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Defining dimension of a complex network

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An important question in Statistical Mechanics is the dependence of model behaviour on the dimension of the system. In this paper we discuss extending the definition of dimension from regular lattices to complex networks. We use the definition to study how the extensive property of the power law potential exponent depends on dimension.

A related article is "Zeta Function and Dimension of Complex Network" (Modern Physics Letters B (To be published), available at "http://www.geocities.com/oshanker/graphzetafunction.pdf")

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1. Introduction

The behaviour of different processes on discrete regular lattices have been studied quite extensively. They show a rich diversity of behaviour, including a non-trivial dependence on the dimension of the regular lattice [1,2,3,4,5,6,7,8,9]. In recent years the study has been extended from regular lattices to complex networks. Thus, it seems useful to define dimension for complex networks.

In this paper I discuss extending the definition of dimension to complex networks. I illustrate the use of the definition with some examples. I then use the definition to study how the extensive property of the power law potential exponent depends on dimension. Finally, I present the conclusions.

2. Dimension of Complex Network

There has been growing interest in extending the study of different processes from regular lattices such as $\mathbf{Z}^d$, where d, a positive integer, is the dimension, to complex networks [10,11,12,13,14,15,16,17]. Processes have been studied for a variety of complex systems, including social networks [15], biochemical networks [18,19,20], and information networks such as the web [21]. We propose here one possible definition for the dimension of complex networks. There are many ways to approach the problem, and the best definition will depend on the nature of the process being studied. For

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systems which arise in physical problems (which is our main focus) one usually can identify some physical space relations among the vertices. Nodes which are linked directly will have more influence on each other than nodes which are separated by several links. Thus, it makes sense to define the distance $r(i, j)$ between nodes i and j as the length of the shortest path connecting the nodes. Our definition of dimension attempts to make use of this property. Secondly, since the dimensions of regular discrete lattices can be easily specified, we also require that our definition reduce to the "obvious" dimensions of discrete regular lattices. These conditions reduce the number of possibilities we need to consider.

Usually, dimension is defined based on the scaling exponent of some property in the appropriate limit. The best property to use for physics applications seems to be the scaling of volume with distance. For regular $\mathbf{Z}^d$ lattices the number of nodes j within a distance $r(i, j)$ of node i scales as $r(i, j)^d$. For complex networks, we can define the volume as the number of nodes j within a distance $r(i, j)$ of node i, averaged over i, and we define the dimension as the exponent which determines the scaling behaviour of the volume with distance. For a vector $\vec{n} = (n_1, \ldots, n_d) \in \mathbf{Z}^d$, where d is a positive integer, the Euclidean norm $\|\vec{n}\|$ is defined as the Euclidean distance from the origin to $\vec{n}$, i.e.,

$$\|\vec{n}\| = \sqrt{n_1^2 + \cdots + n_d^2}. \tag{1}$$

However, the definition which generalises to complex networks is the L^1 norm,

$$\|\vec{n}\|_1 = \|n_1\| + \cdots + \|n_d\|, \tag{2}$$

and we shall use this norm for our study. The scaling properties hold for both the Euclidean norm and the L^1 norm. The scaling relation is

$$V(r) = kr^d, \tag{3}$$

where d is not necessarily an integer for comlex networks. k is a geometric constant which depends on the complex network. If the scaling relation Eqn. 3 holds, then we can also define the surface area $S(r)$ as the number of nodes which are exactly at a distance r from a given node, and $S(r)$ scales as

$$S(r) = kdr^{d-1}. \tag{4}$$

There are many accepted properties[22] that a definition of dimension is expected to satisfy. Let $\mathbf{U}$ be the set for which dimension is being defined. If we can define vectors $\vec{m}$ corresponding to the set $\mathbf{U}$, then we give below some of the usually specified properties.

Definition 1. A function $f : \mathbf{U} \to \mathbf{U}$ is *bi-Lipschitz* if there exists $\alpha, \beta \in (0, \infty)$ such that, for all $\vec{m}$, $\vec{n} \in \mathbf{U}$, $\alpha\|\vec{m} - \vec{n}\| \leq \|f(\vec{m}) - f(\vec{n})\| \leq \beta\|\vec{m} - \vec{n}\|$.

Let $A, B \subseteq \mathbf{U}$. Then the properties that should be satisfied by a definition of dimension include:

(1) Monotonicity: $A \subseteq B$ implies $\dim(A) \leq \dim(B)$.
(2) Stability: $\dim(A \cup B) = \max\{\dim(A), \dim(B)\}$.
(3) Lipschitz invariance: If $f : \mathbf{U} \to \mathbf{U}$ is bi-Lipschitz, then $\dim(f(A)) = \dim(A)$.

When the set $\mathbf{U}$ consists of the nodes of a complex network, it is not straightforward to interpret the above conditions, because the concept of a vector is not obvious for complex networks. However, all the above properties are satisfied for discrete regular lattices $\mathbf{Z}^d$, which is one reason for insisting that our definition of dimension for complex networks reduce to the usual definition for the special case of discrete regular networks. In the next section we show how the definition of the dimension as the scaling exponent of volume with distance can be applied to some particular models.

3. Examples of dimension calculation

In this section we calculate the dimension for a given class of models. The models that we study start with a network built on a one-dimensional regular lattice. One then adds edges to create a low density of "shortcuts" that join remote parts of the lattice to one another. The starting network is a one-dimensional lattice of N vertices with periodic boundary conditions, (a ring). Each vertex is joined to its neighbors on either side, which results in a system with N edges. The network is extended by taking each node in turn and, with probability p, adding an edge to a new location m nodes distant.

The rewiring process allows the model to interpolate between a one-dimensional regular lattice and a two-dimensional regular lattice. When the rewiring probability $p = 0$, we have a one-dimensional regular lattice of size N. When $p = 1$, every node is connected to a new location and the graph is essentially a two-dimensional lattice with m and N/m nodes in each direction. For p between 0 and 1, we have a graph which interpolates between the one and two dimensional regular lattices. The graphs we study are parametrized by:

$$size = N, \tag{5}$$

$$shortcutdistance = m, and \tag{6}$$

$$rewiringprobability = p. \tag{7}$$

For illustrative purposes we have chosen the modest values $N = 3599$ and $m = 59$, and varied p between 0.0 and 0.25, by which value the network behaves pretty much like a two-dimensional system. $N = 3599$ is equal to the product $59 * 61$, and 59 and 61 are primes. We numerically evaluated the dimension of the models using Eqn. 3, and doing a linear regression on a log-log plot. Fig. 1 shows the dependence of the dimension on the probability of shortcuts. The dimension shows the expected interpolating behaviour between one and two dimensional behaviour. The main caution that must be made is that the path lengths over which the scaling behaviour is studied must be small compared to the system size, to avoid finite size

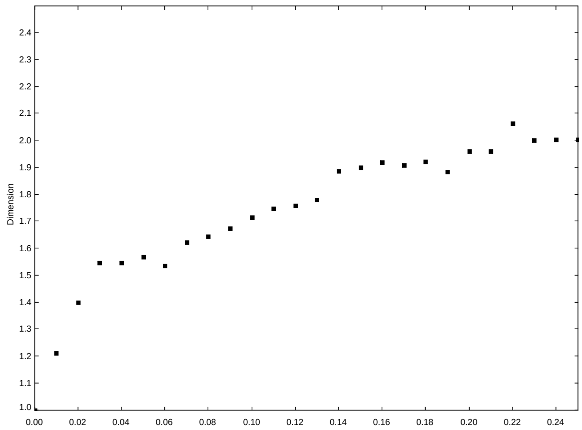

Fig. 1. Dependence of the dimension on the probability of shortcuts

effects. When we average over several randomly chosen points, the scaling behaviour sets in fairly quickly. Thus, we see that the definition of dimension works well even for the relatively modest size of 3599 nodes. The actual networks that are studied in the literature are normally larger than the example chosen here.

4. Application to extensiveness of power law potential

To illustrate the use of the above definition of dimension, we study the power law potential [23,24,25], where the interaction varies with the distance r as $1/r^{\alpha}$. In one dimension the system properties like the free energy do not behave extensively when $0 \leq \alpha \leq 1$, i.e., they increase faster than N as $N \to \infty$, where N is the number of spins in the system.

For definiteness, we will consider the Ising model with the Hamiltonian (with N spins)

$$H = -\frac{1}{2} \sum_{i,j} J(r(i,j)) s_i s_j \tag{8}$$

where s_i are the spin variables, $r(i,j)$ is the distance between node i and node j, and $J(r(i,j))$ are the couplings between the spins. We will specialise to ferromagnetic couplings, i.e., all the J are positive. When the $J(r(i,j))$ have the behaviour $1/r^{\alpha}$, we have the power law potential. For a general complex network we wish to find the condition on the exponent α which preserves extensivity of the Hamiltonian. At zero temperature, the energy per spin is proportional to

$$\rho = \sum_{i,j} J(r(i,j)), \tag{9}$$

and hence extensivity requires that ρ be finite. This requirement is the one which makes $\alpha = 1$ the transition point for the one-dimensional power law potential. For

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one dimensional regular lattices ρ is proportional to the Riemann Zeta function $\zeta(\alpha)$. $\zeta(s)$ is defined[26,27,28,29] for $\mathrm{Re}(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} \left(1 - p^{-s}\right)^{-1} . \tag{10}$$

The product expression over the primes was first given by Euler. Eq. (10) converges for $\mathrm{Re}(s) > 1$. When $s = 1$ the sum diverges logarithmically with the system size N. For a general complex network we can use Eqn. 4 to show that ρ is proportional to $\zeta(\alpha - d + 1)$. Thus, for the potential to be extensive, we require

$$\alpha > d. \tag{11}$$

For the models studied in Section 3 we numerically evaluated the sum at the transition value of the exponent. Table 1 illustrates the logarithmic divergence of ρ when the exponent α takes on the transition point value.

Table 1. Sum ρ for distance $r = 10$.

Probability p	Value of sum ρ	Logarithmic estimate
0.0	5.86	5.95
0.05	4.74	4.73
0.1	5.40	5.44
0.15	5.26	5.46
0.2	5.36	5.65
0.25	5.55	6.09

5. Conclusions

We have defined dimension for a complex network, which reduces to the usual definition for regular discrete lattices. We showed examples of how it can be calculated in some models. We used the definition to study the behaviour of the power law potential as a function of the dimension for complex networks.

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