Proof of Sharp Dimension Transition in Shortcut Model

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Earlier studies of a parametrized class of models, the shortcut models, whose fractal dimension transitions between integer values indicated that the transition occurs infinitely sharply at the parameter value $p = 0$, as the system size increases to infinity. In this work we prove the property. The sharp transition occurs because of the combinatorially large increase in the available number of paths connecting a pair of points as the path length is increased.

Keywords: Complex Networks; Fractal Dimension;

1. Introduction

The shortcut model was introduced [1,2] while studying the dimension [3] of complex networks (graphs) [4,5,6,7,8,9,10,11,12,13]. The model interpolates between discrete regular lattices. It was observed that the fractal dimension apparently transitions sharply [14,15] from 1 to 2 dimensions at the probability of shortcuts $p = 0$. In this work we provide a proof of this property. The sharp transition occurs because of the combinatorially large increase in the available number of paths connecting a pair of points as the path length is increased. We also study the mean path length and consider some generalisations of the model.

The model is defined by starting with a regular discrete lattice of dimension d with periodic boundary conditions, and adding shortcuts between remote vertices in the lattice. If the shortcuts connect vertices a constant distance apart in one of the d dimensions, then the resulting complex network transitions from a d-dimensional regular lattice to a (d+1)-dimensional regular lattice as the number of shortcuts is increased. The case of the transition from a one-dimensional to a two-dimensional lattice has been well-studied.

Section 2 reviews the shortcut model and proposes a possible generalisation. It also reviews the complex network zeta function. Section 3 studies the mean path length. Section 4 presents the proof that the dimension transitions infinitely sharply in the model as p increases from zero. Finally the conclusions are presented.

2. Shortcut Model

In this section we give a brief definition of the shortcut model [1,2], and we specify a possible generalisation. The model has fractal dimension as defined by the complex

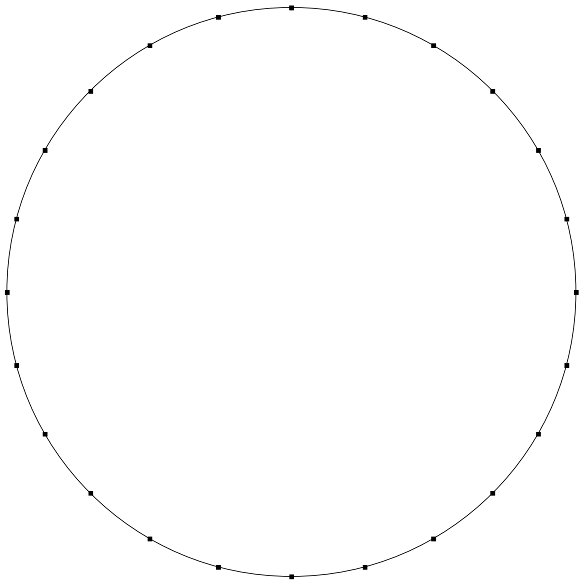

Fig. 1. Shortcut model, $p = 0$.

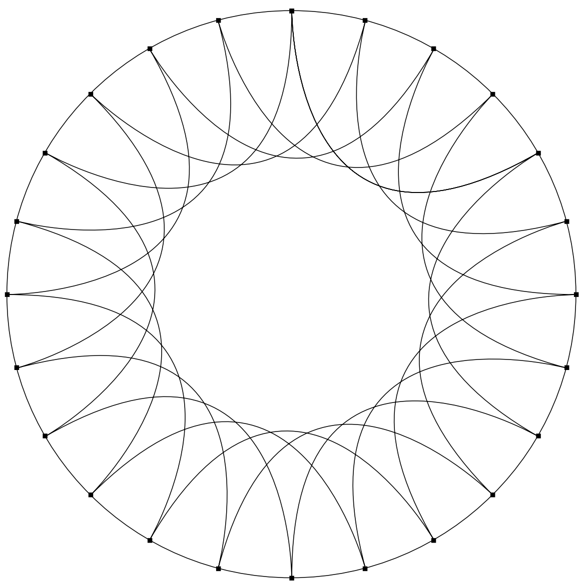

Fig. 2. Shortcut model, $p = 1$.

network zeta function [2], and transitions from a one-dimensional system to a two-dimensional system. The starting network is a one-dimensional lattice of N vertices

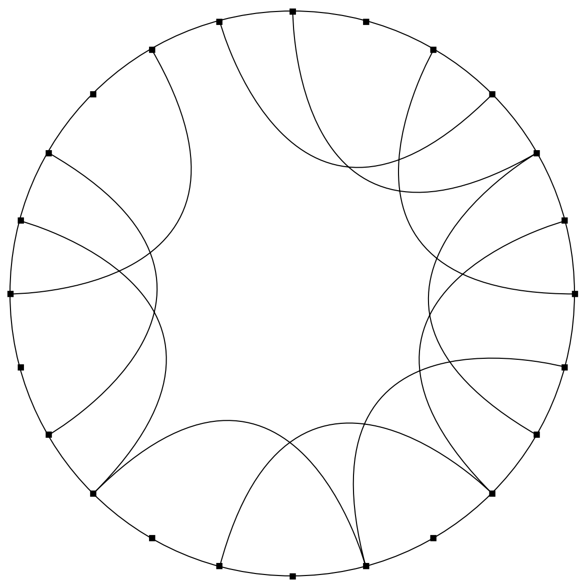

Fig. 3. Shortcut model, $0 < p < 1$.

with periodic boundary conditions. Each vertex is joined to its neighbors on either side, which results in a system with N edges. The network is extended by taking each node in turn and, with probability p, adding an edge to a new location m nodes ahead. We require that $N >> m >> 1$, say $m = \sqrt{N}$. The graphs are parametrized by:

$$size = N, \tag{1}$$

$$shortcut\ distance = m, and \tag{2}$$

$$shortcut\ probability = p. \tag{3}$$

When the shortcut probability $p = 0$ (Figure 1), we have a one-dimensional regular lattice of size N. The nodes are connected by edges represented by the arcs of the circle. When $p = 1$ (Figure 2), every node is connected by a shortcut edge to a new location. Each node now has edges in two directions, the first along the original direction (on the circle), and the second along the shortcut edges. The graph is essentially a two-dimensional graph with m and N/m nodes in each direction. For p between 0 and 1 (Figure 3), we have a graph which interpolates between the one and two dimensional systems.

It is interesting to note the difference between the shortcut model and the "small-world model" of Watts and Strogatz [16,17,18]. In the small world model also one starts with a regular lattice and adds shortcuts with probability p. However, the shortcuts are not constrained to connect to a node a fixed distance ahead. Instead, the other end of the shortcut can connect to any randomly chosen node. As a result, the small world model tends to a random graph rather than a two-dimensional graph as the shortcut probability is increased.

One possible generalisation of the shortcut model is to have a hierarchy of shortcuts. For example, the network can be extended by taking each node in turn and, with probability p_1, adding an edge to a new location m_1 nodes ahead, and with probability p_2, adding an edge to a new location m_2 nodes ahead. We require that $N >> m_1 >> m_2 >> 1$. The dimension of the extended model would lie between d and $d + 2$, where d is the dimension of the starting lattice.

The dimension of a complex network is defined as the value at which the complex network zeta function transitions from non-convergence to convergence in the infinite system limit. For completeness, the complex network zeta function is defined below. The complex network zeta function $\zeta_G(\alpha)$ is defined as

$$\zeta_G(\alpha) := \frac{1}{N} \sum_i \sum_{j \neq i} r_{ij}^{-\alpha}, \tag{4}$$

where N is the graph size, measured by the number of nodes. The definition Eq. 4 can be expressed as a weighted sum over the node distances. The graph surface function, $S(r)$, is defined as the number of nodes which are exactly at a distance r from a given node, averaged over all nodes of the network. This gives the Dirichlet series expression for the complex network zeta function:

$$\zeta_G(\alpha) = \sum_r S(r)/r^{\alpha}. \tag{5}$$

When the exponent α tends to infinity, the sum in Eq. 4 gets contributions only from the nearest neighbours of a node. The other terms tend to zero. Thus, $\zeta_G(\alpha)$ tends to the average vertex degree for the complex network. When α is zero the sum in Eq. 4 gets a contribution of one from each node. This means that $\zeta_G(\alpha)$ is $N - 1$, and hence tends to infinity as the system size increases.

Furthermore, $\zeta_G(\alpha)$ is a decreasing function of α. Thus, if it is finite for any value of α, it will remain finite for all higher values of α. If it is infinite for some value of α, it will remain infinite for all lower values of α. Thus, there is at most one value of α, $\alpha_{transition}$, at which $\zeta_G(\alpha)$ transitions from being infinite to being finite. This is reminiscient of the behaviour of Hausdorff dimension[3]. We define the complex network dimension as the value of the exponent α at which $\zeta_G(\alpha)$ transitions from being infinite to being finite. For regular discrete d-dimensional lattices $\mathbf{Z}^d$ with distance defined using the L^1 norm

$$\|\vec{n}\|_1 = \|n_1\| + \cdots + \|n_d\|, \tag{6}$$

the transition from non-convergence to convergence of the complex network zeta function occurs at $\alpha = d$, as one would expect.

In section 3 we study the mean path length. The average path length is given in terms of the graph surface function or the complex network zeta function by

$$\ell = \sum_r rS(r) / \sum_r S(r) = \lim_{N \to \infty} \zeta_G(-1)/\zeta_G(0). \tag{7}$$

In Eq. 7 we have normalised the mean path length using the number of nodes N in the denominator, rather than the maximum possible number of edges, $N(N-1)/2$. For the scaling behaviour of the mean path length with N, this just results in the scaling exponent increasing by one. We do this merely for the convenience of working with positive scaling exponents.

Table 1. Scaling Exponent for Mean Path Length

Probability	Scaling Exponent	Dimension
0.002	0.95	1.05
0.006	0.86	1.16
0.1	0.74	1.35
0.2	0.62	1.61
0.4	0.48	2.08
0.6	0.50	2.00
0.8	0.50	2.00
1.0	0.51	1.96

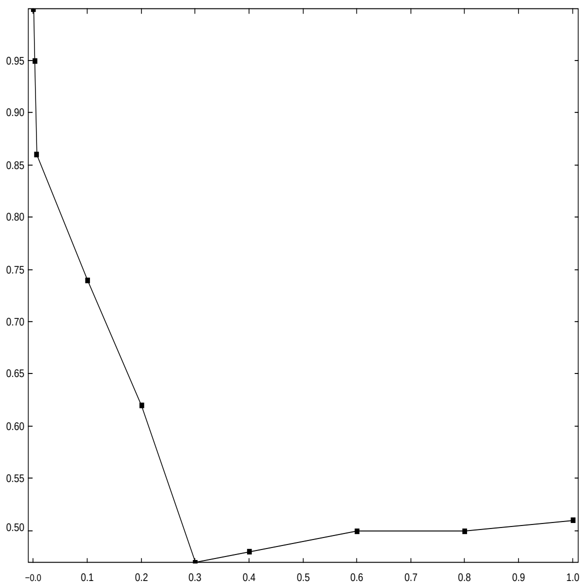

Fig. 4. Scaling exponent for mean path length vs. shortcut probability.

3. Mean Path Length

In this section we calculate the mean path length for the shortcut model. For a d-dimensional system we expect the mean path length ℓ to scale with size N as $N^{1/d}$. We show that the scaling exponent changes sharply as p increases from 0. This provides further evidence for the sharp transition of the dimension of the shortcut model at $p = 0$, a property which we prove in Section 4.

When the shortcut probability $p = 0$ the mean path length scales linearly. We calculated the mean path lengths using Eq. 7 for different p for N varying from 1000 to 16384. Table 1 shows the scaling exponent for ℓ. Figure 4 shows a plot of the scaling exponent vs. shortcut probability. Note, in particular, the rather sharp drop in the exponent from the value 1 as p increases from 0. The scaling exponent shows the change in behaviour from a one dimensional system to a two dimensional system.

In the next section we provide a proof for the sharp transition of the dimension at $p = 0$.

4. Proof of Sharp Transition

In this section we show that the dimension of the shortcut model transitions sharply at $p = 0$. The sharp transition is related to the combinatorially large increase in the available number of paths connecting a pair of points as the path length is increased, as explained below.

For a one-dimensional regular lattice the graph surface function $S_1(r)$ in Eq. 5 is exactly two for all values of r. This is because there are two nearest neighbours, two next-nearest neighbours, etc. Thus, the complex network zeta function $\zeta_G(\alpha)$ of Eq. 4 is equal to $2\zeta(\alpha)$, where $\zeta(\alpha)$ is the usual Riemann zeta function. Thus, the transition from non-convergence to convergence occurs at $\alpha = 1$. The graph surface function $S_d(r)$ for a lattice of dimension d scales aymptotically as $S_d(r) \rightarrow 2^d r^{d-1}/\Gamma(d)$ for large r. $r \rightarrow \infty$ corresponds to $\alpha \rightarrow \alpha_{transition}$. Thus, $\zeta_G(\alpha) \rightarrow 2^d \zeta(\alpha - d + 1)/\Gamma(d)$ as $\alpha \rightarrow \alpha_{transition}$. The largest pole of $\zeta_G(\alpha)$ occurs for $\alpha = d$.

Now consider the shortcut model which starts with a one-dimensional regular lattice. When $p = 0$ the graph surface function is constant as the path length is varied. When $p = 1$ the graph surface function increases linearly with the path length r. The increase in the graph surface function arises because points which were far apart when $p = 0$ are brought closer together by the shortcut edges when $p = 1$. This much is fairly straightforward. The interesting question is the behaviour when p lies between 0 and 1. Let us consider a pair of points which are separated by i shortcut edges and $r-i$ normal edges when $p = 1$. There are $\binom{r}{i}$ possible different paths connecting the two points, corresponding to all the possible ways of choosing the i shortcut edges in the total path of length r. This is a very large number when r and i are large. Consider a particular path from among this large set. When p lies between 0 and 1 the probability that this particular path will have all the i shortcut edges present is given by p^i. The probability that the particular path does not have all the required shortcut edges is $(1 - p^i)$. Thus, any one particular path will have a small probability of having all the required shortcut edges present. However, we need only one from among the huge number of available paths to have all the required shortcuts. The probability that none of the possible paths have the required shortcuts is $(1 - p^i)^{\binom{r}{i}}$. When p lies strictly between 0 and 1 this rapidly goes to zero because of the combinatorially large value of the exponent $\binom{r}{i}$. Thus, the probability that there is at least one path which has all the i required shortcuts is given by

$$1 - (1 - p^i)^{\binom{r}{i}}, \tag{8}$$

which is very close to 1. Thus, when p lies strictly between zero and one, the graph surface function becomes essentially the same as for the case $p = 1$. This accounts for the sudden transition observed empirically in earlier studies for the dimension, when p increases from 0.

5. Conclusions

In this work we showed that the dimension of shortcut model shows a sharp transition at $p = 0$. The sharp transition had been conjectured earlier, based on the study of the complex network zeta function, and on processes like the random walk. The proof of this property is now provided.

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