Hurst Exponent for spectra of Complex Networks

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In this paper we use the rescaled range analysis to study the spacings between the eigenvalues of the adjacency matrices of different types of complex networks. The distribution seems to be of the persistent fractional Brownian motion type. The spacings have a Hurst exponent varying from 0.5 to 0.9 for the networks studied. This range implies a positive correlation between successive increments in the sequence of eigenvalues. For Hurst exponents at the lower end, a change in the parameters could lead to negative correlations.

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I. INTRODUCTION

Recent years have seen much use of complex networks [1–8] in the quantitative study of a variety of complex systems, including social networks [6], biochemical networks [9–11], and information networks such as the web [12]. In this paper we use the rescaled range analysis to study the spacings between the eigenvalues of the adjacency matrix of different types of complex networks. Our motivation comes from an earlier work [13] which found that the application of this method of analysis to the Riemann zeta function [14, 15] and Dirichlet L-functions uncovered several intriguing results. Given the close relation of these functions [16] to the theory of random matrix models [17–25], it seems interesting to study the random matrices which arise in other contexts. This paper investigates the random matrices which arise in the study of complex networks.

II. TYPES OF COMPLEX NETWORKS

In this section we establish the required notation and introduce the types of network models that we have studied. The origin of the theory of networks is often credited to Euler's celebrated solution in 1735 of the Königsberg bridge problem. The availability of powerful computers and communication networks in recent days has resulted in the focus shifting from the analysis of single small graphs to consideration of large-scale statistical properties of graphs. The networks studied in the literature can be classified in many ways, e.g., directed vs. undirected networks, network growth mechanisms, the processes taking place on the networks, etc. In this paper we study the Poisson random graph models of Rapoport [26, 27] and Erdős and Rényi [28, 29], the "small-world model" of Watts and Strogatz [30–32], and the Barabási and Albert [33, 34] model incorporating the mechanisms of growth and preferential attachment and leading to scale-free power law degree distributions.

The random graph models that we study here are the $G_{n,m}$ of Erdős and Rényi, which is the ensemble of all graphs having n vertices and exactly m edges, each possible graph appearing with equal probability. Random graphs are parametrized by:

$$size = N, and \tag{1a}$$

$$edges = m. \tag{1b}$$

Solomonoff and Rapoport [26] and independently Erdős and Rényi [28] proposed the very similar and simple model called $G_{n,p}$ by Erdős and Rényi. This is defined as a network having some number n of vertices, and each pair is connected with probability p (or else the pair is disconnected).

The "small-world model" of Watts and Strogatz is a model with high transitivity. It is useful for networks that may have a geographical component to them (e.g., the vertices are related to positions in space and geographical proximity plays a role in deciding which vertices are connected to which others). The small-world model starts with a network built on a low-dimensional regular lattice. One then moves edges to create a low density of "shortcuts" that join remote parts of the lattice to one another. The best studied case is the one-dimensional lattice. The starting network is a one-dimensional lattice of N vertices with periodic boundary conditions, (a ring). Each vertex is joined to its neighbors k or fewer lattice spacings away, which results in a system with Nk edges. The small-world model is created by taking each edge in turn and, with probability p, moving one end of that edge to a new location chosen uniformly at random from the lattice, except that no double edges or self-edges are created.

The rewiring process allows the small-world model to interpolate between a regular lattice and a network which is similar to a random graph. When the rewiring probability $p = 0$, we have a regular lattice. When $p = 1$, every edge is rewired to a new random location and the graph is almost a random graph. Watts and Strogatz showed by numerical simulation that there exists a region in between these two extremes for which the model

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has both low path lengths and high transitivity. Watts and Strogatz graphs are parametrized by:

$$size = N, \quad (2a)$$
$$neighbours = k, and \quad (2b)$$
$$rewiringprobability = p. \quad (2c)$$

The model of Barabási and Albert starts with a small initial network of m_0 unconnected nodes. New vertices are added to the network with degree m. The other end of each newly added edge is attached to an existing vertex with probability proportional to the degree of that vertex. The edges are undirected. Each vertex in the graph appears with initial degree m, and hence automatically has a non-zero probability of receiving new links.

Let p_k be the fraction of vertices in the network with degree k, so that $\sum_k p_k = 1$. The probability that a new edge attaches to a vertex of degree k is

$$kp_k / \sum_k kp_k = kp_k/2m. \quad (3)$$

The sum in the denominator is equal to the mean degree of the network, which is $2m$, since there are m edges for each vertex added, and each edge contributes two ends to the degrees of network vertices. This type of growth leads to a scale-free power law degree distribution. Watts and Strogatz graphs are parametrized by:

$$size = N, \quad (4a)$$
$$initialnodes = m_0, and \quad (4b)$$
$$degree = m. \quad (4c)$$

We study the spectra of the the adjacency matrix A,

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge joining vertices } i, j, \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

(for undirected networks A is symmetric). We use Householder transformations to convert the matices to Hessenberg form, and we use the QR algorithm to find the eigenvalues. We use the central 70 per cent of the eigenvalues for the analysis. In what follows we apply the rescaled range analysis to study the distribution of the spacings of the adjacency matrix for these three types of graphs.

III. ANALYSIS OF EIGENVALUE DISTRIBUTIONS

In this section we define the notation for rescaled range analysis [35–39] and apply it to study the eigenvlaues of the adjacency matrices of complex networks. This method has been widely used, and it is suited for phenomena which exhibit a combination of random (or pseudo-random) behaviour and regular behaviour. The observed value in the series of observations is commonly denoted by $\xi(t)$, where t is the ordinate of the observation. Rescaled range analysis studies the correlations and the distribution of the ξ by boxing the observed data into bins of different sizes (the bin size being denoted by τ), and by studying the scaling behaviour of the statistical parameters with the bin size τ. The results of the analysis are summarized by the Hurst exponent H, defined below. Most natural phenomenon seem to have a value of about 0.72 for H.

Let the mean value of ξ for a given bin be denoted by $\langle\xi\rangle_\tau$. We define $X(t,\tau)$ as

$$X(t,\tau) = \sum_{u=1}^{t} \left(\xi(u) - \langle\xi\rangle_\tau\right). \quad (6)$$

The range $R(\tau)$ is $Max(X(t,\tau)) - Min(X(t,\tau))$ and $S(\tau)$ denotes the standard deviation of the ξ. Then, as explained by Mandelbrot and Ness [40, 41], under quite general conditions the dimensionless rescaled range R/S varies with τ as $\tau \to \infty$ according to the scaling law

$$(R/S) = (c\tau)^H, \quad (7)$$

where H is defined as the Hurst exponent. One can use linear regression analysis on a log-log plot of R/S against τ to estimate the Hurst exponent H as the best fit slope of the log-log plot. If the ξ behave like normal Brownian motion, then the Hurst exponent is 0.5. Any deviation from this value implies that the values of the observable are not independent of each other. Mandelbrot introduced a generalized form of the Brownian motion model, the fractional Brownian motion [40, 41], as a typical simple family of random functions that models asymptotic dependence. In this model the Hurst exponent lies in the range $0 < H < 1$. H is related to the fractal dimension of the graph of the series of observations. A low value of H implies a large fractal dimension, namely, a curve which shows a lot of detailed structure. There are three types of generalized fractional Brownian motion: (a) the persistent, for values of H in the range $0.5 < H < 1$, (b) the case $H = 0.5$ which corresponds to the independent white noise processes of ordinary Brownian motion, and (c) the anti-persistent, for $0 < H < 0.5$.

A persistent type of fractional Brownian motion implies that the increments persistence is maintained over longer periods of time, depending on the excess of the Hurst exponent value over 0.5. If at some time in the past there was a positive increment i. e., an increase, it is more likely that there will be an increase in the future. A decreasing trend in the past implies the likelihood of a decreasing trend in the future.

In the anti-persistent range any increasing trend in the past makes a decreasing trend in the future more probable, and vice versa. The strength of this anti-correlation depends on the extent to which H is lower than 0.5. The graph for an anti-persistent process shows a lot of jumps. We find that the spacings of the eigenvalues of the adjacency matrix for all the networks studied show a positive correlation, i.e., they are of the persistent type.

We applied the rescaled range analysis to the following networks: a) Erdős and Rényi, b) Watts and Strogatz, and c) Barabási and Albert. Table I shows the

results of the analysis for the three models. For comparision, it may be mentioned that regular Brownian motion leads to an exponent of 0.5, and that most analyses in the literature find a value around 0.72 for many natural phenomenon. Fig. 1 shows the regression analysis for eigenvalues of the Barabasi Albert network with $N = 400, m = m_0 = 5$. The horizontal axis is the log of the bin size ($log_2(\tau)$), and the vertical axis is the log of the mean $R/S(log_2(R/S))$ for the given values of τ. The slope of the best fit line (also shown in the figure) gives the value of the Hurst exponent. In the next section we conclude by comparing the results with the results for the Riemann zeta function.

IV. CONCLUSIONS

Our study of the spectra of the adjacency matrices of different types of complex networks indicates that the distribution seems to be of the persistent fractional Brownian motion type. The spacings have a Hurst exponent H varying from 0.5 to 0.9 for the networks studied. This range implies a positive correlation between successive increments in the sequence of eigenvalues ($H > 0.5$). For Hurst exponents at the lower end, a change in the parameters could lead to negative correlations.

The zeroes of the generalised zeta functions are expected to be closely related to the eigenvalues of random matrices chosen from different ensembles. While those ensembles are different from the ones which arise in complex networks, it is instructive to compare the results and note the differences. Our study for the generalised zeta functions showed that the Hurst exponent is remarkably constant for the different zeta functions. We found a variation in H for the Riemann zeta function from 0.09 to 0.0997, for zeroes covering the range 10^7 to 10^{22} (fifteen orders of magnitude)! The low value of the exponent is interesting. It implies that there is a large degree of antipersistence. There is a significant amount of self-affinity in the distribution of the zeroes. The fractal dimension is very close to 1.9, indicating a very noisy behaviour. The distribution for different Dirichlet-L functions at the intermediate range of zeroes also shows values for the Hurst exponent which are very close to the values found for the Riemann function zeroes. Thus, though the zeros are expected to be related to random matrices, the behaviour is quite different from the behaviour of the random matrices which appear in studying the spectra of complex networks.

TABLE I: Hurst Coefficient for different models.

	Network parameters		Rescaled Range results	
Model	Size N	Other parameters	H	Standard error
Barabási and Albert	400	$m = 5, m_0 = 5$	0.88	0.05
Barabási and Albert	514	$m = 5, m_0 = 5$	0.87	0.09
Erdős and Rényi	200	$m = 400$	0.65	0.04
Erdős and Rényi	200	$m = 2000$	0.57	0.08
Watts and Strogatz	200	$k = 10, p = 0.1$	0.90	0.04
Watts and Strogatz	200	$k = 10, p = 0.3$	0.71	0.04
Watts and Strogatz	200	$k = 10, p = 0.7$	0.62	0.03

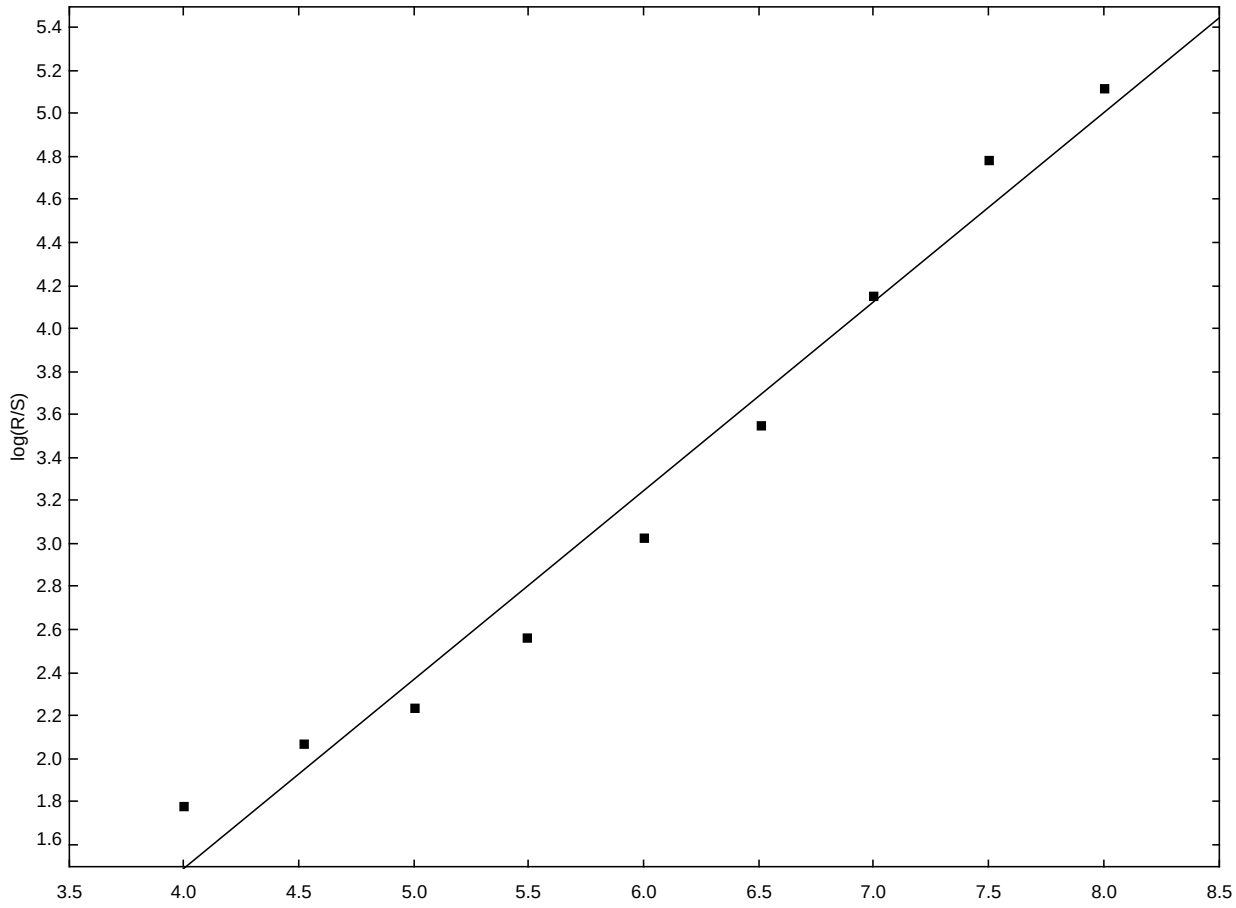

FIG. 1: Rescaled Range Analysis for the Barabasi Albert network with $N = 400, m = m_0 = 5$.

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