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Graph Zeta Function and Dimension of Complex Network

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In a recent paper we had defined the dimension of a complex network in terms of the scaling property of the volume. The question assumes significance because the dependence of system behaviour on dimension is an important topic in statistical mechanics. Hence we consider the definition in more detail, and we propose a more widely applicable definition in this work. This definition has good mathematical properties, and it is based on the definition of a zeta function for complex networks.

Keywords: Complex Networks, Fractal Dimension, Graph Zeta Function

1. Introduction

In a recent paper[1] we defined the dimension of a complex network[2,3,4,5,6,7,8,9] in terms of the scaling behaviour of the volume. We had then applied the defintion to study the transition of a system from extensive to non-extensive thermodynamic behaviour as the exponent of a power law interaction potential is varied. Since the dependence of a system's behaviour on dimension is a key topic of Statistical Mechanics, we propose a more widely applicable definition in this work. The new definition has good mathematical properties. We base the definition on the concept of a zeta function for complex networks. Though our main interest is in physics systems, the new definition proposed here is likely to be useful even in the more general applications of complex networks, because of its well-behaved mathematical structure.

In section 2 we review the volume definition of dimension[1]. We present some further properties of the definition. In section 3 we define the zeta function for a complex network. We apply the zeta function to present a new definition of the dimension. In section 4 we calculate the graph zeta function for some networks. In section 5 we compare the two definitions and present the conclusions.

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2. Volume Dimension

In this section we review the volume definition of dimension[1]. We present some further properties of the definition. The previous paper, and this work, have their main motivation in applications to physics systems [10,11,12,13,14,15,16,17,18,19,20,21]. The volume definition of dimension is

Definition 1. Volume definition of dimension: The distance r_{ij} between nodes i and j is the length of the shortest path connecting the nodes. The volume is the number of nodes j within a distance r_{ij} of node i, averaged over i. The dimension d is the exponent which determines the scaling behaviour of the volume with distance.

One of the requirements that was placed on the volume definition of dimension was that it reduce to the "obvious" dimensions of discrete regular lattices $\mathbf{Z}^d$. The first comment we wish to make is that Definition 1 allows us to set up a metric space corresponding to a complex network, if we specify that the distance of a node from itself is zero. The reason that we can set up the metric space is that Definition 1 satisfies the triangle inequality

$$r_{ik} < r_{ij} + r_{jk} \tag{1}$$

for all j. This opens up the intriguing possibility of defining open sets and other topological structures for a complex network.

We also emphasize that the scaling properties implied by Definition 1 are characteristic of physics systems. In the more general applications of complex networks, the volume often tends to infinity even for a finite distance, as the system size goes to infinity. We illustrate with the example of random graphs, defined as networks having some number N of vertices, and each pair is connected with probability p (or else the pair is disconnected). Such graphs have a diameter of two with probability approaching one, in the infinite limit! To see this, consider two nodes A and B. For any node C different from A or B, the probability that C is not connected to both A and B is $(1 - p^2)$. Thus, the probability that none of the $N - 2$ nodes provides a path of length 2 between nodes A and B is $(1 - p^2)^{(N-2)}$. This goes to zero as the system size goes to infinity, and hence most random graphs have their nodes connected by paths of length at most 2! In physics systems the interest is mainly in systems which have finite volumes for finite distances, as the system size goes to infinity.

Defintion 1 was used[1] to study the transition of a system from extensive to non-extensive thermodynamic behaviour as the exponent of a power law interaction potential is varied. For the power law potential [10,19,20,21], the interaction varies with the distance r as $1/r^{\alpha}$. We found the intriguing result that the transition took place when the power law exponent was exactly equal to the dimension of the system! For the potential to be extensive, we require

$$\alpha > d, \tag{2}$$

where d is the volume dimension of the system. This raises the question of whether this property can be used to define the dimension. In the next section we show that we can indeed base the definition of dimension on this property, and in fact the new definition is applicable even to systems where the volume does not scale as some power of the radius.

3. Zeta Function Dimension

In this section we define the zeta function for a complex network. We apply the zeta function to present a new defintion of the dimension. Zeta functions have been defined for a variety of systems. The earliest and most well-known zeta function is the Riemann Zeta function $\zeta(\alpha)$, defined[22,23,24,25] for $\mathrm{Re}(\alpha) > 1$ by

$$\zeta(\alpha) = \sum_{n=1}^{\infty} n^{-\alpha}. \tag{3}$$

Eq. (3) converges for $\mathrm{Re}(\alpha) > 1$. When $\alpha = 1$ the sum diverges logarithmically with the system size N. $\zeta(\alpha)$ is a decreasing function of α and tends to one as α tends to infinity.

For a complex network, let us define the graph zeta function $\zeta_G(\alpha)$ as

$$\zeta_G(\alpha) := \frac{1}{N} \sum_i \sum_{j \neq i} r_{ij}^{-\alpha}. \tag{4}$$

When the exponent α tends to infinity, the sum in Eq. 4 gets contributions only from the nearest neighbours of a node. The other terms tend to zero. Thus, $\zeta_G(\alpha)$ tends to the average coordination number of the system. Except for special cases like the complete graph (a graph where every node is connected to every other node), for which the average coordination number is $N - 1$, this will be a finite quantity. For the special cases we define the zeta function dimension to be infinite. When α is zero the sum in Eq. 4 gets a contribution of one from each node. This means that $\zeta_G(\alpha)$ is $N - 1$, and hence tends to infinity as the system size increases.

Furthermore, $\zeta_G(\alpha)$ is a decreasing function of α,

$$\zeta_G(\alpha_1) > \zeta_G(\alpha_2), \tag{5}$$

if $\alpha_1 < \alpha_2$. Thus, if it is finite for any value of α, it will remain finite for all higher values of α. If it is infinite for some value of α, it will remain infinite for all lower values of α. Thus, there is at most one value of α, $\alpha_{transition}$, at which $\zeta_G(\alpha)$ transitions from being infinite to being finite. This is reminiscient of the behaviour of Hausdorff dimension[26]. In fact, we can define the complex network dimension as the value of the exponent α at which $\zeta_G(\alpha)$ transitions from being infinite to being finite, i.e.,

$$d_{zeta-function} := \alpha_{transition}. \tag{6}$$

If $\zeta_G(\alpha)$ remains infinite in the large system limit for all values of α, we define the graph dimension to be infinite. This will be case, for example, for complete graphs or for the random graphs considered in Section 2.

Let $S(r)$ denote the number of nodes which are exactly at a distance r from a given node, averaged over all nodes of the network. We can write

$$\zeta_G(\alpha) = \sum_r S(r) r^{-\alpha}. \tag{7}$$

Thus, the graph zeta function definition of the dimension is equivalent to the volume definition[1] when the volume definition is defined. The zeta function dimension corresponds to the ususal definition of dimension for the regular lattices $\mathbf{Z}^d$, since $S(r)$ scales as r^{d-1} for these lattices, and hence $\zeta_G(\alpha)$ diverges when $\alpha \leq d$. For regular lattices $\mathbf{Z}^d$ we use the L^1 norm,

$$\|\vec{n}\|_1 = \|n_1\| + \cdots + \|n_d\|, \tag{8}$$

because this is the norm which generalises to complex networks. The zeta function definition has the advantage that it is defined for all graphs.

There are many accepted properties [26] that a definition of dimension is expected to satisfy. We give below some of the usually specified properties. Let $\mathbf{U}$ be the set of graph nodes for which dimension is being defined. For nodes X, Y in the set $\mathbf{U}$ we define the bi-Lipschitz property as

Definition 2. A function $f : \mathbf{U} \to \mathbf{U}$ is *bi-Lipschitz* if there exists $c_1, c_2 \in (0, \infty)$ such that, for all $X, Y \in \mathbf{U}$, $c_1 r(X,Y) \leq r(f(X), f(Y)) \leq c_2 r(X,Y)$,

where r is the distance between X and Y. Let $E, F \subseteq \mathbf{U}$. The properties that should be satisfied by a definition of dimension include:

(1) Monotonicity: $E \subseteq F$ implies $\dim(E) \leq \dim(F)$.
(2) Stability: $\dim(E \cup F) = \max\{\dim(E), \dim(F)\}$.
(3) Lipschitz invariance: If $f : \mathbf{U} \to \mathbf{U}$ is bi-Lipschitz, then $\dim(f(E)) = \dim(E)$.

From the preceeding discussion it is clear that the zeta function defintion of dimension Eq. 6 satisfies all the conditions.

4. Examples of Graph Zeta Function

In this section we calculate the graph zeta function for some networks. For the complete graph,

$$\zeta_G(\alpha) = N - 1 \tag{9}$$

for all α. For random graphs, the average coordination number will be $p(N-1)$. We recall the discussion in section 2, that for random graphs almost all nodes are at a distance of one or two from any given node. Thus,

$$\zeta_G(\alpha) = p(N-1) + (N - p(N-1)) 2^{-\alpha}. \tag{10}$$

Let us now consider the the regular lattices $\mathbf{Z}^d$. We remind the reader that we use the L^1 norm Eq. 8. We first study $S_d(r)$, the number of nodes which are exactly at a distance r from a given node, averaged over all nodes of the lattice. We can then use Eq. 7 to calculate $\zeta_G(\alpha)$. The $S_d(r)$ satisfy the recursion relation

$$S_{d+1}(r) = 2 + S_d(r) + 2\sum_{i=1}^{r-1} S_d(i). \tag{11}$$

This result follows by choosing a given axis of the lattice and summing over cross-sections for the allowed range of distances along the chosen axis. Asymptotically, $S_d(r) \to O(2^d r^{d-1}/\Gamma(d))$ as $r \to \infty$. $S_1(r)$ is equal to 2. Table 1 gives $S_d(r)$ and $\zeta_G(\alpha)$ for some values of the lattice dimension d. It follows from the recursion relation 11 that $S_d(r)$ is a polynomial of order $d-1$ in r, and hence $\zeta_G(\alpha)$ is the sum of different Riemann zeta functions. In studying higher values of d, the following expression for the sum of positive integers raised to a given power k will be useful:

$$\sum_{i=1}^{r} i^k = \frac{r^{k+1}}{(k+1)} + \frac{r^k}{2} + \sum_{j=1}^{(k+1)/2} \frac{B_{2j} k! r^{k+1-2j}}{(2j)!(k+1-2j)!}, \tag{12}$$

where B_{2j} are the Bernoulli numbers[25]. Eq. 12 results from the application of the Euler-Maclaurin summation formula[25].

5. Conclusions

We have defined the dimension of a complex network in terms of graph zeta functions. This definition is mathematically well-behaved. It is defined for all graphs, and has a close analogy to the Hausdorff dimension. We expect that this definition will be useful for applications beyond the physics domain. We have given calculations of the graph zeta function for some specific networks.

We have shown that the zeta function definition given here is equivalent to the volume definition given in an earler paper[1]. However, the volume definition is defined only for complex networks which have specific scaling properties. The volume definition has the advantage that it can be easily evaluated numerically. Thus, the zeta function definition presented here is useful for mathematical analysis, while the volume definition[1] is useful for numerical studies.

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Table 1. Graph zeta function for different regular lattices.

Dimension d	$S(r)$	$\zeta_G(\alpha)$
1	2	$2\zeta(\alpha)$
2	$4r$	$4\zeta(\alpha-1)$
3	$4r^2+2$	$4\zeta(\alpha-2)+2\zeta(\alpha)$
4	$\frac{8}{3}r^3+\frac{16}{3}r$	$\frac{8}{3}\zeta(\alpha-3)+\frac{16}{3}\zeta(\alpha-1)$
$r\to\infty$	$O(2^d r^{d-1}/\Gamma(d))$	$O(2^d\zeta(\alpha-d+1)/\Gamma(d))$