separate this added money into two countries. This addendum may affect the old optimum ratio. This result depends on the first term of above equation, $-(\alpha 1_2 + \alpha 2_2 - \alpha 1_1 - \alpha 2_1)$.

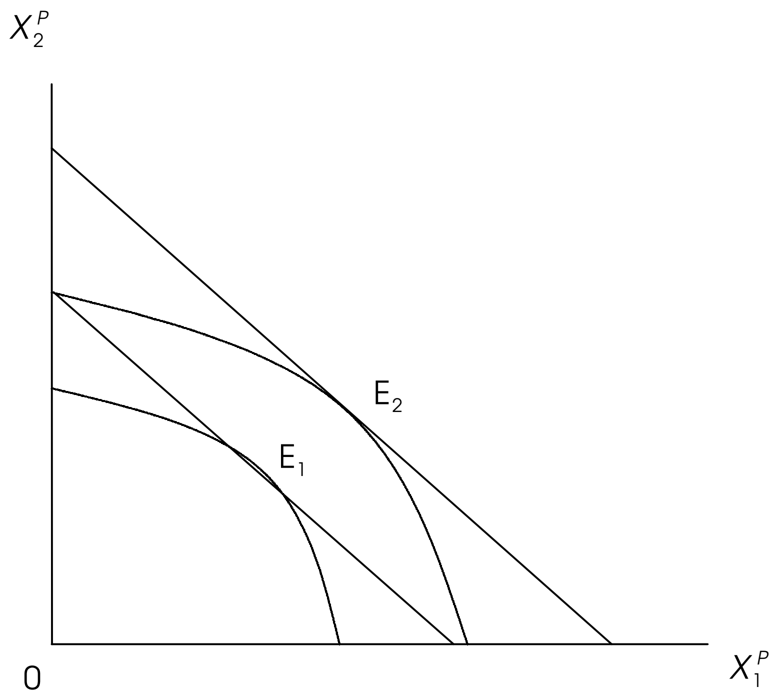
Because of the characteristic of Cobb-Douglas function, the term, $\alpha 1_i + \alpha 2_i$, is the homogeneous degree (HD) of the i^{th} country. Then, the new optimal ratio depends on HD_i of each country. We can consider in three cases.

- If $HD_1 = HD_2$ then $\frac{\partial \gamma_1}{\partial TC} = 0$. The optimum ratio doesn't change.
- If $HD_1 < HD_2$ then $\frac{\partial \gamma_1}{\partial TC} < 0$. The optimum ratio will change because the old ratio doesn't hold the first order condition. Because of $\frac{\partial \gamma_1}{\partial TC} < 0$, the investor will move his money from the first country into the second country.

- If $HD_1 > HD_2$ then $\frac{\partial \gamma_1}{\partial TC} > 0$. The optimum ratio will change, too. The direction of changing is opposite of the upper case

Consider the figure 3.2. The adding investor's capital will affect the CCL, shifted in the northeast because he can produce the goods more. From that, we've got a new equilibrium, E_2.

Figure 3.2 the increasing of total capital



3.3 The transaction cost per unit from the first country to the second one

From Appendix A.6 in equation (A.6.8)

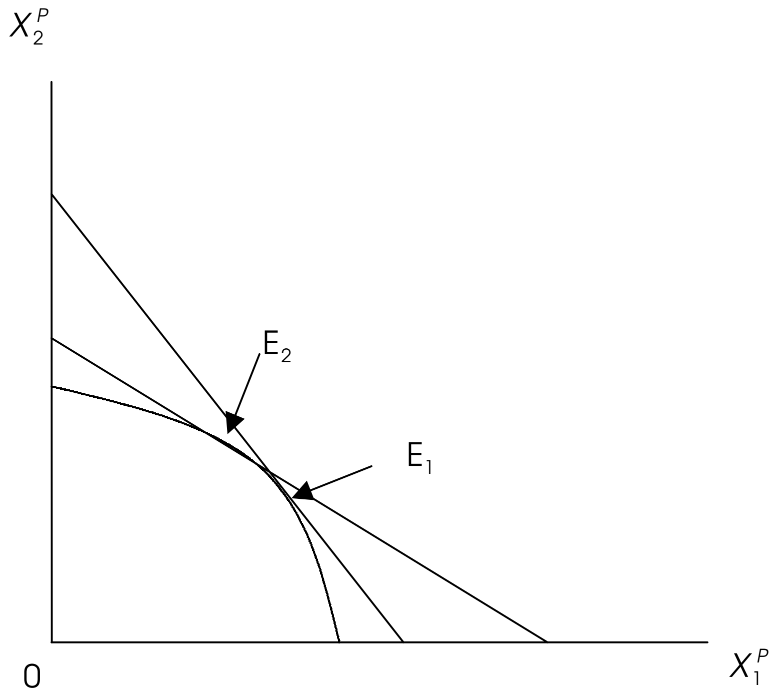
$$\frac{\partial \gamma_1}{\partial t_{12}} = -\left(1 + \frac{B_2}{B_1}\frac{\alpha 1_2 + \alpha 2_2}{\alpha 1_1 + \alpha 2_1} \cdot \frac{X_2^P}{X_1^P}\right)^{-2} \frac{\partial W_1}{\partial t_{12}}$$

$$\frac{\partial \gamma_1}{\partial t_{12}} = -\frac{\alpha 1_j + \alpha 2_j}{\alpha 1_i + \alpha 2_i} \cdot \frac{X_j^P}{X_i^P}\left(\frac{B_i \cdot 0 + B_j \varepsilon_{ij}}{B_i^{\;2}}\right)\left(1 + \frac{B_2}{B_1}\frac{\alpha 1_2 + \alpha 2_2}{\alpha 1_1 + \alpha 2_1} \cdot \frac{X_2^P}{X_1^P}\right)^{-2} < 0$$

From above equation, we found that the increasing of transaction cost per unit affects the marginal benefit, B_1, decrease. Then, the investor will decrease the level of production by moving his capital from the first country to the second country. Moving his capital, the investor will increase the level of production in the second country, and then, the marginal cost of the first country decrease, MC_1, too. From this changing, he can hold the first order condition again.

See the figure 4.3. The increasing of transaction cost will affect the slope of IL, decrease. From that, we've got a new equilibrium, E_2.

Figure 3.3 the increasing of transaction cost per unit



3.4 The price of goods of the first country

From Appendix A.6 in equation (A.6.7)

$$\frac{\partial \gamma_1}{\partial P_1} = -\left(1 + \frac{B_2}{B_1}\frac{\alpha 1_2 + \alpha 2_2}{\alpha 1_1 + \alpha 2_1} \cdot \frac{X_2^P}{X_1^P}\right)^{-2} \frac{\partial W_1}{\partial P_1}$$

$$\frac{\partial \gamma_1}{\partial P_1} = -\frac{\alpha 1_2 + \alpha 2_2}{\alpha 1_1 + \alpha 2_1} \cdot \frac{X_2^P}{X_1^P}\left(\frac{B_1\varepsilon_{12} - B_2(1-\varepsilon_{12})}{B_1^{\ 2}}\right)\left(1 + \frac{B_2}{B_1}\frac{\alpha 1_2 + \alpha 2_2}{\alpha 1_1 + \alpha 2_1} \cdot \frac{X_2^P}{X_1^P}\right)^{-2}$$

From above equation, we found that the increasing of price of goods in the first country, P_1, doesn't temp the investor invest more in the first country but he has to consider the other factors such as the ratio of export, ε_{ij}, the marginal benefit, B_i, etc.

Chapter 4 n x k Model

This chapter will expand the idea in chapter 2, 2x2 model, into the general model, nxk model. I will add new country and factor of production. I parted this chapter in six sections. Those are

Section 1 Separating the capital money

See the figure 4.1. The investor has limited capital, TC, and want to produce goods in each country. He will separate his money into n parts, $TC_i = \gamma_i TC$ by $i \in \{1,2,3,\ldots,n\}$ and $\sum_{i=1}^{n} \gamma_i = 1$.

Section 2 The quantity of produced goods in the i^{th} country

Producing the goods in each country, he faces the limited capital, $TC_i = \gamma_i TC$ and the k factors of production, called $Z_i^1, Z_i^2, \ldots, Z_i^K$. And the price of them was $P_i^{Z1}, P_i^{Z2}, \ldots, P_i^{ZK}$ respectively. He wants to maximize quantity of production. Thus, for $i \in \{1,2,3,\ldots,n\}$

Capital	$TC_i = \gamma_i TC$
Production function	$X_i^P = f_i(Z_i^1, Z_i^2, \ldots, Z_i^K) = A_i \prod_{k=1}^{K} Z_i^{K\,\alpha K_i}$
Price of factors	$P_i^{Z1}, P_i^{Z2}, \ldots, P_i^{ZK}$

From the above, We've got a problem of optimization of one constrain.

Max $X_i^P = f_i(Z_i^1, Z_i^2, \ldots, Z_i^K) = A_i \prod_{k=1}^{K} Z_i^{1\,\alpha K_i}$ s.t. $\gamma_i TC = \sum_{k=1}^{K} P_i^{ZK} Z_i^K$

From Appendix B.1 in equation (B.1.4) and (B.1.5) we've got.

$$Z_i^K = \left(\frac{\alpha K_i}{P_i^{ZK}}\right)\left(\frac{\gamma_i TC}{\sum \alpha K_i}\right)$$

$$X_i^P = A_i \prod_{k=1}^{K} \left(\frac{\alpha K_i}{P_i^{ZK}}\right)^{\alpha K_i} \left(\frac{\gamma_i TC}{\sum \alpha K_i}\right)^{\sum \alpha K_i}$$

Section 3 Exporting the goods form the i^{th} country to the j^{th} country

Like the section 3 in chapter 2, we will export from the i^{th} country to the j^{th} country, supposed X_{ij}^{E} by $X_{ij}^{E} = \varepsilon_{ij} X_{i}^{P}$ and the transaction cost per unit is t_{ij}. Thus $TrC_{ij} = t_{ij} X_{ij}^{E}$

The quantity of demand of each country is parted n parts, owned country and exported from the $n-1$ countries. The same idea, the quantity of supply of each country is parted n parts, owned country and exported from the $n-1$ countries.

$$X_{i}^{P} = \sum_{j=1}^{n} X_{ij}^{E} = X_{ii}^{E} + \sum_{\substack{j \neq i \\ j=1}}^{n} X_{ij}^{E} = \left(1 - \sum_{\substack{j \neq i \\ j=1}}^{n} \varepsilon_{ij} \right) X_{i}^{P} + \sum_{\substack{j \neq i \\ j=1}}^{n} \varepsilon_{ij} X_{i}^{P}$$

$$X_{i}^{S} = X_{ii}^{E} + \sum_{\substack{j \neq i \\ j=1}}^{n} X_{ji}^{E} = \left(1 - \sum_{\substack{j \neq i \\ j=1}}^{n} \varepsilon_{ij} \right) X_{i}^{P} + \sum_{\substack{j \neq i \\ j=1}}^{n} \varepsilon_{ji} X_{j}^{P}$$

Section 4 Net total revenue in each country

Like the section 4 in chapter 3, we get

$$TR_{i} = P_{i}^{X} X_{i}^{S} - \sum_{\substack{j \neq i \\ j=1}}^{n} TrC_{ij} = P_{i} \left(1 - \sum_{\substack{j \neq i \\ j=1}}^{n} \varepsilon_{ij} \right) X_{i}^{P} + P_{i} \sum_{\substack{j \neq i \\ j=1}}^{n} \varepsilon_{ji} X_{j}^{P} - \sum_{\substack{j \neq i \\ j=1}}^{n} t_{ij} \varepsilon_{ij} X_{i}^{P}$$

From $TR = \sum_{i=1}^{n} TR_{i}$

$$TR = \sum_{i=1}^{n} \left[P_{i} \left(1 - \sum_{\substack{j \neq i \\ j=1}}^{n} \varepsilon_{ij} \right) X_{i}^{P} + P_{i} \sum_{\substack{j \neq i \\ j=1}}^{n} \varepsilon_{ji} X_{j}^{P} - \sum_{\substack{j \neq i \\ j=1}}^{n} t_{ij} \varepsilon_{ij} X_{i}^{P} \right]$$

$$TR = \sum_{i=1}^{n} \left[P_{i} \left(1 - \sum_{\substack{j \neq i \\ j=1}}^{n} \varepsilon_{ij} \right) + \sum_{\substack{j \neq i \\ j=1}}^{n} P_{j} \varepsilon_{ij} - \sum_{\substack{j \neq i \\ j=1}}^{n} t_{ij} \varepsilon_{ij} \right] X_{i}^{P}$$

$$TR = \sum_{i=1}^{n} \left[P_{i} \left(1 - \sum_{\substack{j \neq i \\ j=1}}^{n} \varepsilon_{ij} \right) + \sum_{\substack{j \neq i \\ j=1}}^{n} \varepsilon_{ij} \left(P_{j} - t_{ij} \right) \right] X_{i}^{P}$$

$$B_{i} = \frac{\partial TR}{\partial X_{i}^{P}} = P_{i} \left(1 - \sum_{\substack{j \neq i \\ j=1}}^{n} \varepsilon_{ij} \right) + \sum_{\substack{j \neq i \\ j=1}}^{n} \varepsilon_{ij} \left(P_{j} - t_{ij} \right)$$

$$TR = \sum_{i=1}^{n} B_{i} X_{i}^{P}$$

Section 5 The optimal ratio of investment for each country

From $\pi = \sum_{i=1}^{n} B_i X_i^P - TC$

Max π s.t. $\sum_{i=1}^{n} \gamma_i = 1$

From Appendix B.3 in equation (B.3.4)

$$\gamma_i = \left(1 + \sum_{\substack{i \neq j \\ i=1}}^{n} \left\{ \frac{B_j}{B_i} \cdot \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P} \right\} \right)^{-1}$$

Like the analysis in chapter 2 section 5, we get two interested propositions. From Appendix B.4, too.

Proposition 1 $\frac{MC_i}{B_i} = \frac{MC_j}{B_j}$ for $\forall i, j \in \{1, 2, \ldots, n\}$

Because of the first order condition of maximization in Appendix B.4 in equation (A.4.2), we've got the well-known condition, $\frac{MC_i}{B_i} = \frac{MC_j}{B_j}$ for $\forall i, j \in \{1, 2, \ldots, n\}$. It's the necessary condition for maximal profit. To get the maximal profit, the investor should separate his capital like this condition.

Proposition 2 $\sum_{k=1}^{K} \alpha K_i < 1$ for $\forall i, j \in \{1, 2, \ldots, n\}$

Because of the second order condition in Appendix B.5 in equation (B.5.4), we've got the other condition, $\sum_{k=1}^{K} \alpha K_i < 1$. It's the sufficient condition for maximal profit. To hold this condition, the production function of each country must be strict character, called decreasing return to scale. If the characteristic of production function of any country isn't hold, the investor will move all of his capital into that country.

Section 6 The change of investment

This section looks like the chapter 3. We will look over the effect of changing of some factors. We consider only four factors. Those are

1. The price of the k^{th} factor of production of the i^{th} country
2. The total capital
3. The transaction cost per unit from the i^{th} country to the j^{th} one
4. The price of goods of the i^{th} country

From Appendix B.6 in equation (B.6.1) and (B.6.2)

$$d\gamma_i = -\left(1 + \sum_{\substack{j \neq i \\ j=1}}^{n} \left(\frac{B_j}{B_i} \cdot \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P} \right) \right)^{-2} d\left(\sum_{\substack{j \neq i \\ j=1}}^{n} \left(\frac{B_j}{B_i} \cdot \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P} \right) \right)$$

$$d\gamma_i = -\left(1 + \sum_{\substack{j \neq i \\ j=1}}^{n} \left(\frac{B_j}{B_i} \cdot \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P} \right) \right)^{-2} \left(\sum_{\substack{j \neq i \\ j=1}}^{n} dW_i \right)$$

1. The price of the k^{th} factor of production of the i^{th} country

From Appendix B.6 in equation (B.6.4)

$$\frac{\partial \gamma_i}{\partial P_i^{ZK}} = -\left(1 + \sum_{\substack{j \neq i \\ j=1}}^{n} \left(\frac{B_j}{B_i} \cdot \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P} \right) \right)^{-2} \left(\sum_{\substack{j \neq i \\ j=1}}^{n} \frac{\partial W_i}{\partial P_i^{ZK}} \right)$$

$$\frac{\partial \gamma_i}{\partial P_i^{ZK}} = -(n-1)\alpha K_i \frac{W_i}{P_i^{ZK}} \left(1 + \sum_{\substack{j \neq i \\ j=1}}^{n} \left(\frac{B_j}{B_i} \cdot \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P} \right) \right)^{-2} < 0$$

This result looks like in chapter 3, too. Increasing of the price of factor will make the optimum ratio for that country decreased, $\frac{\partial \gamma_i}{\partial P_i^{ZK}} < 0$.

2.The total capital

From Appendix B.6 in equation (B.6.5)

$$\frac{\partial \gamma_i}{\partial TC} = -\left(1 + \sum_{\substack{j \neq i \\ j=1}}^{n} \left(\frac{B_j}{B_i} \cdot \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P} \right) \right)^{-2} \left(\sum_{\substack{j \neq i \\ j=1}}^{n} \frac{\partial W_i}{\partial TC} \right)$$

$$\frac{\partial \gamma_i}{\partial TC} = -\left(1 + \sum_{\substack{j \neq i \\ j=1}}^{n} \left(\frac{B_j}{B_i} \cdot \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P} \right) \right)^{-2} \left(\sum_{\substack{j \neq i \\ j=1}}^{n} \left(\sum \alpha K_j - \sum \alpha K_i\right) \frac{W_i}{TC} \right)$$

More generally the result in chapter 3, we found that we have to consider all of homogeneous degree. We can't compare HD only two countries but also HD of the rest, too.

We have to consider the last term of above equation. I simplify it.

$$\sum_{\substack{j \neq i \\ j=1}}^{n} \left(\sum \alpha K_j - \sum \alpha K_i\right) = \sum_{\substack{j \neq i \\ j=1}}^{n} \sum \alpha K_j - (n-1)\sum \alpha K_i$$

$$= \sum_{j=1}^{n} \sum \alpha K_j - n\sum \alpha K_i$$

$$= n\left(\frac{\sum_{j=1}^{n}\sum \alpha K_j}{n} - \sum \alpha K_i\right)$$

$$= n\left(\overline{HD} - HD_i\right)$$

From above equation, To know the sign of this term, $\frac{\partial \gamma_i}{\partial TC}$, we should compare two things. The former is the homogeneous degree of that country, HD_i, and the latter is the average of homogeneous degree of every country, $\overline{HD}$. Because of that, we can part in three cases.

- If $HD_i = \overline{HD}$ then $\frac{\partial \gamma_i}{\partial TC} = 0$. The optimum ratio in the i^{th} country doesn't change. However, the ratio of the rest may change.
- If $HD_i < \overline{HD}$ then $\frac{\partial \gamma_i}{\partial TC} < 0$. It means that the productivity of the i^{th} country is less than the other countries, comparatively. Because of that, the investor will move his money out from the i^{th} country into the rest.
- If $HD_i > \overline{HD}$ then $\frac{\partial \gamma_i}{\partial TC} > 0$. The investor will move his money in the i^{th} country.

3.The transaction cost per unit from the i^{th} country to the j^{th} one

From Appendix B.6 in equation (B.6.7)

$$\frac{\partial \gamma_i}{\partial t_{ij}} = -\left(1 + \sum_{\substack{j \neq i \\ j=1}}^{n}\left(\frac{B_j}{B_i} \cdot \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P}\right)\right)^{-2}\left(\sum_{\substack{j \neq i \\ j=1}}^{n} \frac{\partial W_i}{\partial t_{ij}}\right)$$

$$\frac{\partial \gamma_i}{\partial t_{ij}} = -\left(1 + \sum_{\substack{j \neq i \\ j=1}}^{n}\left(\frac{B_j}{B_i} \cdot \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P}\right)\right)^{-2}\left(\sum_{\substack{j \neq i \\ j=1}}^{n} \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P}\left(\frac{B_i \cdot 0 + \varepsilon_{ij} B_j}{B_i^{\,2}}\right)\right) < 0$$

This result looks like in the chapter 3. If the transaction cost per unit increases, the optimum ratio of investment will decrease, $\frac{\partial \gamma_i}{\partial t_{ij}} < 0$.

4.The price of goods of the i^{th} country

From Appendix B.6 in equation (A.6.6)

$$\frac{\partial \gamma_i}{\partial P_i} = -\left(1+\sum_{\substack{j\neq i\\ j=1}}^{n}\left(\frac{B_j}{B_i}\cdot\frac{\sum \alpha K_j}{\sum \alpha K_i}\cdot\frac{X_j^P}{X_i^P}\right)\right)^{-2}\left(\sum_{\substack{j\neq i\\ j=1}}^{n}\frac{\partial W_i}{\partial P_i}\right)$$

$$\frac{\partial \gamma_i}{\partial P_i} = -\left(1+\sum_{\substack{j\neq i\\ j=1}}^{n}\left(\frac{B_j}{B_i}\cdot\frac{\sum \alpha K_j}{\sum \alpha K_i}\cdot\frac{X_j^P}{X_i^P}\right)\right)^{-2}\left(\sum_{\substack{j\neq i\\ j=1}}^{n}\left(\frac{\sum \alpha K_j}{\sum \alpha K_i}\cdot\frac{X_j^P}{X_i^P}\left(\frac{B_i\varepsilon_{ij}-B_j\left(1-\sum_{\substack{j\neq i\\ j=1}}^{n}\varepsilon_{ij}\right)}{B_i^{\,2}}\right)\right)\right)$$

This result looks like in the chapter 3, too. We can't find the direction of changing.

Chapter 5 Conclusion

5.1 The factor that determine the optimum ratio of investment

From the studying in five chapters, we found that the optimum ratio of investment is depended on many factors. We can part them into two parts. The former is cost factor and the latter is benefit factor. However, the investor has to manage his money in order to hold the first order condition, $\frac{MC_i}{B_i} = \frac{MC_j}{B_j}$ for $\forall i,j \in \{1,2,\ldots,n\}$ This is the necessary condition for maximization. To simplify this idea, I create two curves the Iso revenue Line (IL) and Constant Cost Line (CCL).

From the model, the investor should separate his money, efficiency. He has to concern about any factors that determine cost and benefit.

From the maximization process, the second order condition told us that the homogeneous degree of the production function of every country is to less than 1, $HD_i < 1$ for $\forall i,j \in \{1,2,\ldots,n\}$, or is decreasing return to scale function in order to diminishing in some levels of production.

5.2 The direction of investment

This paper use comparative static analysis to consider the direction of the optimum ratio of investment when four factors changed.

5.2.1 The price of factor of production

Increasing of the price of factor in the i^{th} country affects the ratio of investment in that country decrease, $\frac{\partial \gamma_i}{\partial P_i^{ZK}} < 0$. Because the price of factor affects the cost in production, the investor will move his money to other country.

From the figure 3.1, increasing of P_1^{Z1} affect the CCL, rotated clockwise because level of production in the first country decrease. From that, we get a new equilibrium, E_2.

5.2.2 The total capital

Adding the total capital, TC , the investor should separate this added money into n countries. This addendum may affect the old optimum ratio of investment.

To know the sign of this term, $\frac{\partial \gamma_i}{\partial TC}$, we should compare two things. The former is the homogeneous degree of that country, HD_i, and the latter is the average of homogeneous degree of every country, $\overline{HD}$. If $HD_i = \overline{HD}$ then

$\frac{\partial \gamma_i}{\partial TC} = 0$. The optimum ratio in the i^{th} country doesn't change. If $HD_i < \overline{HD}$ then $\frac{\partial \gamma_i}{\partial TC} < 0$. It means that the productivity of the i^{th} country is less than the other countries, comparatively. Because of that, the investor will move his money out from the i^{th} country into the rest.

From the figure 3.2, increasing of TC affect the CCL, shifted northeast. From that, we get a new equilibrium, E_2.

5.2.3 The transaction cost per unit

Increasing of transaction cost per unit makes the marginal benefit decreased. The investor will decrease the ratio of investment in that country.

From the figure 4.3, The increasing of transaction cost will affect the slope of IL , decrease. From that, we get a new equilibrium, E_2.

5.2.4 the price of goods

We found that the increasing of price of goods doesn't allure the investor but he has to consider the other factors such as the ratio of export, ε_{ij}, the marginal benefit, B_i, etc.

5.3 The further research

Because this paper is only the part of subject in Bachelor Degree, I try to simplify it by many restrictions. Though the investor may not apply it in the real world, he should bring some ideas to understand the idea of economics.

The further paper should concern about these my restrictions.

5.3.1 The investor is the price taker

From equation, I assume the constant price of goods. It's not realistic for some investors. If you are big enough, you can control the market. It doesn't affect the first order condition, $\frac{MC_i}{MB_i} = \frac{MC_j}{MB_j}$ for $\forall i, j \in \{1,2,\ldots,n\}$ because this condition looks like the rule of microeconomics. In the sight of economist, if we have many choices, this condition will hold forever.

5.3.2 Cobb-Douglas production function

The Cobb-Douglas function has 3 restrictions.

5.3.2.1 Unique Solution

In the real world, the multi equilibrium may exist.

5.3.2.2 The Elasticity of factor of production

From calculating the elasticity of factor of production, we get a constant value, $\frac{\partial X_i^P}{\partial Z_i^K}\frac{Z_i^K}{X_i^P} = \alpha K_i$. In the real world, the elasticity may not be constant.

5.3.2.3 Elastic of substitution

From calculating the elasticity of substitution, we always get 1.

5.3.3 Perfect capital mobility

Surely, there are some restrictions for investors in every country called imperfect capital mobility.

Appendix A 2x2 Model

A.1 Maximize Quantity

Max $X_i^P = f(Z_i^1, Z_i^2) = A_i {Z_i^1}^{\alpha 1_i} {Z_i^2}^{\alpha 2_i}$ s.t. $\gamma_i TC = P_i^{Z1} Z_i^1 + P_i^{Z2} Z_i^2$

Create the Lagrange function [1]

$$L^* = A_i {Z_i^1}^{\alpha 1_i} {Z_i^2}^{\alpha 2_i} + \lambda^* \left(\gamma_i TC - \left(P_i^{Z1} Z_i^1 + P_i^{Z2} Z_i^2\right)\right)$$

$$\frac{\partial L^*}{\partial Z_i^1} = \frac{\alpha 1_i}{Z_i^1} X_i^P - \lambda^* P_i^{Z1} = 0 \qquad \text{………. (A.1.1)}$$

$$\frac{\partial L^*}{\partial Z_i^2} = \frac{\alpha 2_i}{Z_i^2} X_i^P - \lambda^* P_i^{Z2} = 0 \qquad \text{………. (A.1.2)}$$

$$\frac{\partial L^*}{\partial \lambda^*} = \gamma_i TC - P_i^{Z1} Z_i^1 + P_i^{Z2} Z_i^2 = 0 \qquad \text{………. (A.1.3)}$$

(A.1.1)/(A.1.2)

$$\frac{\alpha 1_i}{\alpha 2_i} \frac{Z_i^2}{Z_i^1} = \frac{P_i^{Z1}}{P_i^{Z2}} \qquad \text{………. (A.1.4)}$$

From (A.1.3) and (A.1.4)

$$\gamma_i TC = P_i^{Z1} Z_i^1 \left(1 + \frac{\alpha 2_i}{\alpha 1_i}\right) = P_i^{Z2} Z_i^2 \left(1 + \frac{\alpha 1_i}{\alpha 2_i}\right)$$

Then $Z_i^1 = \left(\frac{\alpha 1_i}{P_i^{Z1}}\right)\left(\frac{\gamma_i TC}{\alpha 1_i + \alpha 2_i}\right)$ ………. (A.1.5)

$$Z_i^2 = \left(\frac{\alpha 2_i}{P_i^{Z2}}\right)\left(\frac{\gamma_i TC}{\alpha 1_i + \alpha 2_i}\right) \qquad \text{………. (A.1.6)}$$

Replace in the production function

$$X_i^P = A_i \left(\frac{\alpha 1_i}{P_i^{Z1}}\right)^{\alpha 1_i} \left(\frac{\alpha 2_i}{P_i^{Z2}}\right)^{\alpha 2_i} \left(\frac{\gamma_i TC}{\alpha 1_i + \alpha 2_i}\right)^{\alpha 1_i + \alpha 2_i} \qquad \text{………. (A.1.7)}$$

[1] Mathematics for Economics analysis,p.654-655

A.2 Differentiation

From A.1.7 $$X_i^P = A_i\left(\frac{\alpha 1_i}{P_i^{Z1}}\right)^{\alpha 1_i}\left(\frac{\alpha 2_i}{P_i^{Z2}}\right)^{\alpha 2_i}\left(\frac{\gamma_i TC}{\alpha 1_i + \alpha 2_i}\right)^{\alpha 1_i+\alpha 2_i}$$

$$\frac{\partial X_i^P}{\partial \gamma_i} = \frac{(\alpha 1_i + \alpha 2_i)}{\gamma_i} X_i^P \qquad \text{....... (A. 2.1)}$$

$$\ln X_i^P = \ln A_i + \alpha 1_i\left(\ln \alpha 1_i - \ln P_i^{Z1}\right) + \alpha 2_i\left(\ln \alpha 2_i - \ln P_i^{Z2}\right)$$
$$+ (\alpha 1_i + \alpha 2_i)\left(\ln \gamma_i TC - \ln(\alpha 1_i + \alpha 2_i)\right)$$

$$\frac{\partial \ln X_i^P}{\partial \ln P_i^{Z1}} = -\alpha 1_i \qquad \text{....... (A. 2.2a)}$$

$$\frac{\partial X_i^P}{\partial P_i^{Z1}} = -\alpha 1_i \frac{X_i^P}{P_i^{Z1}} \qquad \text{....... (A. 2.2b)}$$

$$\frac{\partial \ln X_i^P}{\partial \ln P_i^{Z2}} = -\alpha 2_i \qquad \text{....... (A. 2.3a)}$$

$$\frac{\partial X_i^P}{\partial P_i^{Z2}} = -\alpha 2_i \frac{X_i^P}{P_i^{Z2}} \qquad \text{....... (A. 2.3b)}$$

$$\frac{\partial \ln X_i^P}{\partial \ln \gamma_i TC} = (\alpha 1_i + \alpha 2_i) \qquad \text{....... (A. 2.4a)}$$

$$\frac{\partial X_i^P}{\partial \gamma_i TC} = (\alpha 1_i + \alpha 2_i)\frac{X_i^P}{\gamma_i TC} \qquad \text{....... (A. 2.4b)}$$

$$\frac{\partial \ln X_i^P}{\partial \ln TC} = (\alpha 1_i + \alpha 2_i) \qquad \text{....... (A. 2.5a)}$$

$$\frac{\partial X_i^P}{\partial TC} = (\alpha 1_i + \alpha 2_i)\frac{X_i^P}{TC} \qquad \text{....... (A. 2.5b)}$$

$$dX_i^P = A_i\left(\frac{\alpha 1_i}{P_i^{Z1}}\right)^{\alpha 1_i}\left(\frac{\alpha 2_i}{P_i^{Z2}}\right)^{\alpha 2_i}\left(\frac{TC}{\alpha 1_i + \alpha 2_i}\right)^{\alpha 1_i+\alpha 2_i}(\alpha 1_i + \alpha 2_i)\gamma_i^{\alpha 1_i+\alpha 2_i-1} d\gamma_i$$

Give $Y_i = A_i\left(\frac{\alpha 1_i}{P_i^{Z1}}\right)^{\alpha 1_i}\left(\frac{\alpha 2_i}{P_i^{Z2}}\right)^{\alpha 2_i}\left(\frac{TC}{\alpha 1_i + \alpha 2_i}\right)^{\alpha 1_i+\alpha 2_i}(\alpha 1_i + \alpha 2_i)$

Then $dX_i^P = Y_i \gamma_i^{\alpha 1_i+\alpha 2_i-1} d\gamma_i$

$$\frac{dX_2^P}{dX_1^P} = \frac{Y_2 \gamma_2^{\alpha 1_2+\alpha 2_2-1} d\gamma_2}{Y_1 \gamma_1^{\alpha 1_1+\alpha 2_1-1} d\gamma_1} = -\frac{Y_2 \gamma_2^{\alpha 1_2+\alpha 2_2-1}}{Y_1 \gamma_1^{\alpha 1_1+\alpha 2_1-1}} < 0 \qquad \text{....... (A. 2.6)}$$

From $B_i = P_i(1-\varepsilon_{ij}) + \varepsilon_{ij}(P_j - t_{ij})$

$$\frac{\partial B_i}{\partial P_i} = (1-\varepsilon_{ij}) \qquad \text{....... (A. 2.7)}$$

$$\frac{\partial B_i}{\partial P_j} = \varepsilon_{ij} \qquad \text{....... (A. 2.8)}$$

$$\frac{\partial B_i}{\partial t_{ij}} = -\varepsilon_{ij} \qquad \text{....... (A. 2.9)}$$

A.3 Maximize Profit

From $\pi = TR - TC$

$$\pi = B_1 X_1^P + B_2 X_2^P - TC$$

Max π s.t. $\gamma_1 + \gamma_2 = 1$

Create the Lagrange function

$$L^{**} = B_1 X_1^P + B_2 X_2^P - TC + \lambda^{**}\left(1 - (\gamma_1 + \gamma_2)\right)$$

$$\frac{\partial L^{**}}{\partial \gamma_1} = B_1 \frac{\partial X_1^P}{\partial \gamma_1} - \lambda^{**} = 0$$

From A.2.1

$$\lambda^{**} = B_1 \frac{\partial X_1^P}{\partial \gamma_1} = B_1 \frac{\alpha 1_1 + \alpha 2_1}{\gamma_1} X_1^P \qquad \ldots\ldots\ (A.\ \ 3.1)$$

$$\frac{\partial L^{**}}{\partial \gamma_2} = B_2 \frac{\partial X_2^P}{\partial \gamma_2} - \lambda^{**} = 0$$

From A.2.1

$$\lambda^{**} = B_2 \frac{\partial X_2^P}{\partial \gamma_2} = B_2 \frac{\alpha 1_2 + \alpha 2_2}{\gamma_2} X_2^P \qquad \ldots\ldots\ (A.\ \ 3.2)$$

$$\frac{\partial L^{**}}{\partial \lambda^{**}} = 1 - \gamma_1 - \gamma_2 = 0 \qquad \ldots\ldots\ (A.\ \ 3.3)$$

(A.3.1) equal (A.3.2)

$$B_1 \frac{\alpha 1_1 + \alpha 2_1}{\gamma_1} X_1^P = B_2 \frac{\alpha 1_2 + \alpha 2_2}{\gamma_2} X_2^P$$

$$\frac{\gamma_1}{\gamma_2} \cdot \frac{\alpha 1_2 + \alpha 2_2}{\alpha 1_1 + \alpha 2_1} \cdot \frac{X_2^P}{X_1^P} = \frac{B_1}{B_2} \qquad \ldots\ldots\ (A.\ \ 3.4)$$

$$\gamma_1 = \frac{B_1}{B_2} \frac{\alpha 1_1 + \alpha 2_1}{\alpha 1_2 + \alpha 2_2} \cdot \frac{X_1^P}{X_2^P} \gamma_2$$

Replace in (A.3.3)

$$\frac{B_1}{B_2} \frac{\alpha 1_1 + \alpha 2_1}{\alpha 1_2 + \alpha 2_2} \cdot \frac{X_1^P}{X_2^P} \gamma_2 + \gamma_2 = 1$$

$$\gamma_2 = \left(1 + \frac{B_1}{B_2} \frac{\alpha 1_1 + \alpha 2_1}{\alpha 1_2 + \alpha 2_2} \cdot \frac{X_1^P}{X_2^P}\right)^{-1}$$

Then $\gamma_i = \left(1 + \frac{B_j}{B_i} \frac{\alpha 1_j + \alpha 2_j}{\alpha 1_i + \alpha 2_i} \cdot \frac{X_j^P}{X_i^P}\right)^{-1}$ .... (A. 3.5)

A.4 First order condition

From (A.2.4b)

$$\frac{\partial X_i^P}{\partial TC_i} = (\alpha 1_i + \alpha 2_i) \cdot \frac{X_i^P}{\gamma_i TC}$$

Then $$\frac{\frac{\partial X_1^P}{\partial TC_1}}{\frac{\partial X_2^P}{\partial TC_2}} = \frac{(\alpha 1_1 + \alpha 2_1) \cdot \frac{X_1^P}{\gamma_1 TC}}{(\alpha 1_2 + \alpha 2_2) \cdot \frac{X_2^P}{\gamma_2 TC}} = \frac{\gamma_2}{\gamma_1} \cdot \frac{\alpha 1_1 + \alpha 2_1}{\alpha 1_2 + \alpha 2_2} \cdot \frac{X_1^P}{X_2^P} \qquad \text{....... (A.4.1)}$$

From (A.3.4) and (A.4.1)

$$\frac{\frac{\partial X_1^P}{\partial TC_1}}{\frac{\partial X_2^P}{\partial TC_2}} = \frac{B_1}{B_2}$$

$$\frac{\frac{\partial TC_1}{\partial X_1^P}}{B_1} = \frac{\frac{\partial TC_2}{\partial X_2^P}}{B_2}$$

$$\frac{MC_1}{B_1} = \frac{MC_2}{B_2} \qquad \text{....... (A.4.2)}$$

A.5 Second order condition

From $\pi = B_1X_1^P + B_2X_2^P - TC$

$$\frac{\partial^2\pi}{\partial\gamma_i\partial\gamma_i} = B_i\frac{\partial^2 X_i^P}{\partial\gamma_i\partial\gamma_i} \quad = B_iA_i\left(\frac{\alpha 1_i}{P_i^{Z1}}\right)^{\alpha 1_i}\left(\frac{\alpha 2_i}{P_i^{Z2}}\right)^{\alpha 2_i}\left(\frac{TC}{\alpha 1_i + \alpha 2_i}\right)^{\alpha 1_i + \alpha 2_i}$$

$$\times(\alpha 1_i + \alpha 2_i)(\alpha 1_i + \alpha 2_i - 1)\gamma_i^{\alpha 1_i + \alpha 2_i - 2}$$

$$\frac{\partial^2\pi}{\partial\gamma_i\partial\gamma_i} = B_iY_i(\alpha 1_i + \alpha 2_i - 1)\gamma_i^{\alpha 1_i + \alpha 2_i - 2} \qquad \text{.......} \ (A.5.1)$$

$$\frac{\partial^2\pi}{\partial\gamma_i\partial\gamma_j} = 0 \qquad \text{.......} \ (A.5.2)$$

From
$$\left|\overline{H}\right| = \begin{vmatrix} 0 & \frac{\partial g}{\partial\gamma_1} & \frac{\partial g}{\partial\gamma_2} \\ \frac{\partial g}{\partial\gamma_1} & \frac{\partial^2\pi}{\partial\gamma_1\partial\gamma_1} & \frac{\partial^2\pi}{\partial\gamma_1\partial\gamma_2} \\ \frac{\partial g}{\partial\gamma_2} & \frac{\partial^2\pi}{\partial\gamma_2\partial\gamma_1} & \frac{\partial^2\pi}{\partial\gamma_2\partial\gamma_2} \end{vmatrix} = \begin{vmatrix} 0 & -1 & -1 \\ -1 & \frac{\partial^2\pi}{\partial\gamma_1\partial\gamma_1} & 0 \\ -1 & 0 & \frac{\partial^2\pi}{\partial\gamma_2\partial\gamma_2} \end{vmatrix}$$

The condition for maximization is[2]

$$\left|\overline{H}_2\right| = -\frac{\partial^2\pi}{\partial\gamma_1\partial\gamma_1} - \frac{\partial^2\pi}{\partial\gamma_2\partial\gamma_2} > 0 \qquad \text{.......} \ (A.5.3)$$

Replace (A.5.1),(A.5.2) in (A.5.3)

$$B_1Y_1(\alpha 1_1 + \alpha 2_1 - 1)\gamma_1^{\alpha 1_1 + \alpha 2_1 - 2} + B_2Y_2(\alpha 1_2 + \alpha 2_2 - 1)\gamma_2^{\alpha 1_2 + \alpha 2_2 - 2} < 0$$

Then $\alpha 1_1 + \alpha 2_1 - 1 < 0$

$\alpha 1_2 + \alpha 2_2 - 1 < 0$

Then $\alpha 1_i + \alpha 2_i - 1 < 0$ .... (A.5. 4)

[2] Microeconomics and decision model of the firm, p.68

A.6 The direction of γ_i

From (A.3.5)

$$\gamma_i = \left(1 + \frac{B_j}{B_i}\frac{\alpha 1_j + \alpha 2_j}{\alpha 1_i + \alpha 2_i}\cdot\frac{X_j^P}{X_i^P}\right)^{-1}$$

$$d\gamma_i = -\left(1 + \frac{B_j}{B_i}\frac{\alpha 1_j + \alpha 2_j}{\alpha 1_i + \alpha 2_i}\cdot\frac{X_j^P}{X_i^P}\right)^{-2} d\left(\frac{B_j}{B_i}\frac{\alpha 1_j + \alpha 2_j}{\alpha 1_i + \alpha 2_i}\cdot\frac{X_j^P}{X_i^P}\right) \quad \text{....... (A. 6.1)}$$

Give $W_i = \left(\frac{B_j}{B_i}\frac{\alpha 1_j + \alpha 2_j}{\alpha 1_i + \alpha 2_i}\cdot\frac{X_j^P}{X_i^P}\right)$ .... (A. 6.2)

$$\ln W_i = \ln B_j - \ln B_i + \ln\frac{\alpha 1_j + \alpha 2_j}{\alpha 1_i + \alpha 2_i} + \ln X_j^P - \ln X_i^P \quad \text{....... (A. 6.3)}$$

$$\frac{\partial \ln W_i}{\partial \ln P_i^{Z1}} = -\frac{\partial \ln X_i^P}{\partial \ln P_i^{Z1}}$$

From Appendix A.2 in equation (A.2.2a)

$$\frac{\partial \ln W_i}{\partial \ln P_i^{Z1}} = \alpha 1_i$$

$$\frac{\partial W_i}{\partial P_i^{Z1}} = \alpha 1_i \frac{W_i}{P_i^{Z1}} \quad \text{....... (A. 6.4)}$$

$$\frac{\partial \ln W_i}{\partial \ln P_i^{Z2}} = -\frac{\partial \ln X_i^P}{\partial \ln P_i^{Z2}}$$

From Appendix A.2 in equation (A.2.3a)

$$\frac{\partial \ln W_i}{\partial \ln P_i^{Z2}} = \alpha 2_i$$

$$\frac{\partial W_i}{\partial P_i^{Z2}} = \alpha 2_i \frac{W_i}{P_i^{Z2}} \quad \text{....... (A. 6.5)}$$

$$\frac{\partial \ln W_i}{\partial \ln TC} = \frac{\partial \ln X_j^P}{\partial \ln TC} - \frac{\partial \ln X_i^P}{\partial \ln TC}$$

From Appendix A.2 in equation (A.2.5a)

$$\frac{\partial \ln W_i}{\partial \ln TC} = (\alpha 1_j + \alpha 2_j) - (\alpha 1_i + \alpha 2_i)$$

$$\frac{\partial W_i}{\partial TC} = (\alpha 1_j + \alpha 2_j - \alpha 1_i - \alpha 2_i)\frac{W_i}{TC} \quad \text{....... (A. 6.6)}$$

From (A.6.2)

$$\frac{\partial W_i}{\partial P_i} = \frac{\alpha 1_j + \alpha 2_j}{\alpha 1_i + \alpha 2_i}\cdot\frac{X_j^P}{X_i^P}\left(\frac{B_i\frac{\partial B_j}{\partial P_i} - B_j\frac{\partial B_i}{\partial P_i}}{B_i^2}\right)$$

From Appendix A.2 in equation (A.2.7) and (A.2.8)

$$\frac{\partial W_i}{\partial P_i} = \frac{\alpha 1_j + \alpha 2_j}{\alpha 1_i + \alpha 2_i}\cdot\frac{X_j^P}{X_i^P}\left(\frac{B_i\varepsilon_{ij} - B_j(1-\varepsilon_{ij})}{B_i^2}\right) \quad \text{....... (A. 6.7)}$$

From (A.6.2)

$$\frac{\partial W_i}{\partial t_{ij}} = \frac{\alpha 1_j + \alpha 2_j}{\alpha 1_i + \alpha 2_i} \cdot \frac{X_j^P}{X_i^P} \left(\frac{B_i \frac{\partial B_j}{\partial t_{ij}} - B_j \frac{\partial B_i}{\partial t_{ij}}}{B_i^2} \right)$$

From Appendix A.2 in equation (A.2.9)

$$\frac{\partial W_i}{\partial t_{ij}} = \frac{\alpha 1_j + \alpha 2_j}{\alpha 1_i + \alpha 2_i} \cdot \frac{X_j^P}{X_i^P} \left(\frac{B_i \cdot 0 + B_j \varepsilon_{ij}}{B_i^2} \right) \qquad \text{....... (A. 6.8)}$$

Appendix B nxk Model

B.1 Maximize Quantity

Max $X_i^P = f_i(Z_i^1, Z_i^2, \ldots, Z_i^K) = A_i \prod_{k=1}^{K} Z_i^{K^{\alpha K_i}}$ s.t. $\gamma_i TC = \sum_{k=1}^{K} P_i^{ZK} Z_i^K$

Create the Lagrange function

$$L^{\#} = A_i \prod_{k=1}^{K} Z_i^{K^{\alpha K_i}} + \lambda^{\#}\left(\gamma_i TC - \sum_{k=1}^{K} P_i^{ZK} Z_i^K\right)$$

$$\frac{\partial L^{\#}}{\partial Z_i^K} = \frac{\alpha K_i}{Z_i^K} X_i^P - \lambda^{\#} P_i^{ZK} = 0 \qquad \text{....... (B.1.1)}$$

$$\frac{\partial L^{\#}}{\partial \lambda^{\#}} = \gamma_i TC - \sum_{k=1}^{K} P_i^{ZK} Z_i^K = 0 \qquad \text{....... (B.1.2)}$$

From (B.1.1)

$$\frac{\alpha 1_i}{Z_i^1}\frac{X_i^P}{P_i^{Z1}} = \frac{\alpha 2_i}{Z_i^2}\frac{X_i^P}{P_i^{Z2}} = \ldots = \frac{\alpha K_i}{Z_i^K}\frac{X_i^P}{P_i^{ZK}} \qquad \text{....... (B.1.3)}$$

From (B.1.2) and (B.1.3)

$$\gamma_i TC = P_i^{Z1} Z_i^1 \left(1 + \frac{\alpha 2_i}{\alpha 1_i} + \frac{\alpha 3_i}{\alpha 1_i} + \ldots + \frac{\alpha K_i}{\alpha 1_i}\right) = P_i^{Z1} Z_i^1 \left(\frac{\sum \alpha K_i}{\alpha 1_i}\right)$$

$$Z_i^1 = \left(\frac{\alpha 1_i}{P_i^{Z1}}\right)\left(\frac{\gamma_i TC}{\sum \alpha K_i}\right)$$

$$Z_i^K = \left(\frac{\alpha K_i}{P_i^{ZK}}\right)\left(\frac{\gamma_i TC}{\sum \alpha K_i}\right) \qquad \text{....... (B.1.4)}$$

Replace in the production function

$$X_i^P = A_i \prod_{k=1}^{K} \left(\frac{\alpha K_i}{P_i^{ZK}}\right)^{\alpha K_i} \left(\frac{\gamma_i TC}{\sum \alpha K_i}\right)^{\Sigma \alpha K_i} \qquad \text{....... (B.1.5)}$$

B.2 Differentiation

From $X_i^P = A_i \prod_{k=1}^{K} \left(\frac{\alpha K_i}{P_i^{ZK}} \right)^{\alpha K_i} \left(\frac{\gamma_i TC}{\sum \alpha K_i} \right)^{\sum \alpha K_i}$

$$\frac{\partial X_i^P}{\partial \gamma_i} = \frac{\left(\sum \alpha K_i\right)}{\gamma_i} X_i^P \qquad \text{....... (B. 2.1)}$$

$$\ln X_i^P = \ln A_i + \sum_{k=1}^{K} \alpha K_i \left(\ln \alpha K_i - \ln P_i^{ZK} \right) + \left(\sum \alpha K_i \right) \left(\ln \gamma_i TC - \ln \sum \alpha K_i \right)$$

$$\frac{\partial \ln X_i^P}{\partial \ln P_i^{Zk}} = -\alpha k_i \qquad \text{....... (B. 2.2a)}$$

$$\frac{\partial X_i^P}{\partial P_i^{ZK}} = -\alpha K_i \frac{X_i^P}{P_i^{ZK}} \qquad \text{....... (B. 2.2b)}$$

$$\frac{\partial \ln X_i^P}{\partial \ln \gamma_i TC} = \sum \alpha K_i \qquad \text{....... (B. 2.3a)}$$

$$\frac{\partial X_i^P}{\partial \gamma_i TC} = \sum \alpha K_i \frac{X_i^P}{\gamma_i TC} \qquad \text{....... (B. 2.3b)}$$

$$\frac{\partial \ln X_i^P}{\partial \ln TC} = \sum \alpha K_i \qquad \text{....... (B. 2.4a)}$$

$$\frac{\partial X_i^P}{\partial TC} = \sum \alpha K_i \frac{X_i^P}{TC} \qquad \text{....... (B. 2.4b)}$$

From $B_i = P_i \left(1 - \sum_{\substack{j \neq i \\ j=1}}^{n} \varepsilon_{ij} \right) + \sum_{\substack{j \neq i \\ j=1}}^{n} \varepsilon_{ij} \left(P_j - t_{ij} \right)$

$$\frac{\partial B_i}{\partial P_i} = 1 - \sum_{\substack{j \neq i \\ j=1}}^{n} \varepsilon_{ij} \qquad \text{....... (B. 2.5)}$$

$$\frac{\partial B_i}{\partial P_j} = \varepsilon_{ij} \qquad \text{....... (B. 2.6)}$$

$$\frac{\partial B_i}{\partial t_{ij}} = -\varepsilon_{ij} \qquad \text{....... (B. 2.7)}$$

B.3 Maximize Profit

From $\pi = TR - TC$

$$\pi = \sum_{i=1}^{n} B_i X_i^P - TC$$

Create the Lagrange function

Max π s.t. $\sum_{i=1}^{n} \gamma_i = 1$

$$L^{\#\#} = \sum_{i=1}^{n} B_i X_i^P - TC + \lambda^{\#\#}\left(1 - \sum_{i=1}^{n} \gamma_i\right)$$

$$\frac{\partial L^{\#\#}}{\partial \gamma_i} = B_i \frac{\partial X_i^P}{\partial \gamma_i} - \lambda^{\#\#} = 0$$

$$\lambda^{\#\#} = B_i \frac{\partial X_i^P}{\partial \gamma_i} = B_i \frac{\sum \alpha K_i}{\gamma_i} X_i^P \qquad \text{.......} (B.\ 3.1)$$

$$\frac{\partial L^{\#\#}}{\partial \lambda^{\#\#}} = 1 - \sum \gamma_i = 0 \qquad \text{.......} (B.\ 3.2)$$

From (B.3.1) $\forall i, j \in \{1, 2, \ldots, n\}$

$$\forall i, j \in \{1, 2, \ldots, n\} \quad B_i \frac{\sum \alpha K_i}{\gamma_i} X_i^P = B_j \frac{\sum \alpha K_j}{\gamma_j} X_j^P$$

$$\frac{\gamma_i}{\gamma_j} \cdot \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P} = \frac{B_i}{B_j} \qquad \text{.......} (B.\ 3.3)$$

From (B.3.2) and (B.3.3) $\forall i, j \in \{1, 2, \ldots, n\}$

$$\gamma_i + \sum_{\substack{j \neq i \\ j=1}}^{n} \left(\frac{B_j}{B_i} \cdot \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P}\right) \gamma_i = 1$$

$$\gamma_i = \left(1 + \sum_{\substack{j \neq i \\ j=1}}^{n} \left(\frac{B_j}{B_i} \cdot \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P}\right)\right)^{-1} \qquad \text{.......} (B.\ 3.4)$$

B.4 First order condition

From Appendix B.2 in equation (B.2.3b) $\forall i, j \in \{1,2,\ldots,n\}$

$$\frac{\partial X_i^P}{\partial TC_i} = \sum \alpha K_i \cdot \frac{X_i^P}{TC_i} = \sum \alpha K_i \cdot \frac{X_i^P}{\gamma_i TC}$$

Then $$\frac{\dfrac{\partial X_i^P}{\partial TC_i}}{\dfrac{\partial X_j^P}{\partial TC_j}} = \frac{\sum \alpha K_i \cdot \dfrac{X_i^P}{\gamma_i TC}}{\sum \alpha K_j \cdot \dfrac{X_j^P}{\gamma_j TC}} = \frac{\gamma_j}{\gamma_i} \cdot \frac{\sum \alpha K_i}{\sum \alpha K_j} \cdot \frac{X_i^P}{X_j^P} \qquad \ldots\ldots\ (B.\ 4.1)$$

From (B.3.3) and (B.4.1) $\forall i, j \in \{1,2,\ldots,n\}$

$$\frac{\dfrac{\partial X_i^P}{\partial TC_i}}{\dfrac{\partial X_j^P}{\partial TC_j}} = \frac{B_j}{B_i}$$

$$\frac{\dfrac{\partial TC_i}{\partial X_i^P}}{B_i} = \frac{\dfrac{\partial TC_j}{\partial X_j^P}}{B_j}$$

$$\frac{MC_i}{B_i} = \frac{MC_j}{B_j} \qquad \ldots\ldots\ (B.\ 4.2)$$

B.5 Second order condition

From $\pi = \sum_{i=1}^{n} B_i X_i^P - TC$

$$\frac{\partial^2\pi}{\partial\gamma_i\partial\gamma_i} = B_i A_i \prod_{k=1}^{K}\left(\frac{\alpha K_i}{P_i^{ZK}}\right)^{\alpha K_i}\left(\frac{TC}{\sum\alpha K_i}\right)^{\sum\alpha K_i}\sum\alpha K_i\left(\sum\alpha K_i - 1\right)\gamma_i^{\sum\alpha K_i - 2}$$

$$\frac{\partial^2\pi}{\partial\gamma_i\partial\gamma_i} = B_i Y_i\left(\sum\alpha K_i - 1\right)\gamma_i^{\sum\alpha K_i - 2} \qquad \text{....... (B.5.1)}$$

$$\frac{\partial^2\pi}{\partial\gamma_i\partial\gamma_j} = 0 \qquad \text{....... (B.5.2)}$$

From
$$\left|\overline{H}\right| = \begin{vmatrix} 0 & \frac{\partial g}{\partial\gamma_1} & \cdots & \frac{\partial g}{\partial\gamma_K} \\ \frac{\partial g}{\partial\gamma_1} & \frac{\partial^2\pi}{\partial\gamma_1\partial\gamma_1} & \cdots & \frac{\partial^2\pi}{\partial\gamma_1\partial\gamma_K} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g}{\partial\gamma_K} & \frac{\partial^2\pi}{\partial\gamma_K\partial\gamma_1} & \cdots & \frac{\partial^2\pi}{\partial\gamma_K\partial\gamma_K} \end{vmatrix} = \begin{vmatrix} 0 & -1 & \cdots & -1 \\ -1 & \frac{\partial^2\pi}{\partial\gamma_1\partial\gamma_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & \frac{\partial^2\pi}{\partial\gamma_K\partial\gamma_K} \end{vmatrix}$$

The condition for maximization is[3]

$$\left|\overline{H}_2\right| = -\frac{\partial^2\pi}{\partial\gamma_1\partial\gamma_1} - \frac{\partial^2\pi}{\partial\gamma_2\partial\gamma_2} > 0$$

$$\left|\overline{H}_3\right| = -\left(\frac{\partial^2\pi}{\partial\gamma_1\partial\gamma_1}\frac{\partial^2\pi}{\partial\gamma_2\partial\gamma_2}\right) - \left(\frac{\partial^2\pi}{\partial\gamma_1\partial\gamma_1}\frac{\partial^2\pi}{\partial\gamma_3\partial\gamma_3}\right) - \left(\frac{\partial^2\pi}{\partial\gamma_2\partial\gamma_2}\frac{\partial^2\pi}{\partial\gamma_3\partial\gamma_3}\right) < 0$$

$$\forall K \in \{1,2,\ldots,K\} \qquad \left|\overline{H}_K\right| = (-1)^K \sum_{i=1}^{n}\left(\frac{\prod_{k=1}^{K}\frac{\partial^2\pi}{\partial\gamma_K\partial\gamma_K}}{\frac{\partial^2\pi}{\partial\gamma_i\partial\gamma_i}}\right) > 0 \qquad \text{....... (B.5.3)}$$

Replace (B.5.1) and (B.5.2) in (B.5.3)

$$\forall K \in \{1,2,\ldots,K\} \qquad \sum_{k=1}^{K}\alpha K_i - 1 < 0$$

$$\sum_{k=1}^{K}\alpha K_i < 1 \qquad \text{....... (B.5. 4)}$$

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[3]Fundamental Method of Mathematics Economics, p.385

B.6 The direction of γ_i

From (B.3.4)

$$\gamma_i = \left(1 + \sum_{\substack{j \neq i \\ j=1}}^{n} \left(\frac{B_j}{B_i} \cdot \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P} \right)\right)^{-1}$$

$$d\gamma_i = -\left(1 + \sum_{\substack{j \neq i \\ j=1}}^{n} \left(\frac{B_j}{B_i} \cdot \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P} \right)\right)^{-2} d\left(\sum_{\substack{j \neq i \\ j=1}}^{n} \left(\frac{B_j}{B_i} \cdot \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P} \right)\right) \qquad \text{....... (B. 6.1)}$$

Give $$W_i = \frac{B_j}{B_i} \cdot \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P} \qquad \text{....... (B. 6.2)}$$

$$\ln W_i = \ln B_j - \ln B_i + \ln \frac{\sum \alpha K_j}{\sum \alpha K_i} + \ln X_j^P - \ln X_i^P \qquad \text{....... (B. 6.3)}$$

$$\frac{\partial \ln W_i}{\partial \ln P_i^{ZK}} = -\frac{\partial \ln X_i^P}{\partial \ln P_i^{ZK}}$$

From Appendix B.2 in equation (B.2.2a)

$$\frac{\partial \ln W_i}{\partial \ln P_i^{ZK}} = \alpha K_i$$

$$\frac{\partial W_i}{\partial P_i^{ZK}} = \alpha K_i \frac{W_i}{P_i^{ZK}} \qquad \text{....... (B. 6.4)}$$

$$\frac{\partial \ln W_i}{\partial \ln TC} = \frac{\partial \ln X_j^P}{\partial \ln TC} - \frac{\partial \ln X_i^P}{\partial \ln TC}$$

From Appendix B.2 in equation (B.2.4a)

$$\frac{\partial \ln W_i}{\partial \ln TC} = \sum \alpha K_j - \sum \alpha K_i$$

$$\frac{\partial W_i}{\partial TC} = \left(\sum \alpha K_j - \sum \alpha K_i\right) \frac{W_i}{TC} \qquad \text{....... (B. 6.5)}$$

From (B.6.2)

$$\frac{\partial W_i}{\partial P_i} = \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P} \left(\frac{B_i \frac{\partial B_j}{\partial P_i} - B_j \frac{\partial B_i}{\partial P_i}}{B_i^{\,2}} \right)$$

From Appendix B.2 in equation (B.2.5) and (B.2.6)

$$\frac{\partial W_i}{\partial P_i} = \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P} \left(\frac{B_i \varepsilon_{ij} - B_j \left(1 - \sum_{\substack{j \neq i \\ j=1}}^{n} \varepsilon_{ij} \right)}{B_i^{\,2}} \right) \qquad \text{....... (B. 6.6)}$$

From (B.6.2)

$$\frac{\partial W_i}{\partial t_{ij}} = \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P} \left(\frac{B_i \frac{\partial B_j}{\partial t_{ij}} + B_j \frac{\partial B_i}{\partial t_{ij}}}{B_i^2} \right)$$

From Appendix B.2 in equation (B.2.7)

$$\frac{\partial W_i}{\partial t_{ij}} = \frac{\sum \alpha K_j}{\sum \alpha K_i} \cdot \frac{X_j^P}{X_i^P} \left(\frac{B_i \cdot 0 + \varepsilon_{ij} B_j}{B_i^2} \right) \qquad \text{....... (B. 6.7)}$$

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