Problem Book

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Contents

Part I

Classical Electrodynamics

Chapter 1

Electrostatics

1.1 Problem. Pr. 1.1

A bidimensional electrostatic field varies with x and y. Show that the average of the potential along any circle is the same that the potential at the center of the circle, bucause there are not charges on the region.

Solution

Consider a circle centred at the origin of coordinates, the radius is r_0. For this set up,

$$\langle \Phi \rangle_{r_0} = \frac{1}{2\pi r_0} \int r_0 d\theta \Phi(r_0, \theta) = \frac{1}{2\pi} \int d\theta \Phi(r_0, \theta). \tag{1.1}$$

Since $\langle \Phi \rangle_{r_0}$ does not depend on r_0, it follows that

$$\frac{d}{dr} \langle \Phi \rangle_{r_0} = 0. \tag{1.2}$$

The last equation holds even for $r_0 \to 0$, then

$$\boxed{\langle \Phi \rangle_{r_0} = \langle \Phi \rangle_{r=0} = \Phi(0).} \tag{1.3}$$

If one considers a circle centred at $\vec{R} - \vec{r}$,

$$\langle \Phi \rangle_{r_0} = \frac{1}{2\pi r_0} \int r_0 d\theta \Phi(\vec{R}, \theta) = \frac{1}{2\pi} \int d\theta \Phi(\vec{R}, \theta). \tag{1.4}$$

Since $\langle \Phi \rangle_{r_0}$ does not depend on r_0, it follows that

$$\frac{d}{dr} \langle \Phi \rangle_{r_0} = 0. \tag{1.5}$$

The last equation holds even for $r_0 \to 0$, then

$$\boxed{\langle\Phi\rangle_{r_0} = \langle\Phi\rangle_{r=0} = \Phi(\vec{R} - \vec{r}).} \tag{1.6}$$

1.2 Problem. Pr. 1.2

A piar of parallel infinite plates are separated by a distance s and they are kept at potentials 0 and V_0. Use the Poisson equation to finding the potencial on the region between the plates, where the charge density is $\rho = \rho_0 x/s$. The distance x is measured from the grounded plate.What are the surface charge density on the plates?

Solution

It's well known that

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} = -\frac{\rho_0}{\epsilon_0 s}x, \tag{1.7}$$

then,

$$V = -\frac{\rho_0}{6\epsilon_0 s}x^3 + V_1' x + V_1. \tag{1.8}$$

Since the boundary conditions are $V(x = 0) = 0$ and $V(x = s) = V_0$, it follows that

$$V_1 = 0. \qquad V_1' = \frac{1}{s}\left(V_0 + \frac{\rho_0}{6\epsilon_0}s^2\right). \tag{1.9}$$

Finally,

$$\boxed{V(x) = -\frac{\rho_0}{6\epsilon_0 s}x^3 + \frac{1}{s}\left(V_0 + \frac{\rho_0}{6\epsilon_0}s^2\right)x.} \tag{1.10}$$

In order to get the charge densities, one can use the result, derived from Gauß' law,

$$\sigma_i = -\epsilon_0 \left.\frac{\partial V}{\partial n}\right|_{x_i}. \tag{1.11}$$

Therefore,

$$\boxed{\sigma(x = 0) = -\frac{\epsilon_0 V_0}{s} + \frac{\rho_0}{6}s,} \tag{1.12}$$

and

$$\boxed{\sigma(x = s) = \frac{\rho_0}{3}s - \frac{\epsilon_0 V_0}{s}.} \tag{1.13}$$

1.3 Problem. Pr. 1.3

The axial electric field E_z on the axis of an accelerated tube in a kind of ionic accelerator is given by $E_z = E_{z_0} + kz^2$, where z is measured from the centre of the tube along the axis. The component E_ϕ is zero. Show that $E_r = -kzr$, assume that the charge density is zero.

Draw the force field.

Which is the maximun charge density that can be tolerated if the above radial field has an accuracy of 5% at the tube extrama? $L = 1\ m$, $E_{z_0} = 7.5 * 10^5\ V/m$ and $k = 10^6\ V/m^3$.

Solution

Since $E_z = E_{z_0} + kz^2$ and $E_\phi = 0$, by Gauß' law, it follows that

$$\vec{\nabla} \cdot \vec{E} = 0, \tag{1.14}$$

so that the set up is considered in the vacuum.

In general,

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\sqrt{g}} \partial_a (\sqrt{g} E^a), \tag{1.15}$$

then, in cylindric coordinates, it yields

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \frac{1}{r}\partial_r(rE_e) + \partial_\phi E_\phi + \partial_z E_z \\ &= \frac{1}{r}\partial_r(rE_r) + \partial_z E_z. \end{aligned} \tag{1.16}$$

It follows from (1.14) and (1.16) that

$$\frac{1}{r}\partial_r(rE_r) + \partial_z E_z = 0. \tag{1.17}$$

Since

$$r\partial_z E_z = 2kzr, \tag{1.18}$$

then

$$\partial_r(rE_r) = E_r + r\partial_r E_r = -2kzr. \tag{1.19}$$

Equation (1.19) looks like a Euler differential equation, then one can make and ansatz $E_r = cr$, with c a constant. Substituting into (1.19), one gets

$$2cr = -2kzr \quad \Rightarrow \quad c = -kz. \tag{1.20}$$

Finally,

$$\boxed{\boxed{E_r = -kzr.}} \tag{1.21}$$

In order to determine the induced chage density at the extrema, up to 5%, the radial electric field, change to $E_r \rightarrow E'_r = (1 + 1/20)E_r$. Therefore,

$$\vec{\nabla} \cdot \vec{E}' = \frac{1}{r}\partial_r(rE'_r) + \partial_z E_z = \mp\frac{kz}{10} = \frac{\rho_{in}}{\epsilon_0}. \tag{1.22}$$

At the extrema,

$$\rho(z = L/2) = \mp\frac{kL\epsilon_0}{20}, \tag{1.23}$$

and

$$\rho(z = -L/2) = \pm\frac{kL\epsilon_0}{20}. \tag{1.24}$$

One just should introduce the numerical values given in the problem,

$$\rho(z = L/2) = -4.425 * 10^{-7}\frac{C}{m^3}, \tag{1.25}$$

$$\rho(z = -L/2) = 4.425 * 10^{-7}\frac{C}{m^3}. \tag{1.26}$$

1.4 Problem. Pr. 1.4

A potential V is applied between two coaxil cylinders of radii r_1 and r_2 respectively. Show that the electric field has a minimum value when $r_1 = r_2/e$.

Solution

By Gauß' law, one know that

$$E(r_1 < r < r_2) = \frac{q_{in}}{2\pi\epsilon_0 rh} = \frac{\sigma}{\epsilon_0}\left(\frac{r_1}{r}\right). \tag{1.27}$$

Since the cylinders are kept to a constant potential, V,it follows that,

$$V = -r_1\frac{\sigma}{\epsilon_0}\int_{r_2}^{r_1}\frac{dr}{r} = -\frac{\sigma r_1}{\epsilon_0}\ln\left(\frac{r_1}{r_2}\right), \tag{1.28}$$

therefore,

$$\frac{V}{r_1} = \frac{\sigma}{\epsilon_0}\ln\left(\frac{r_2}{r_1}\right). \tag{1.29}$$

Next, consider a change on the inner radius, $r_1 \rightarrow r_1 + \Delta r_1$. Since V is still a constant, the superficial charge density must change, $\sigma \rightarrow \sigma'$. Then

$$-V = \frac{\sigma r_1}{\epsilon_0}\ln\left(\frac{r_1}{r_2}\right) = \frac{\sigma' r'_1}{\epsilon_0}\ln\left(\frac{r'_1}{r_2}\right). \tag{1.30}$$

In roder to see how σ changes with a variation of r_1, expand the logaritm

$$\ln\left(\frac{r'_1}{r_2}\right) = \ln\left(\frac{r_1}{r_2}\right) + \frac{\Delta r_1}{r_1}. \tag{1.31}$$

Inserting the last equation in (1.30), one obtains

$$\begin{aligned}
\frac{\sigma'}{\epsilon_0}(r_1 + \Delta r_1)\left[\ln\left(\frac{r_1}{r_2}\right) + \frac{\Delta r_1}{r_1}\right] &= \frac{\sigma r_1}{\epsilon_0}\ln\left(\frac{r_1}{r_2}\right), \\
\Rightarrow\ \frac{\sigma' - \sigma}{\epsilon_0} r_1 \ln\left(\frac{r_1}{r_2}\right) + \frac{\sigma'}{\epsilon_0}\Delta r_1\left[\ln\left(\frac{r_1}{r_2}\right) + 1\right] &= 0, \\
\Rightarrow\ \sigma = \sigma' + \sigma'\frac{\Delta r_1}{r_1}\left[1 + \frac{1}{\ln\left(\frac{r_1}{r_2}\right)}\right]. & \qquad (1.32)
\end{aligned}$$

From (1.32) one can obtain the derivative,

$$\frac{d\sigma}{dr_1} = \lim_{\Delta r_1 \to 0} \frac{\sigma' - \sigma}{\Delta r_1} = -\frac{1}{r_1}\left[1 + \frac{1}{\ln\left(\frac{r_1}{r_2}\right)}\right]. \tag{1.33}$$

In order for $\vec{E}$ been a minimun, (1.33) should be equal to zero, thus,

$$1 = -\ln\left(\frac{r_1}{r_2}\right) = \ln\left(\frac{r_2}{r_1}\right) \quad \Rightarrow \quad \boxed{\boxed{r_1 = \frac{r_2}{e}.}} \tag{1.34}$$

1.5 Problem. Pr. 1.5

A sheet conductor of arbitrary form carries a charge Q, the density on a region is σ. Gauß' law states that just out of the surface the electric field is σ/ϵ_0. Show that if a small hole is made, the electric field at the hole is $\sigma/2\epsilon_0$.

Solution

Gauß' law states

$$\oint_S d\vec{S} \cdot \vec{E} = \frac{q_{in}}{\epsilon_0}. \tag{1.35}$$

In the inner space of a conductor, the electric field vanishes, so by (1.35), one gets

$$E * A = \frac{\sigma A}{\epsilon_0} \quad \Rightarrow \quad E = \frac{\sigma}{\epsilon_0}. \tag{1.36}$$

Nonetheless, if a small hole is made on the conductor, due to the linear property of the equations, the new set up can be realized like a conductor with charge density σ plus a disk with charge density $-\sigma$. When the electric field $\vec{E}$ is measure at a distance $d \to 0$ from the disk, it seems to be an infinite plate therefore,

$$E' * 2A = -\frac{\sigma A}{\epsilon_0} \quad \Rightarrow \quad E' = -\frac{\sigma}{2\epsilon_0}. \tag{1.37}$$

Hence,

$$\boxed{E_t = E + E' = \frac{\sigma}{2\epsilon_0}.} \tag{1.38}$$

1.6 Problem. Pr. 1.6

Calculate the dipolar momentum of a spherical sheet with a charge density $\sigma = \sigma_0 \cos\theta$, with θ is the polar angle.

Solution

It's well known that the dipolar momentum is given by

$$\vec{p} = \int_{V'} d^3x' \; \vec{x}'\rho(x'). \tag{1.39}$$

Since the considered configuration is

$$\sigma = \sigma_0 \cos\theta, \quad \Rightarrow \quad \rho = \sigma_0 \cos\theta \frac{\delta(r-a)}{a^2}. \tag{1.40}$$

Then,

$$\begin{aligned} p_x &= a\sigma_0 \int_0^{\pi} \int_0^{2\pi} \sin\theta d\theta d\varphi \sin\theta \cos\varphi \cos\theta \\ &= 0 \end{aligned} \tag{1.41}$$

$$\begin{aligned} p_y &= a\sigma_0 \int_0^{\pi} \int_0^{2\pi} \sin\theta d\theta d\varphi \sin\theta \sin\varphi \cos\theta \\ &= 0 \end{aligned} \tag{1.42}$$

$$\begin{aligned} p_z &= 2\pi a\sigma_0 \int_0^{\pi} \sin\theta d\theta \cos\theta \cos\theta \\ &= -2\pi a\sigma_0 \int_0^{\pi} d(\cos\theta) \cos^2\theta \\ &= -\frac{2}{3}\pi a\sigma_0(-1-1) \\ &= \frac{4}{3}\pi a\sigma_0. \end{aligned} \tag{1.43}$$

From eqs. (1.41),(1.42) and (1.43), one finally obtains,

$$\boxed{\vec{p} = \frac{4}{3}\pi a\sigma_0\hat{k}.} \tag{1.44}$$

1.7 Problem. Pr. 1.7

Find the required time for a pair of plate of a plane capacitor join, if the mass density is m_0 and they are separated by an initial distance x_0.

1. When the plates are charged with a density σ and then isolated.

2. When the plates are kept to a constant potential V.

Solution

A pair of charged plane (and infinete) plates are considered. Assume the charge density is σ, and their mass by unit area is m_0.

Hence, on a characteristic area element,

$$d\vec{F} = dq\vec{E}, \tag{1.45}$$

with $\vec{E}$ the electric field generated by the opposite plate.

From (1.35) and (1.37), (1.45) can be written like

$$d\vec{F} = dq\frac{\sigma}{2\epsilon_0} = -\frac{\sigma^2 a^2}{2\epsilon_0} = m_0 a^2\ddot{x}. \tag{1.46}$$

Now, by kinematics,

$$\vec{x}_0 = -\frac{1}{2}\ddot{a}t^2. \tag{1.47}$$

Thus,

$$\begin{aligned} t &= \sqrt{-\frac{2x_0}{\ddot{x}}} \\ &= \frac{2}{\sigma}\sqrt{x_0 m_0\epsilon_0}. \end{aligned} \tag{1.48}$$

On the other hand, if one id interested on a couple of plates kept at constant potential, on must consider the change of the capacitance.

$$C = \frac{q}{V} = \frac{\epsilon_0 A}{d}, \tag{1.49}$$

i.e.,

$$\sigma(x) = \frac{q}{A} = \frac{\epsilon_0 V_0}{x}. \tag{1.50}$$

By Gauß' law,

$$\vec{E} = \frac{\sigma}{2\epsilon_0} = \frac{V_0}{x}. \tag{1.51}$$

Hence,

$$d\vec{F} = -\frac{a^2 V_0^2 \epsilon_0}{2x^2} = m_0 a^2 \ddot{x}, \tag{1.52}$$

then,

$$m_0 \ddot{x}\dot{x} = -\frac{V_0^2 \epsilon_0}{2} \frac{\dot{x}}{x^2}, \tag{1.53}$$

thus,

$$\frac{d}{dt}\left(m_0 \dot{x}^2 - \frac{V_0^2 \epsilon_0}{x}\right) = 0. \tag{1.54}$$

This yields,

$$\frac{dx}{dt} = \sqrt{\frac{V_0^2 \epsilon_0}{m_0}\left(\frac{1}{x} + c\right)}. \tag{1.55}$$

Since $\dot{x}(t = 0) = 0$ at $x(t = 0) = x_0$,

$$\frac{dx}{dt} = \sqrt{\frac{V_0^2 \epsilon_0}{m_0}\left(\frac{1}{x} + \frac{1}{x_0}\right)} = \sqrt{\frac{V_0^2 \epsilon_0}{m_0 x_0}\left(\frac{x_0 - x}{x}\right)}, \tag{1.56}$$

Therefore,

$$\int_{x_0}^{0} \frac{dx\sqrt{x}}{\sqrt{x_0} - \sqrt{x}} = \sqrt{\frac{V_0^2 \epsilon_0}{m_0 x_0}} t. \tag{1.57}$$

Changing $\sqrt{x} = -\sqrt{x_0}\sin\theta$,

$$\int_{x_0}^{0} \frac{dx\sqrt{x}}{\sqrt{x_0} - \sqrt{x}} = 2x_0 \int_{0}^{\pi/2} d\theta \sin^2\theta = x_0\left(\frac{\pi}{2} + 1\right), \tag{1.58}$$

Thus,

$$t = x_0\left(\frac{\pi}{2} + 1\right)\sqrt{\frac{m_0 x_0}{\epsilon_0 V_0^2}}. \tag{1.59}$$

1.8 Problem. Pr. 1.8

A variable capacitor consists on a couple of coaxial cylinders of radii a and b with $b - a << a$, free of moving along the coaxial direction. Use energy methods to calculate the magnitud and direction of the force on the inner cylinder if it's displaced respect the exterior cylinder.

Solution

The energy carried by a capacitor is

$$U = \frac{q^2}{2C}, \tag{1.60}$$

with q the charge on the capacitor and C its capacitance.

For a cylinder capacitor,

$$C = \frac{2\pi\epsilon_0 L}{\ln(b/a)}. \tag{1.61}$$

If one moves the inner cylinder a distance x, the effective cylinder capacitor has length $L - x$, thus

$$C(x) = \frac{2\pi\epsilon_0 (L - x)}{\ln(b/a)}. \tag{1.62}$$

Moreover, the force on that cylinder is given by

$$\begin{aligned} \vec{F} &= -\vec{\nabla} U \\ &= -\frac{q^2}{4\pi\epsilon_0} \ln\left(\frac{b}{a}\right) \frac{\hat{x}}{(L - x)^2}. \end{aligned} \tag{1.63}$$

1.9 Problem. Pr. 1.9

Verify that the charge on a grounded infinite conductive plane, induced by a point charge, q at a distance a from the plane, is $-q$.

Solution

By the image method, the position of the image charge is $(-a, 0, 0)$ and its charge is $-q$,because $\Phi(0, y, z) = 0$. Moreover,

$$\Phi(x, y, z) = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{\sqrt{(x - a)^2 + y^2 + z^2}} - \frac{1}{\sqrt{(x + a)^2 + y^2 + z^2}} \right). \tag{1.64}$$

From Gauß' law,

$$\begin{aligned} \oint_S \vec{E} \cdot d\vec{S} &= -\oint_S \vec{\nabla}\Phi \cdot \hat{n} dS \\ &= -\oint_S \frac{\partial \Phi}{\partial n} dS \\ &= \frac{\sigma S}{\epsilon_0}, \end{aligned} \tag{1.65}$$

then,

$$\sigma_S = -\epsilon_0 \left.\frac{\partial \Phi}{\partial n}\right|_S . \tag{1.66}$$

Finally,

$$\sigma(x=0) = -\frac{qa}{2\pi(a^2+y^2+z^2)^{3/2}}. \tag{1.67}$$

Hence,

$$\begin{aligned}
q_t &= \int dydx\sigma(y,z) \\
&= -\frac{qa}{2\pi}\int rdrd\theta\frac{1}{(a^2+r^2)^{3/2}} \\
&= -qa\int\frac{rdr}{(a^2+r^2)^{3/2}} \\
&= qa\left.\left(\frac{1}{(a^2+r^2)^{1/2}}\right)\right|_0^\infty \\
&= -q.
\end{aligned} \tag{1.68}$$

1.10 Problem. Pr. 1.10

Find $\Phi(\vec{r})$ generated by a point charge, q, placed at a distance d from a conducting sphere of radius R.

Solution

Since the set up has axial symmetry, the image charge must life on the axis that joins the center of the sphere with the charge q. Consider the image charge,$-q'$, placed at a distance d' from the center.

By definition of conductor, $\Phi|_S = 0$. The most straightforware of getting this condition on the sphere is applying for the points colineals with the axis. Then,

$$\begin{aligned}
\frac{q}{4\pi\epsilon_0}\frac{1}{d-R} &= \frac{q'}{4\pi\epsilon_0}\frac{1}{R-d'} \\
\Rightarrow \quad \frac{q}{4\pi\epsilon_0 d}\frac{1}{1-R/d} &= \frac{q'}{4\pi\epsilon_0 R}\frac{1}{1-d'/R},
\end{aligned} \tag{1.69}$$

in order to hold the equality,

$$q' = q\frac{R}{d} \tag{1.70}$$

$$d' = \frac{R^2}{d}. \tag{1.71}$$

1.11 Problem. Pr. 1.11

Consider a fluid conduct of rectangular form, delimited by the plane points, $(0,0)$, $(a,0)$, $(0,b)$ and (a,b). The edges of the rectangle are a constant temperature T_1 and T_2 and the front edges have equal temperature. Find the temperature, $T(x,y)$, for all point inside the conduct.

Solution

Consider the Laplacian equation in rectangular coordinates,

$$\nabla^2 T = 0. \tag{1.72}$$

One can consider homogeneous boundary conditions at $x = 0$ and $x = a$. After the separation of variables,

$$X'' + m^2 X = 0 \quad \Rightarrow \quad X = A_m \cos mx + B_m \sin mx, \tag{1.73}$$

$$Y'' - m^2 Y = 0 \quad \Rightarrow \quad Y = C_m \cosh my + D_m \sinh my, \tag{1.74}$$

with $m > 0$. By the boundary condition,

$$\begin{aligned} X(x) &= B_n \sin \frac{n\pi x}{a} \\ Y(x) &= D_n \sinh \frac{n\pi(y-b)}{a}, \end{aligned} \tag{1.75}$$

or

$$\begin{aligned} X(x) &= B_n \sin \frac{n\pi x}{a} \\ Y(x) &= D_n \sinh \frac{n\pi y}{a}, \end{aligned} \tag{1.76}$$

where $n = \frac{ma}{\pi}$. (1.75) for homogeneous B.C. at $y = b$ and (1.76) for homogeneous B.C. at $y = 0$.

One can consider homogeneous boundary conditions at $y = 0$ and $y = b$. After the separation of variables,

$$Y'' + m^2 Y = 0 \quad \Rightarrow \quad Y = A_m \cos my + B_m \sin my, \tag{1.77}$$

$$X'' - m^2 X = 0 \quad \Rightarrow \quad X = C_m \cosh mx + D_m \sinh mx, \tag{1.78}$$

with $m > 0$. By the boundary condition,

$$\begin{aligned} Y(x) &= B_n \sin \frac{n\pi y}{b} \\ X(x) &= D_n \sinh \frac{n\pi(x-a)}{b}, \end{aligned} \tag{1.79}$$

or

$$\begin{aligned} Y(x) &= B_n \sin \frac{n\pi y}{b} \\ X(x) &= D_n \sinh \frac{n\pi x}{b}, \end{aligned} \tag{1.80}$$

where $n = \frac{ma}{\pi}$. (1.79) for homogeneous B.C. at $x = a$ and (1.80) for homogeneous B.C. at $x = 0$.

By using the relations

$$\int_0^a dx \sin^2 \frac{n\pi x}{a} = \frac{a}{2}, \tag{1.81}$$

$$\int_0^a dx \sin \frac{n\pi x}{a} = -\frac{a}{n\pi}((-1)^n - 1), \tag{1.82}$$

and the linelity of the Laplace equation, the coordinate temperature is

$$\begin{aligned} T(x,y) &= -\frac{4T_1}{\pi} \sum_{n\ odd} \frac{1}{n} \frac{\sin \frac{n\pi x}{a}}{\sinh \frac{n\pi b}{a}} \left(\sinh \frac{n\pi}{a}(y-b) + \sinh \frac{n\pi}{a} y \right) \\ &= -\frac{4T_2}{\pi} \sum_{n\ odd} \frac{1}{n} \frac{\sin \frac{n\pi y}{b}}{\sinh \frac{n\pi a}{b}} \left(\sinh \frac{n\pi}{b}(x-a) + \sinh \frac{n\pi}{b} x \right). \end{aligned} \tag{1.83}$$

1.12 Problem. Pr. 1.12

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Solution

1.13 Problem. Pr. 1.13

Verify that $\widetilde{\phi}$ satisfies the Laplace eq. by using the chain rule.

Solution

One begins with the Laplacian equation

$$\nabla^2 \phi = 0. \tag{1.84}$$

If one would like to change (x, y) to (u, v) coordinates with $u = u(x, y)$ and $v = v(x, y)$ given by the real and imaginary part of an analytic function, i.e., each of them satisfy the Laplace equation on the (x, y) coordinates.

By the chain rule, it follows that,

$$\phi_{xx} = \phi_u u_{xx} + \phi_v v_{xx} + \phi_{uu} u_x^2 + \phi_{vv} v_x^2 + 2\phi_{uv} u_x v_x, \tag{1.85}$$
$$\phi_{yy} = \phi_u u_{yy} + \phi_v v_{yy} + \phi_{uu} u_y^2 + \phi_{vv} v_y^2 + 2\phi_{uv} u_y v_y, \tag{1.86}$$

Then, adding both terms,

$$0 = \phi_u \nabla^2_{(x,y)} u + \phi_v \nabla^2_{(x,y)} v + \phi_{uu}(u_x^2 + u_y^2) + \phi_{vv}(v_x^2 + v_y^2). \tag{1.87}$$

Next, one uses the Cauchy equations

$$u_x = v_y, \tag{1.88}$$
$$u_y = -v_x, \tag{1.89}$$

then, by changing v_x and v_y by u's, (1.87) becomes

$$0 = (\phi_{uu} + \phi_{vv})(u_x^2 + u_y^2). \tag{1.90}$$

Since u_x and u_y are not vanishing quantities, it follows that

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\phi(u, v) = 0. \tag{1.91}$$

1.14 Problem. Pr. 1.14

How do $\vec{p}$ and Q_{ij} transform under a traslation on the coordinate system?

Solution

By definition,

$$p_i = \int_V r_i \rho(\vec{r}) d^3 r, \tag{1.92}$$

and

$$Q_{ij} = \int_V d^3 r \rho(\vec{r})(3x_i x_j - \delta_{ij}\vec{r}^2). \tag{1.93}$$

Under $r \mapsto r' = r + a$,

$$p_i \mapsto p_i' = \int_V r_i' \rho(\vec{r}') d^3 r' = \int_V (r_i + a_i)\rho(\vec{r}') d^3 r' = p_i + a_i q_t, \tag{1.94}$$

where

$$\int_V r_i \rho(\vec{r}') d^3 r' = p_i, \qquad \int_V (r_i + a_i)\rho(\vec{r}') d^3 r' = q_t. \tag{1.95}$$

This happens because V is the volume where the charge density is placed.

Similarly,

$$\begin{aligned} Q_{ij} \mapsto Q'_{ij} &= \int_V d^3r' \rho(\vec{r}')(3x'_i x'_j - \delta_{ij}\vec{r}'^2) \\ &= \int_V d^3r \rho(\vec{r}) \left(3(x_i + a_i)(x_j + a_j) - \delta_{ij}(\vec{r} + \vec{a})^2\right) \\ &= \int_V d^3r \rho \left(\vec{r})(3(x_i x_j + x_i a_j + a_i x_j + a_i a_j) - \delta_{ij}(\vec{r}^2 + 2\vec{r} \cdot \vec{a} + \vec{a}^2)\right) \\ &= Q_{ij} + 3(a_i p_j + a_j p_i) + q_t(3a_i a_j + \vec{a}^2 \delta_{ij}). \end{aligned} \tag{1.96}$$

Chapter 2

Electrostatic II

2.1 Problem Pr. 2.1

The energy of a dipole p in a field $\vec{E}$ is $W = -\vec{p} \cdot \vec{E}$. Use the method of images to find the energy of a dipole p at a distance r from an infinite, grounded, conducting plane when the angle between p and the normal to the plane is θ.

Find the force and torque on the dipole due to the induced charges on the plane.

Solution

The electric field prodeced by a dipole is

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{\vec{r}+\vec{r}'+\vec{l}}{|\vec{r}+\vec{r}'+\vec{l}|^3} - \frac{\vec{r}+\vec{r}'}{|\vec{r}+\vec{r}'|^3}\right). \tag{2.1}$$

By expanding the denominator,

$$|\vec{r}+\vec{r}'+\vec{l}|^{-3} = |\vec{r}+\vec{r}'|^{-3}\left(1 + 3\frac{(\vec{r}-\vec{r}')\cdot\vec{l}}{|\vec{r}+\vec{r}'|^2} + \ldots\right), \tag{2.2}$$

then,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(3\frac{(\vec{r}-\vec{r}')\cdot\vec{p}}{|\vec{r}+\vec{r}'|^5}(\vec{r}-\vec{r}') - \frac{\vec{p}}{|\vec{r}+\vec{r}'|^3}\right). \tag{2.3}$$

Putting $\vec{r}' = 0$, $\vec{r} = -2r\hat{i}$ and $\vec{p} = -p(\cos\theta\hat{i} + \sin\theta\hat{j})$, the electric field due to the image dipole at the position of the dipole is

$$\vec{E}_{ext} = \frac{p}{32\pi\epsilon_0 r^3}(-2\cos\theta\hat{i} + \sin\theta\hat{j}). \tag{2.4}$$

Since $W = -\vec{p} \cdot \vec{E}_{ext}$ and $\vec{p} = p(-\cos\theta\hat{i} + \sin\theta\hat{j})$, it follows that

$$\boxed{\boxed{W = -\frac{p^2}{32\pi\epsilon_0 r^3}(1 + \cos^2\theta).}} \tag{2.5}$$

Since the net charge is zero, and the dipoles are punctuals, there isn't net force.

$$\boxed{\boxed{\vec{F} = 0.}} \tag{2.6}$$

2.2 Problem Pr. 2.2

Show that a harmonic function F is uniquely determined at all the points within a given region by its value at the boundary.

Show also that F if identically equal to zero throughout the region if it vanishes at all points of its boundary.

Solution

In order to show that the solution is unique, assume that there exist two different functions F_1 and F_2 s.t. both satisfy the B.C. $F_i|_S = F_s$.

Since the Laplace equation is linear, the function $F = F_1 - F_2$ satisfies $\nabla^2 F = 0$ with homogeneous B.C.

The Green's identity states that

$$\int_S u\vec{\nabla}v \cdot d\vec{\sigma} = \int_V (u\nabla^2 v + \vec{\nabla}u \cdot \vec{\nabla}v)d\tau. \tag{2.7}$$

For the case $u = v = F$, since $F|_S = 0$ and $\nabla^2 F = 0$, Green'sidentity yields

$$\int_V (\vec{\nabla}F)^2 d\tau = 0. \tag{2.8}$$

This implies that $\vec{\nabla}F = 0$ and then, $F_1 = F_2 + const$. but the B.C. implies that $const = 0$, i.e.,

$$\boxed{\boxed{F_1 = F_2.}}$$

Moreover, since the solution of the Laplace equation with constant boundary conditions cannot have maxima or minima inside the boundary,it follows that if $F|_S = 0$, then

$$\boxed{\boxed{F|_V = 0}}.$$

2.3 Problem Pr. 2.3

A small hemispherical bump, of radius a, is raised on the inner surface of one plate of a parallel-plate capacitor, separated by a distance d. Find the resulting potential between the plates.

Solution

In order to solve the problem, one sholud solve the Laplace equation inside the capacitor. Note that the problem has azimutal symmetry, then

$$\Phi(r,\theta) = A_0 + \sum_{l=1}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right)P_l(\cos\theta). \tag{2.9}$$

the coefficient $B_0 = 0$ because the innersolution is regular and the logarithmic part of the solution is dropped.

The boundary conditions are

$$\begin{aligned}\Phi(r=a) &= \Phi_0, &(2.10)\\ \Phi(\theta=\pi/2) &= \Phi_0, &(2.11)\\ \Phi(r=d/\cos\theta) &= 0, &(2.12)\\ \Phi(r>>a) &= \frac{\sigma}{\epsilon_0}\cos\theta.. &(2.13)\end{aligned}$$

Then, by finiteness, $A_{l>1} = 0$, also equation (2.13) the coefficients $A_1 = \frac{\sigma}{\epsilon_0}$ and B_1 are the only ones allowed, 'ceuse $P_1(\cos\theta) = \cos\theta$. Then

$$\Phi(r,\theta) = A_0 + \frac{\sigma}{\epsilon_0}r\cos\theta + \frac{B_1}{r^2}\cos\theta. \tag{2.14}$$

From (2.11),

$$\Phi(\theta=\pi/2) = A_0 = \Phi_0, \Rightarrow \quad \Phi(r,\theta) = \Phi_0 + \frac{\sigma}{\epsilon_0}r\cos\theta + \frac{B_1}{r^2}\cos\theta. \tag{2.15}$$

From (2.10),

$$\Phi(a,\theta) = \Phi_0 + \left(\frac{\sigma a}{\epsilon_0} + \frac{B_1}{a^2}\right)\cos\theta = \Phi_0, \tag{2.16}$$

thus,

$$B_1 = -\frac{\sigma a^3}{\epsilon_0} \quad \Rightarrow \quad \Phi(r,\theta) = \Phi_0 + \left(\frac{\sigma}{\epsilon_0}r - \frac{\sigma a^3}{\epsilon_0 r^2}\right)\cos\theta. \tag{2.17}$$

Finally, condition (2.12) yields

$$\Phi(r=d/\cos\theta) = \Phi_0 + \frac{\sigma}{\epsilon_0}\left(d - \frac{a^3}{d^2}\cos^3\theta\right) = 0, \tag{2.18}$$

then, since $a << d$,

$$\sigma = -\frac{\Phi_0 \epsilon_0}{d}. \tag{2.19}$$

Therefore,

$$\boxed{\Phi(r,\theta) = \Phi_0 \left(1 + \frac{r^3 - a^3}{r^2 d} \cos\theta\right).} \tag{2.20}$$

2.4 Problem Pr. 2.4

A charge Q is uniformly distributed throughout the volume of an ellipsoid of revolution whose semi-major axis is a and whose semi-minor axis is b. Find the electrostatic potential at any in the space outside the ellipsoid.

Solution

The most useful coordinate system for solving the problem is cylindric coordinates. Then, the potential due to a differential of volume is

$$d\Phi(h) = \frac{\rho}{4\pi\epsilon_0} \frac{r dr d\theta dz}{\sqrt{r^2 + (h-z)^2}}, \tag{2.21}$$

the potential of the ellipsoid is therefore,

$$\Phi(h) = \frac{\rho}{4\pi\epsilon_0} \int_0^{2\pi} d\theta \int_{-a}^{a} dz \int_0^{R} \frac{r dr}{\sqrt{r^2 + (h-z)^2}}, \tag{2.22}$$

with

$$R^2 = b^2 \left(1 - \frac{z^2}{a^2}\right). \tag{2.23}$$

Then, (2.22) yields,

$$\Phi(h) = \frac{\rho}{2\epsilon_0} \int_{-a}^{a} dz \left(\sqrt{R^2 + (h-z)^2} - h + z \right). \tag{2.24}$$

Call I and II the first and second integral of the RHS of (2.24),then,

$$II = \int_{-a}^{a} (-h + z) = -2ah. \tag{2.25}$$

The argument of the square-root of I can be written as

$$\left(z\sqrt{1 - \frac{b^2}{a^2}} - \frac{h}{\sqrt{1 - \frac{b^2}{a^2}}} \right)^2 + b^2 + h^2 - \frac{h^2 a^2}{a^2 - b^2}. \tag{2.26}$$

After defining

$$u = z\sqrt{1-\frac{b^2}{a^2}} - \frac{h}{\sqrt{1-\frac{b^2}{a^2}}}, \quad J^2 = b^2 + h^2 - \frac{h^2a^2}{a^2-b^2}, \tag{2.27}$$

one can change the integral on z by an integral on u, with

$$du = dz\sqrt{1-\frac{b^2}{a^2}},$$

thus,

$$\begin{aligned} I &= \frac{1}{\sqrt{1-\frac{b^2}{a^2}}}\int_{u_-}^{u_+} du\sqrt{u^2+J^2} \\ &= \frac{1}{\sqrt{1-\frac{b^2}{a^2}}}\left(\frac{u\sqrt{u^2+J^2}}{2} + \frac{J^2}{2}\ln(u+\sqrt{u^2+J^2})\right)\Bigg|_{u_-}^{u_+}. \end{aligned} \tag{2.28}$$

On the above equations, the integration limits are

$$u_+ = a\sqrt{1-\frac{b^2}{a^2}} - \frac{h}{\sqrt{1-\frac{b^2}{a^2}}} \tag{2.29}$$

$$u_- = -a\sqrt{1-\frac{b^2}{a^2}} - \frac{h}{\sqrt{1-\frac{b^2}{a^2}}} \tag{2.30}$$

After a lot of algebraic manipulation, one gets

$$\begin{aligned} I = &\frac{1}{\sqrt{1-\frac{b^2}{a^2}}}\left[h\left(b^2-3a^2-\frac{h^2a^2}{a^2-b^2}\right)\right. \\ &\left.+\frac{(b^2+h^2)(a^2-b^2)-h^2a^2}{2(a^2-b^2)}\ln\left(\frac{(a^2-b^2)^2+a(a^2-b^2)+h(ha^2-2a^3+2ab^2-a^2+b^2)}{(a^2-b^2)^2+a(a^2-b^2)+h(ha^2+2a^3-2ab^2+a^2-b^2)}\right)\right]. \end{aligned} \tag{2.31}$$

Finally, the potential on the z axis is

$$\begin{aligned} \Phi = &\frac{\rho}{2\epsilon_0}\left\{-2ah + \frac{1}{\sqrt{1-\frac{b^2}{a^2}}}\left[h\left(b^2-3a^2-\frac{h^2a^2}{a^2-b^2}\right)\right.\right. \\ &\left.\left.+\frac{(b^2+h^2)(a^2-b^2)-h^2a^2}{2(a^2-b^2)}\ln\left(\frac{(a^2-b^2)^2+a(a^2-b^2)+h(ha^2-2a^3+2ab^2-a^2+b^2)}{(a^2-b^2)^2+a(a^2-b^2)+h(ha^2+2a^3-2ab^2+a^2-b^2)}\right)\right]\right\}. \end{aligned} \tag{2.32}$$

2.5 Problem Pr. 2.5

Find the potential distribution inside a hollow conducting cylinder of radius a if the cylinder has a length L and the two ends are closed by plates which are held at the potentials $\Phi = \Phi_0$ and $\Phi = 0$, respectively. Also find the charge distribution on the surface of the cylinder.

Solution

In order to find the potential, one must solve the Laplace equation with the given boundary conditions,

$$\Phi(z = 0) = 0, \tag{2.33}$$
$$\Phi(z = L) = \Phi_0, \tag{2.34}$$
$$\Phi(r = a) = \Phi_1. \tag{2.35}$$

Using cylindric coordinates, it follows that

$$\Phi(\rho, \theta, z) = f(\rho)g(\theta)h(z), \tag{2.36}$$

with

$$f(\rho) = J_m(k\rho), \tag{2.37}$$
$$g(\theta) = A_m \sin(m\theta) + B_m \cos(m\theta), \tag{2.38}$$
$$h(z) = \sinh(kz), \tag{2.39}$$

with $m \in \mathbb{Z}$.

For having a well-possed problem, i.e., determine completely all the arbitrary constants, $\Phi_1 = 0$, then

$$J_m(ka) = 0, \rightarrow \quad k = \frac{\lambda_n^{(m)}}{a}, \tag{2.40}$$

with $n \in \mathbb{N}$ and $\lambda_n^{(m)}$ is the n-th zero of the m-th Bessel function.

Moreover, due to the azimutal symmetry, $m = 0$, then

$$\Phi(\rho, \theta, z) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{\lambda_n^{(0)}}{a} z\right) J_m\left(\frac{\lambda_n^{(0)}}{a}\rho\right). \tag{2.41}$$

Now, by (2.34) and the identities

$$\int_0^a d\rho\rho J_n(x_{nm}x/a)J_n(x_{nm'}x/a) = \frac{a^2}{2}J_{n+1}(x_{nm})\delta_{mm'}, \tag{2.42}$$
$$\int_0^{2\pi} \sin^2 x = \pi, \tag{2.43}$$

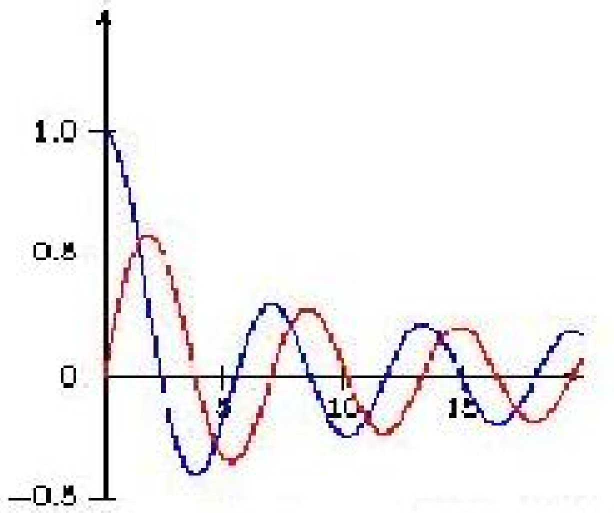


Figure 2.1: It shows the Bessel functions J_0 (in blue) and J_1 (in red).

it follows that

$$A_n = \frac{2}{a^2}\frac{\Phi_0 cosech(\lambda_n^{(0)}L/a)}{J_1^2(\lambda_n^{(0)})}\int_0^a d\rho\rho J_0(\lambda_n^{(0)}\rho/a). \tag{2.44}$$

Since,

$$\int d\rho\rho^n J_{n-1}(\rho) = \rho^n J_n(\rho), \tag{2.45}$$

and $J_{n>1}(\rho = 0) = 0$, the coefficients are

$$A_n = \frac{2\Phi_0}{\lambda_n^{(0)}}\frac{cosech(\lambda_n^{(0)}L/a)}{J_1(\lambda_n^{(0)})}. \tag{2.46}$$

Finally,

$$\boxed{\Phi(\rho,\theta,z) = \sum_{n=1}^{\infty}\frac{2\Phi_0}{\lambda_n^{(0)}}\frac{cosech(\lambda_n^{(0)}L/a)}{J_1(\lambda_n^{(0)})}\sinh\left(\frac{\lambda_n^{(0)}}{a}z\right)J_m\left(\frac{\lambda_n^{(0)}}{a}\rho\right).} \tag{2.47}$$

2.6 Problem Pr. 2.6

A spherical distribution of radius R and constant charge density ρ contains a hole of radius r inside. Determine energy of this configuration.

Solution

2.7 Problem Pr. 2.7

Calculate the exterior potential of a infinite, conducting cylinder of radius R on a uniform electric field E_0perpendicular to the axis of it.

Solution

Since the cylinder is infinite, the problem is nothing but 2-dimensional, so, the general solution is

$$\Phi(r,\theta) = A_0 + B_0 \ln r + \sum_n (A_n r^n B_n r^{-n})(C_n \sin(n\theta) + D_n \cos(n\theta)). \tag{2.48}$$

The B.C. are

$$\vec{E}(r \to \infty) = E_0 \hat{k}, \tag{2.49}$$
$$\Rightarrow \quad \Phi(r \to \infty) = -E_0 r \cos\theta + const. \tag{2.50}$$
$$\Phi(r = R) = \Phi_0. \tag{2.51}$$

At $r \to \infty$, the B.C.implies

$$\Phi(r,\theta) = A_0 + A_1 r \cos\theta + \frac{B_1}{r}\cos\theta = -E_0 r \cos\theta + const., \tag{2.52}$$

then

$$A_1 = -E_0.$$

At $r = R$, the B.C. implies,

$$\Phi(R,\theta) = A_0 - E_0 r \cos\theta + \frac{B_1}{r}\cos\theta = \Phi_0, \tag{2.53}$$

then,

$$A_0 = \Phi_0 \qquad B_1 = E_0 R^2.$$

Finally,

$$\boxed{\Phi(r,\theta) = \Phi_0 - E_0 R\left(1 - \frac{R}{r}\right)\cos\theta.} \tag{2.54}$$

2.8 Problem Pr. 2.8

Find the corrents and equipotential of a cylinder, of radius a immerced on a fluid. For that use a conformal map given by

$$z = \zeta + \frac{a^2}{\zeta}, \tag{2.55}$$

with $\zeta = \xi + \imath\eta$ is the complexification of the original variales (ξ, η) and $z = x + \imath y$ is the complexification of the maped coordinates.

Solution

Since $\zeta = \xi + \imath\eta$,

$$x + \imath y = \xi + \imath\eta + \frac{a^2}{\xi^2 + \eta^2}(\xi - \imath\eta), \tag{2.56}$$

then,

$$x = \xi\left(1 + \frac{a^2}{\xi^2 + \eta^2}\right), \tag{2.57}$$

$$y = \eta\left(1 + \frac{a^2}{\xi^2 + \eta^2}\right). \tag{2.58}$$

The conformal map maps the B.C. at infinity to the same condition but in the z-plane, but the circle is maped to a segment on the x axis. Thus, the velocity in the xy-plane is $\vec{v} = v\hat{x}$ or $\Phi = vx$.

From (2.57), it follows that

$$\boxed{\Phi = v\xi\left(1 + \frac{a^2}{\xi^2 + \eta^2}\right).} \tag{2.59}$$

So that this is a conformal map, all the machinany of the complex variable applies, so, the Cauchy conditions imply that

$$\boxed{\Psi = vy = v\eta\left(1 + \frac{a^2}{\xi^2 + \eta^2}\right),} \tag{2.60}$$

are the perpendicular lines to Φ.

Φ gives the currents, and Ψ gives the equipotentials.

2.9 Additional Problem

Find a relatioin between the multipoles expansion and the Legendre polynomial, of course for an axial charge distribution.

Solution

The outside solution of electric potentials with azimutal symmetry can be written as

$$\begin{aligned}\Phi(r,\theta) &= \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} \\ &= \frac{1}{4\pi\epsilon_0}\left(\frac{Q_t}{r} + \frac{p\cos\theta}{r^2} + \frac{1}{2r^5}Q_{ij}x^i x^j + \dots\right)\end{aligned} \tag{2.61}$$

For a symmetric configuration the quadrupolar tensor is given by

$$[Q] = \begin{pmatrix} Q & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & -2Q \end{pmatrix}, \tag{2.62}$$

thus,

$$Q_{ij}x^i x^j = Qr^2(1 - 3\cos^2\theta) = -2Qr^2 P_1(\cos\theta). \tag{2.63}$$

Therefore, by comparison,

$$\boxed{B_0 = \frac{Q_t}{4\pi\epsilon_0}, \quad B_1 = \frac{p}{4\pi\epsilon_0}, \quad B_2 = -\frac{Q}{4\pi\epsilon_0}.} \tag{2.64}$$

Chapter 3

Dielectrics

3.1 Problem Pr. 3.2

A holed dielectric sphere with inner and outer radii a and $2a$ rspectively, and dielectric coefficient $K_e = 3$, is placed in a previously uniform electric field E_0.Show that in the hole

$$\vec{E} = \frac{27}{34}\vec{E}_0. \tag{3.1}$$

Solution

The problem has azymutal symmetry, so, as there aren't free charges, the potential solves the Laplace eq. $\nabla^2\phi = 0$. Therefore, the solution can be expand in Legendre polynomials.

Call

$$\begin{aligned} &\phi_i(\vec{r}) && \text{If } r < a \\ &\phi_m(\vec{r}) && \text{If } a < r < 2a \\ &\phi_e(\vec{r}) && \text{If } r > 2a. \end{aligned} \tag{3.2}$$

Since the potential is continuous, the B.C. are

B.C. $r = a$

$$\phi_i(a) = \phi_m(a), \tag{3.3}$$

$$\epsilon_0 \frac{\partial \phi_i}{\partial r}(a) = \epsilon \frac{\partial \phi_m}{\partial r}(a). \tag{3.4}$$

B.C. $r = 2a$

$$\phi_e(2a) = \phi_m(2a), \tag{3.5}$$
$$\epsilon_0 \frac{\partial \phi_e}{\partial r}(2a) = \epsilon \frac{\partial \phi_m}{\partial r}(2a). \tag{3.6}$$

Due to finiteness of the potential for $r \to 0$ or $r \to \infty$, it follows that,

$$\phi_i = \sum_l A_l r^l P_l(\cos\theta) \tag{3.7}$$
$$\phi_m = \sum_l \left(B_l r^l + \frac{C_l}{r^{l+1}} \right) P_l(\cos\theta) \tag{3.8}$$
$$\phi_e = -E_0 r \cos\theta + \sum_l D_l \frac{1}{r^{l+1}} P_l(\cos\theta). \tag{3.9}$$

Then, the B.C. are clearly not compatible, unless $l = 1$, therefore,
Continuity $r = a$

$$A_1 a = B_1 a + \frac{C_1}{a^2}, \tag{3.10}$$

Gauß $r = a$

$$\epsilon_0 A_1 = \epsilon \left(B_1 - 2\frac{C_1}{a^3} \right), \tag{3.11}$$

Continuity $r = 2a$

$$-2aE_0 + \frac{D_1}{4a^2} = 2aB_1 + \frac{C_1}{4a^2}, \tag{3.12}$$

Gauß $r = 2a$

$$\epsilon_0 \left(-E_0 - \frac{D_1}{4a^2} \right) = \epsilon \left(B_1 - 2\frac{C_1}{a^3} \right). \tag{3.13}$$

The system of equations (3.10)-(3.13) can be solved, and yield

$$A_1 = -\frac{27}{34} E_0, \tag{3.14}$$
$$B_1 = -\frac{21}{34} E_0, \tag{3.15}$$
$$C_1 = -\frac{3a^3}{17} E_0, \tag{3.16}$$
$$D_1 = \frac{49a^3}{17} E_0. \tag{3.17}$$

In the hole, one needs just the value of A_1, so

$$\phi_i(r,\theta) = -\frac{27}{34} E_0 r \cos\theta = -\frac{27}{34} E_0 z, \tag{3.18}$$

thus,

$$\boxed{\vec{E}_i = -\vec{\nabla}\phi_i = \frac{27}{34}\vec{E}_0.} \tag{3.19}$$

3.2 Problem Pr. 3.3

Use Biot-Savart law for calculating $\vec{B}$ in the center of a square loop carrying a current I.

Solution

Boit-Savart law states

$$\vec{B} = \frac{I}{4\pi\epsilon_0 c^2}\oint \frac{\vec{dl}\times\vec{r}}{|\vec{r}|^3}. \tag{3.20}$$

For a square loop, the total contribution in the center is four times the contributions of a single side, so

$$d\vec{B} = \frac{Ia}{4\pi\epsilon_0 c^2}\hat{k}\int_{-a}^{a}\frac{dl}{\sqrt{a^2+l^2}^3}. \tag{3.21}$$

By a trigonometric change, the integral can be solved then one multiply by 4, and the result is

$$\boxed{\vec{B} = \sqrt{2}\frac{Ia}{\pi\epsilon_0 c^2}\hat{k}.} \tag{3.22}$$

3.3 Problem Pr. 3.4

A plane conductor of width $2a$ carries a current I. Show that

$$B_x = -\frac{\mu_0 I}{4\pi a}\theta \tag{3.23}$$

$$B_x = \frac{\mu_0 I}{4\pi a}\ln\frac{r_2}{r_1} \tag{3.24}$$

in the first coordinate quarter. The conductor's borders are placed at $x = \pm a$, the current flows in direction z. $r_{1,2}$ are the distances between $x = \pm a$ and the point where the measure is made. The angle θ is the one between r_1 and r_2 measure in the direction $1 \to 2$.

Compute $\vec{B}$ at $26cm$ from the center of the conductor for an angle of 72^o if the conductor band has $10cm$ of width and carries a current of $5.76A$.

Find $\vec{B}$ at a distace D, from the origin, placed on the axis x and y. Compare both results for $D \gg a$.

Solution

Theough the problem the angles between r_1 and r_2 w.r.t. the x axis are called α and φ. Also, the angle of an arbitrary point placed on the conductor is called ϑ, and the modulus (from there to the measure point) is d.

Begin by consider the magnetic field produced by an infinitesimal wire composing the conducting band, so,

$$d_x = r_1 \cos\alpha - x = d\cos\vartheta, \tag{3.25}$$
$$d_y = r_1 \sin\alpha = d\sin\vartheta, \tag{3.26}$$
$$d^2 = d_x^2 + d_y^2. \tag{3.27}$$

Similarly,

$$B_x = -B\sin\vartheta, \tag{3.28}$$
$$B_y = B\cos\vartheta, \tag{3.29}$$

where, by Amperè's law,

$$B = \frac{\mu_0 I}{4\pi a}\frac{dx}{d}. \tag{3.30}$$

Therefore, the total magnetic field components are

$$B_x = -\frac{\mu_0 I r_1 \sin\alpha}{4\pi a}\int_{-a}^{a} \frac{dx}{\left(r_1^2\sin^2\alpha + (r_1\cos\alpha - x)^2\right)}, \tag{3.31}$$
$$B_y = \frac{\mu_0 I}{4\pi a}\int_{-a}^{a} \frac{(r_1\cos\alpha - x)dx}{\left(r_1^2\sin^2\alpha + (r_1\cos\alpha - x)^2\right)}. \tag{3.32}$$

Consider the integral on (3.31),

$$\int_{-a}^{a} \frac{dx}{\left(r_1^2\sin^2\alpha + (r_1\cos\alpha - x)^2\right)} = \frac{1}{r_1^2\sin^2\alpha}\int_{-a}^{a} \frac{dx}{\left(1 + \left(\frac{r_1\cos\alpha - x}{r_1\sin\alpha}\right)^2\right)}, \tag{3.33}$$

by changing

$$u = \frac{r_1\cos\alpha - x}{r_1\sin\alpha}, \tag{3.34}$$

the integral is solved

$$\int_{-a}^{a} \frac{dx}{\left(r_1^2\sin^2\alpha + (r_1\cos\alpha - x)^2\right)} = -\frac{1}{r_1\sin\alpha}\left[\arctan(u(a)) - \arctan(u(-a))\right], \tag{3.35}$$

with

$$u(\pm a) = \frac{r_1\cos\alpha \mp a}{r_1\sin\alpha}. \tag{3.36}$$

Thus

$$B_x = \frac{\mu_0 I}{4\pi a}\left[\arctan(u(a)) - \arctan(u(-a))\right]. \tag{3.37}$$

Since $r_1 \sin\alpha = r_2 \sin\varphi$ and $\varphi + \theta = \alpha$, it follows that

$$u(a) = \tan\varphi, \tag{3.38}$$

$$u(-a) = \tan\alpha, \tag{3.39}$$

So,

$$\boxed{\therefore \quad B_x = \frac{\mu_0 I}{4\pi a}\theta.} \tag{3.40}$$

The integral (3.32) is made through the change $u = r_1^2 \sin^2\alpha + (r_1 \cos\alpha - x)^2$, and using the substitutions $u(a) = r_1^2$ and $u(-a) = r_2^2$. So

$$\boxed{B_y = \frac{\mu_0 I}{4\pi a} \ln \frac{r_2}{r_1}.} \tag{3.41}$$

For the given data, the corresponding radii and angles are

$\alpha =$	86.05
$r_1 =$	0.2517 m
$\varphi =$	64.96
$r_2 =$	0.2771 m
θ = 21.09= 0.368 rad	

Thus,

$$\boxed{\begin{aligned} B_x &= -4.24 * 10^{-6} N/m \\ B_y &= 1.11 * 10^{-6} N/m \\ B &= 4.38 * 10^{-6} N/m \end{aligned}}$$

On the x axis,

$$B_x = 0, \tag{3.42}$$

$$B_y = \frac{\mu_0 I}{4\pi a} \ln\left(1 + \frac{2a}{D}\right). \tag{3.43}$$

On the y axis,

$$B_x = -\frac{\mu_0 I}{4\pi a} 2 \arctan\left(\frac{a}{D}\right), \tag{3.44}$$

$$B_y = 0. \tag{3.45}$$

By expanding to first order on a, i.e. $D \gg a$,

$$\ln(1+x) = x + \cdots \quad (3.46)$$

$$\arctan x = x + \cdots\,, \quad (3.47)$$

one obtains,

$$\boxed{|B_a| = |B_b| = \frac{\mu_0 I}{\pi D}.} \quad (3.48)$$

This is 'cause for $D \gg a$, the band looks like a wike.

3.4 Problem Pr. 3.5

A pair of Helmholtz coils consist in a pair of identical coaxial circle loops,they maximize $\vec{B}$'s uniformity in the intediate region. Find the optimal separation between the loops.

Solution

For a single loop,

$$\vec{B}(z) = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}, \quad (3.49)$$

thus, on the Helmholtz coils,

$$\vec{B}(z) = \frac{\mu_0 I}{2}R^2\left[\frac{1}{(R^2+z^2)^{3/2}} + \frac{1}{\left(R^2+\left(z-\frac{d}{2}\right)^2\right)^{3/2}}\right]. \quad (3.50)$$

From the last equation, it follows that

$$\boxed{\frac{\partial \vec{B}}{\partial z}\left(\frac{d}{2}\right) = 0.} \quad (3.51)$$

It's not a bad result, but still does not give a relation between the radii and separation of the coils.

By taking the second derivative one get

$$\frac{\partial^2 \vec{B}}{\partial z^2}\left(\frac{d}{2}\right) \propto \left[-\frac{6R^2}{\left(R^2+\left(\frac{d}{2}\right)^2\right)^{7/2}} + \frac{4}{\left(R^2+\left(\frac{d}{2}\right)^2\right)^{5/2}} + \frac{4\left(\frac{d}{2}\right)^2}{\left(R^2+\left(\frac{d}{2}\right)^2\right)^{7/2}}\right], \quad (3.52)$$

hence, in order to maximize the uniformity of $\vec{B}$ it must vanish, so the condition reached is

$$\boxed{R = d.} \quad (3.53)$$

3.5 Problem Pr. 3.6

Find $\vec{B}$ inside a long straight wire of radius a carrying a current density $\vec{J}$. Assume the electron density through the wire is N and their velocity is $\vec{u}$, find the direction and magnitud of the force acting on the movil charges. Is the assuption of a uniform charge density realistic?

Solution

Assuming that the current density is uniform, allows using the eq

$$\oint \vec{B} \cdot \vec{dl} = \frac{I_{enc}}{\epsilon_0 c^2}, \tag{3.54}$$

so,

$$\boxed{\vec{B}(\vec{r}) = \frac{Jr}{2\epsilon_0 c^2}\hat{\theta}.} \tag{3.55}$$

Now, since $\vec{J} = N\vec{u}$, it follows that

$$\vec{F} = -e\left(\vec{u} \times \frac{Nur}{2\epsilon_0 c^2}\hat{\theta}\right), \tag{3.56}$$

or

$$\boxed{\vec{F} = \frac{eu^2 N}{2\epsilon_0 c^2}\vec{r}.} \tag{3.57}$$

The direction is of course is pointing outside the wire, this implies that after a little time the electrons will flow not by the bulk of the wire but through the surface. That fact spoils the model under consideration.

3.6 Problem Pr. 3.7

Calculate the electric polarization $\vec{P}$ in an infinite dielectric cylinder of radiu a and dielectric constant K_e spinning with an angular velocity ω around its axis, in presence of an axial magnetic field. Calculate the polarized charge densities.

Solution

Assuming that the dielectric material is homogeneous and linear. Therefore,

$$\vec{P} = \epsilon_0 \chi \vec{E} = \epsilon_0 (K - 1)\vec{E}, \tag{3.58}$$

further,

$$\sigma_{pol} = \vec{P} \cdot \hat{n}|_{sup}, \tag{3.59}$$

and

$$\rho_{pol} = -\vec{\nabla} \cdot \vec{P}. \tag{3.60}$$

Hence, the problem reduce to finding the electric field.

Now, from the Lorentz force

$$\vec{E} = \frac{\vec{F}}{q} = \vec{v} \times \vec{B} = (\vec{\omega} \times \vec{r}) \times \vec{B}. \tag{3.61}$$

In cylindrical coordinates

$$\vec{\omega} = \omega\hat{z}, \tag{3.62}$$

$$\vec{r} = r\hat{r}, \tag{3.63}$$

$$\vec{B} = B_0\hat{z}, \tag{3.64}$$

then,

$$(\vec{\omega} \times \vec{r}) \times \vec{B} = B_0\omega\vec{r}. \tag{3.65}$$

Morever, in cylindrical coordinates

$$\vec{\nabla} \cdot \vec{r} = 2,$$

then,

$$\boxed{\boxed{\vec{P} = \epsilon_0(K-1)B_0\omega\vec{r}.}} \tag{3.66}$$

Thus,

$$\boxed{\boxed{\rho_{pol} = -2\epsilon_0(K-1)B_0\omega,}} \tag{3.67}$$

and

$$\boxed{\boxed{\sigma_{pol} = \epsilon_0(K-1)B_0\omega a.}} \tag{3.68}$$

Note: The above densities are constant,as long as ω is constant, and

$$V\rho_{pol} = -S\sigma_{pol}, \tag{3.69}$$

which agrees with the fact that $\rho_{free} = 0$ initially.

3.7 Problem Pr. 3.8

Show that $\vec{B}$ inside a torus formed by a coil wrapped N times carrying a current I is equal to the one generated by a current NI through a wire on the torus axis.

Solution

By taking a path through the circular axis of the torus, one gets from the Amperè's law

$$\boxed{B_T = \frac{NI}{\epsilon_0 c^2 l}.} \tag{3.70}$$

Similarly, taking the same path, but around the straight wire, one gets

$$\boxed{B_w = \frac{NI}{\epsilon_0 c^2 l},} \tag{3.71}$$

where $l = 2\pi R$ with R is the radius of the path.

$$\boxed{\therefore \quad B_T = B_w.} \tag{3.72}$$

3.8 Problem Pr. 3.9

Show that the average of $\vec{B}$, inside the volume a sphereof radius R, generated by a small loop carrying a magnetic dipolar momentum $\vec{m}$ placed whereever insde the sphere, is given by

$$\vec{B} = \frac{\mu_0 m}{2\pi R^3}.$$

Solution

In order to simplify the computations, consider the magnetic dipole oriented in the axis z.

So that the magnetic field is

$$\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}, \tag{3.73}$$

and

$$\int_V \vec{\nabla} \times \vec{A} d^3 r = -\oint_S \vec{A} \times d\vec{a}, \tag{3.74}$$

then

$$\begin{aligned} \left\langle \vec{B} \right\rangle &= -\frac{1}{V} \oint_S \vec{A}(r) \times d\vec{a} \\ &= -\frac{\mu_0}{4\pi V} \oint_S \left(\int \frac{\vec{J}(r') d^3 r'}{\left| \vec{r} - \vec{r}' \right|} \right) \times d\vec{a} \\ &= -\frac{\mu_0}{4\pi V} \int \vec{J}(r') d^3 r' \times \oint_S \frac{d\vec{a}}{\left| \vec{r} - \vec{r}' \right|}. \end{aligned} \tag{3.75}$$

Since,

$$\oint_S \frac{d\vec{a}}{|\vec{r} - \vec{r}'|} = 2\pi R^2 \int_0^\pi \frac{\sin\theta\cos\theta d\theta}{\sqrt{r^2 + z'^2 - 2Rz'\cos\theta}} = \frac{4\pi}{3} z\hat{k}, \tag{3.76}$$

and the definition

$$\int d^2r' \vec{J}(r') \times \vec{r}' = 2\vec{m}, \tag{3.77}$$

finally one obtains

$$\boxed{\langle \vec{B} \rangle = \frac{\mu_0 \vec{m}}{2\pi R^3}.} \tag{3.78}$$

3.9 Problem Pr. 3.11

An infinite wire with a linear charge density λ is placed, at a distance d, parallel to a dielectric plane of a given dielectric constant. Determine the force per unit of length acting on the wire.

Solution

By placing a charged wire in front of a dielectric wall, a polarization cherge is induced on the wall, and since the width of the wal is infinite, the other charges won't be consider as part of the set up.

Gauß' law states that,

$$\epsilon_0 \left\{ E_w + \frac{\sigma_{pol}}{2\epsilon_0} \right\} = \epsilon \left\{ E_w - \frac{\sigma_{pol}}{2\epsilon_0} \right\}, \tag{3.79}$$

then

$$\begin{aligned} \frac{\sigma_{pol}}{2\epsilon_0} &= E_w \left(\frac{K-1}{K+1} \right) \\ &= \frac{\lambda}{2\pi\epsilon_0} \left(\frac{K-1}{K+1} \right). \end{aligned} \tag{3.80}$$

Hence, on the RHS., the total electric field is

$$E_T = \frac{\lambda}{2\pi\epsilon_0 r} \left(1 + \frac{K-1}{K+1} \right), \tag{3.81}$$

which can be interpretedas an image wire placed at the same distance but with charge density

$$\lambda' = -\lambda \left(\frac{K-1}{K+1} \right).$$

Therefore, the electric field of the image wire is

$$\vec{E}' = \frac{\lambda'}{8\pi\epsilon_0 d}, \tag{3.82}$$

and finally

$$\boxed{\vec{F} = -\frac{\lambda^2}{8\pi\epsilon_0 d}\left(\frac{K-1}{K+1}\right).} \tag{3.83}$$

3.10 Problem: Aharanov-Bohm vector potential.

Find the vector potential for the Aharanov–Bohm experiment.

Solution

Consider the f.e.m.,

$$\Phi = \int_S \vec{B}\cdot\hat{n} = \int_S \vec{\nabla}\times\vec{A}\cdot\hat{n} = \oint_\gamma \vec{A}\cdot d\vec{l}. \tag{3.84}$$

So, by analogy, with the equation

$$\oint_\gamma \vec{A}\cdot d\vec{l} = \Phi \longleftrightarrow \oint_\gamma \vec{B}\cdot d\vec{l} = \frac{I_{enc}}{4\pi\epsilon_0 c^2}, \tag{3.85}$$

and using the magnetic field for an infinite solenoid,

$$\vec{B} = \begin{cases} B_0\hat{k} & ;\ r < R \\ 0 & ;\ r > R \end{cases}. \tag{3.86}$$

By symmetry, $\vec{A} \sim \hat{\varphi}$, therefore, for $r < R$,

$$\oint_\gamma \vec{A}\cdot d\vec{l} = A2\pi r = B_0\pi r^2, \tag{3.87}$$

thus,

$$\boxed{\vec{A} = \frac{B_0 r}{2}\hat{\varphi}.} \tag{3.88}$$

For $r > R$,

$$\oint_\gamma \vec{A}\cdot d\vec{l} = A2\pi r = B_0\pi R^2, \tag{3.89}$$

thus,

$$\boxed{\vec{A} = \frac{B_0 R^2}{2r}\hat{\varphi}.} \tag{3.90}$$

NOTE: $\vec{A}$ is continuous at $r = R$.

Chapter 4

Electrodynamics I

4.1 Problem Pr. 4.1

A superconductor has the property that inside it both, $\vec{E}$ and $\vec{B}$ vanish.

1. For the electric field the boundary condition is that at the surface it must be just normal. What is the analogous boundary condition for $\vec{B}$?
2. A small loop with magnetic moment μ oriented with an angle θ respect to the normal, is located in front of an infinite, superconductor wall. How can the magnetic field be found?
3. Find the torque on the magnetic dipole as a funtion of the angle. Deduce the equilibrium points, stables or not.
4. Find the force as a function of the angle.

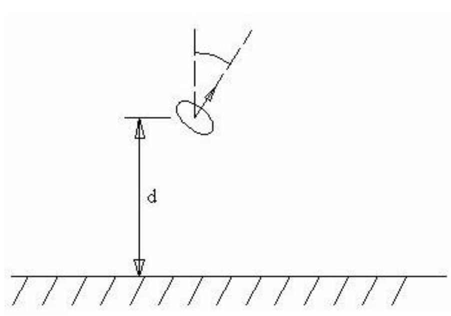


Figure 4.1: Magnetic dipole in front a superconductor.

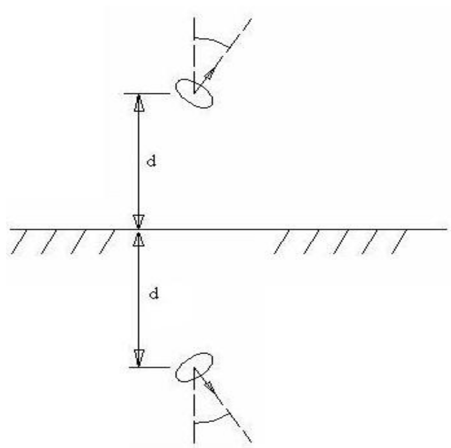
Solution

At the boundary, there is not free charge, so, since $\vec{\nabla} \cdot \vec{B} = 0$, it follows that

$$\boxed{\vec{B}_{\perp} = 0 \text{ At the boundary.}} \tag{4.1}$$

So the magnetic field must be parallel to the surface of the superconductor.

In order to solve the problem of a magnetic dipole placed in front of a (plain) superconductor, one must use the image methods. For satisfying the B.C. of the magnetic field, the image is placed just like a mirror image and the moduli of the magnetic momenta are equal. $\left|\vec{m}\right| = \left|\vec{m}'\right|$.



The torque can be found by

$$\vec{\tau} = \vec{m} \times \vec{B}_{ext}, \tag{4.2}$$

and the force is

$$\vec{F} = \left(\vec{m} \cdot \vec{\nabla}\right) \vec{B}_{ext}. \tag{4.3}$$

Now, the magnetic field of a point magnetic dipole is

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi r^3} \left[3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}\right], \tag{4.4}$$

since $\vec{m}_i = m(\sin\theta, -\cos\theta, 0)$ and $\hat{r} = (0, 1, 0)$, yields

$$\vec{B}(r\hat{y}) = -\frac{\mu_0 m}{4\pi r^3}(\sin\theta, 2\cos\theta, 0). \tag{4.5}$$

Therefore, the torque is

$$\boxed{\vec{\tau} = -\frac{\mu_0 m^2}{64\pi d^3} \sin 2\theta \hat{k}.} \tag{4.6}$$

From here,

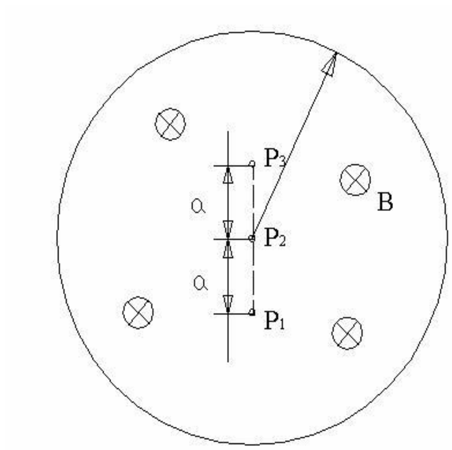
θ	equilibrium
0	unstable
$\pi/2$	stable
π	unstable
$3\pi/2$	stable

On the other hand, the force among this two point magnetic dipoles vanish,

$$\boxed{\boxed{\vec{F} = 0.}} \tag{4.7}$$

4.2 Problem Pr. 4.2

the figure shows an uniform magnetic field $\vec{B}$ confined to a cylinder of radius r. The magnetic field decreace in magnitud to a constant rate of 100 gauss/s. What is the instantaneous acceleration experimented by an electron placed on P_1, P_2 and P_3?. Assume $a = 5cm$.



Solution

Obviously one solves the problem with the help of

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{4.8}$$

or what's the same,

$$\oint \vec{E} \cdot d\vec{l} = A \frac{dB}{dt}. \tag{4.9}$$

The set up has a symmetry and the symmetry left a fixed point (in this case is not just a point but a line... the axis of the cylinder), so one takes as path circumference centered at the origin, thus, the enclosed area is

$$A(p_1) = A(p_3) = \pi a^2 \tag{4.10}$$
$$A(p_2) = 0. \tag{4.11}$$

Axial symmetry assures that $\left|\vec{E}\right|$ is constant along the chosen path, so

$$E = \frac{A}{2\pi r}\frac{dB}{dt}. \tag{4.12}$$

Hence, $E_{p_2} = 0$ and

$$E_{p_1,p_3} = \frac{\pi r^2}{2\pi r}\frac{dB}{dt} = \frac{r}{2}\frac{dB}{dt}, \tag{4.13}$$

in the $-\hat{\theta}$ direction.

Thus,

$$\boxed{\vec{a} = \frac{er}{2m_e}\frac{dB}{dt}\hat{\theta}.} \tag{4.14}$$

Now, since 100 Gaußare $10^{-2}T$, $m_e = 9.1 * 10^{-31}Kg$, $e = 1.6 * 10^{-19}C$, then

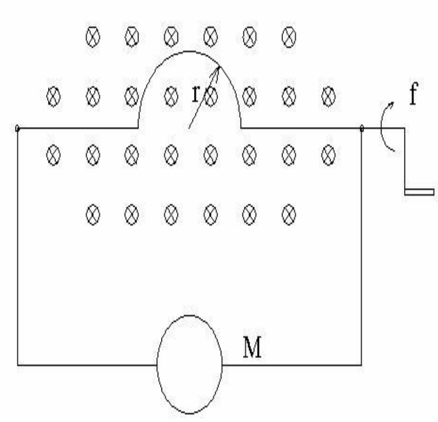
$$\boxed{\vec{a} \simeq 5 * 10^7 m/s^2.} \tag{4.15}$$

4.3 Problem Pr. 4.3

A stiff wire in form of semi-circle of radius r is rotated with a frequency ω on an uniform magnetic field. What?are the frequencies and amplitudes of

- the induced voltage,
- the induced current,

if the internal resistor of the galvanometer is R and any other resistor in the circuit is zero?.



Solution

Since

$$\Phi(t) = BA(t) = BA_0 \cos(\omega t). \tag{4.16}$$

It follows that

$$\varepsilon = -\frac{\partial \Phi}{\partial t} = \omega BA_0 \sin(\omega t). \tag{4.17}$$

This implies that

$$\boxed{\boxed{Amp(V) = \omega BA_0,}} \tag{4.18}$$

and

$$\boxed{\boxed{freq(V) = \omega.}} \tag{4.19}$$

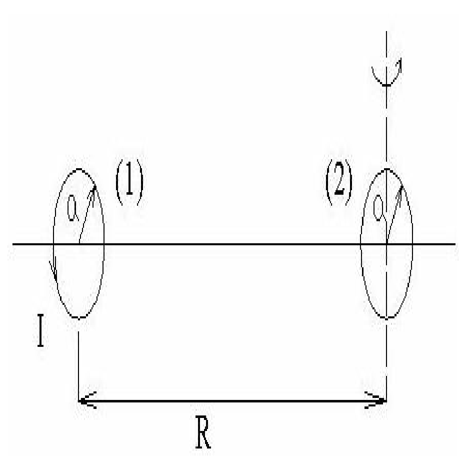
By Ohm's law, $V = IR$, one get

$$I = \frac{\omega BA_0}{R} \sin(\omega t). \tag{4.20}$$

$$\boxed{\boxed{\therefore \begin{array}{rl} Amp(I)= & \frac{\omega BA_0}{R}, \\ freq(I)= & \omega. \end{array}}} \tag{4.21}$$

4.4 Problem Pr. 4.4

A small loop of radius a carries a constant current I. Other loop of radiua a, is placed on the axis of the first one at a distance R, with $R \gg a$. the planes of the loops are parallel. The second loop is rotated with an angular velocity ω through one of its diameters. If the second loop isopened so that no current can flows through it, What's the genereted e.m.f.?



Solution

The second loop is spinning into the magnetic field generated by the first one, so the induced e.m.f. is

$$\varepsilon = -\frac{d\Phi}{dt} = -B\frac{dA}{dt}, \tag{4.22}$$

where A is understands like a given area whose boundary is the loop.

Nonetheless, as the second loop has been cutted, the induced current cannot be generated and the end-points behave like a capacitor, which produce a potential in the inverse direction of ε, so, the total e.m.f. vanish,

$$\boxed{\boxed{\varepsilon = 0.}} \tag{4.23}$$

Other way of seeing the above result is that once one has cutted the loop there isn't a close path, so the definition of e.m.f. loose its sense.

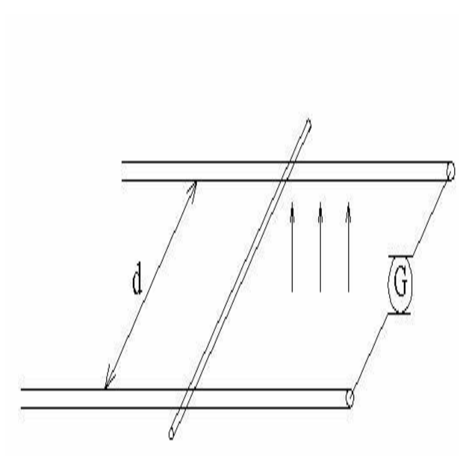
4.5 Problem Pr. 4.5

A metal wire of mass m slides without friction on two rails separated a distance d. There is an uniform magnetic field B.

1. A current I flows from a generator G throuth a rail, croos the wire and return through the other rail. Find the velocity as a funtion of time. Assume the wire is at rest at $t = 0$.

2. Thegenerator is replaced by a battery with an e.m.f. ξ. The volocity of the wire now approximates to a final value. What is the final speed? How does it approximate to its final value, as a funtion of time?

3. What's the current of the second part when the wire reach its final velocity?



Solution

In the first case,since the current is constant, one can write the Lorentz force as

$$\vec{F} = m\vec{a} = l\vec{I} \times \vec{B}, \tag{4.24}$$

or what's equal, so that $\vec{I} \perp \vec{B}$,

$$a = \frac{lIB}{m} = const., \tag{4.25}$$

so,

$$\boxed{v = \frac{lIB}{m}t.} \tag{4.26}$$

In the second case, the generator is changed by a battery whose e.m.f. is ε, by Ohm's law, $I = \frac{(\varepsilon - \varepsilon_{ind})}{R}$. Faraday's yields

$$\varepsilon_{ind} = - - \frac{\partial \Phi}{\partial t} = -Blv, \tag{4.27}$$

therefore,

$$\begin{aligned} \frac{dv}{dt} &= \frac{lIB}{m} \\ &= \frac{lB}{mR}\left(\varepsilon - lBv\right). \end{aligned} \tag{4.28}$$

One can solve this equation by separation of variables, then

$$v(t) = \frac{\varepsilon}{lB} - \frac{A}{lB}e^{-\frac{l^2B^2}{mR}t}. \tag{4.29}$$

Finally, the initial condition $v(t = 0) = 0$, yields

$$\boxed{v(t) = \frac{\varepsilon}{lB}\left(1 - e^{-\frac{l^2B^2}{mR}t}\right).} \tag{4.30}$$

The asymptotic limit of the velocity is $v = \frac{\varepsilon}{lB}$, and again by Ohm's law,

$$\boxed{I = 0.} \tag{4.31}$$

4.6 Problem Pr. 4.6

A toroidal solenoid of N turns has a square transversal section. Each side of the square has length a and the inner radius is b.

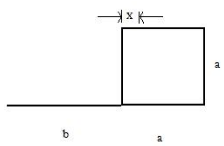
1. Show that the self-inductance is

$$L = \frac{N^2 a}{2\pi\epsilon_0 c^2} \ln\left(1 + \frac{a}{b}\right).$$

2. Express in similar terms the mutual inductance of a system formed by the solenoid and a straight wire placed on the axis of symmetry of the solenoid.

3. Find the rate of the self- and mutual inductances.

Solution



Since ,

$$W = \frac{1}{2}LI^2 = \frac{1}{2\mu_0}\int B^2 d^3 r, \tag{4.32}$$

and for the torus magnetic field,

$$B_x = \frac{NI}{2\pi\epsilon_0 c^2}\frac{1}{b+x}. \tag{4.33}$$

In cylindric coordinates,

$$\begin{aligned} \frac{1}{2\mu_0}\int B^2 d^3 r &= \frac{1}{2\mu_0}\int_b^{a+b} dr\, r \int_0^{2\pi} d\theta \int_0^a dz\, \frac{\mu_0^2 N^2 I^2}{4\pi^2}\frac{1}{r^2} \\ &= \frac{N^2 I^2 a}{4\pi\epsilon_0 c^2}\int_b^{a+b}\frac{dr}{r} \\ &= \frac{N^2 I^2 a}{4\pi\epsilon_0 c^2}\ln\left(1+\frac{a}{b}\right). \end{aligned} \tag{4.34}$$

Therefore,

$$\boxed{L = \frac{N^2 a}{2\pi\epsilon_0 c^2} \ln\left(1 + \frac{a}{b}\right).} \tag{4.35}$$

Similarly,

$$MI_1I_2 = \frac{1}{\mu_0}\int \vec{B}_1 \cdot \vec{B}_2 d^3r, \tag{4.36}$$

and so that $B_1 \neq 0$ just inside T^2,it follows that

$$B_1 = \frac{\mu_0 N I_1}{2\pi r}\hat{\theta} \quad \text{and} \quad B_2 = \frac{\mu_0 I_2}{2\pi r}\hat{\theta}, \tag{4.37}$$

then,

$$\boxed{M = \frac{Na}{2\pi\epsilon_0 c^2}\ln\left(1 + \frac{a}{b}\right).} \tag{4.38}$$

Finally,

$$\boxed{\frac{L}{M} = N.} \tag{4.39}$$

4.7 Problem Pr. 4.7

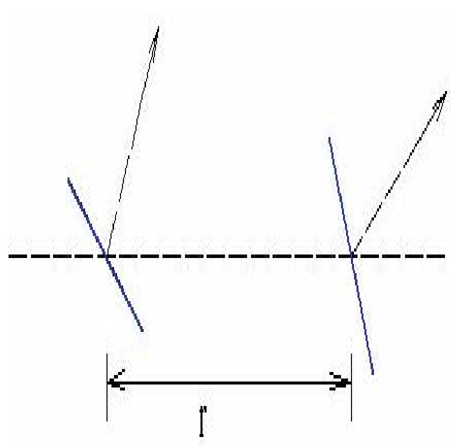
A pair of plane loops, each one of area A and carring a current I, are separated a distance r.

The normals of the loops form angles α_1 and α_2 respect to the line which joints the loops and these angles live on the same plane.

1. Find the mutual inductance. Assume that radii are much more smaller than the separation.

2. Using the expression for M, find the magnitud and direction of the force among the loops.

3. How would be the force if one reverse the current in one or both loops?



Solution

Themagnetic field of a magnetic dipole is

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right]. \tag{4.40}$$

If one choose the origin of coordinates at the place of the first dipole, s.t. the y axis coincide with the jointing line of the dipoles, then

$$\vec{m} = m(0, \cos\alpha_1, \sin\alpha_1), \tag{4.41}$$

and

$$\hat{r} = (0, 1, 0). \tag{4.42}$$

Therefore,

$$\vec{B}(R) = \frac{\mu_0 m}{4\pi R^3}\left[2\cos\alpha_1\hat{y} - \sin\alpha_1\hat{z}\right]. \tag{4.43}$$

Additionally, the vector corresponding to the area of the second dipole is

$$\vec{A}_2 = A(0, \cos\alpha_2, \sin\alpha_2), \tag{4.44}$$

thus, the flux of the magnetic field generated by (1) through the surface of (2), is

$$\Phi_2 = \vec{B}_1(R)\cdot\vec{A}_2 = \frac{\mu_0 mA}{4\pi R^3}\left(2\cos\alpha_1\cos\alpha_2 - \sin\alpha_1\sin\alpha_2\right). \tag{4.45}$$

Then,

$$\boxed{M = \frac{\mu_0 A^2}{4\pi R^3}\left(2\cos\alpha_1\cos\alpha_2 - \sin\alpha_1\sin\alpha_2\right).} \tag{4.46}$$

Furthermore, the force between the to loopsis given by

$$\vec{F} = I^2\vec{\nabla}_1 M, \tag{4.47}$$

by using the gradient in spherical coordinates, it follows that,

$$\boxed{\vec{F} = -\frac{\mu_0 I^2 A^2}{4\pi R^4}\left[3(2\cos\alpha_1\cos\alpha_2 - \sin\alpha_1\sin\alpha_2)\hat{r} + (2\cos\alpha_1\sin\alpha_2 + \sin\alpha_2\cos\alpha_2)\hat{\theta}\right].} \tag{4.48}$$

Obviously, if one reverse one (or both) current(s), is just like changing the corresponding angle $\theta_i \to \theta_i + \pi$ in the equation (4.48).

4.8 Problem Pr. 4.8

A metal rod is magnetized along the azimutal direction. What dependence on the radius can M have so that the net magnetic charge in the system vanish?

Solution

So, one have thet $\vec{M} = f(r)\hat{\theta}$.

If the material is "uniformly" magnetized, it must satisfy that

$$\vec{\nabla} \cdot \vec{M} = \vec{\nabla} \cdot \vec{M} = 0. \tag{4.49}$$

In cylindrical coordinates the only components that contributes are

$$\vec{\nabla} \cdot \vec{M} = \frac{1}{r}\partial_\theta(rf(r)) = 0, \tag{4.50}$$

$$\vec{\nabla} \cdot \vec{M} = \frac{1}{r}\partial_\theta(rf(r)) = 0. \tag{4.51}$$

Equation (4.50) doesn't make a constraint on f , but (4.51) impose that

$$\boxed{f(r) = \frac{1}{r}.} \tag{4.52}$$

4.9 Problem Pr. 4.9

A long wire of radius a carries a current I and it's surrounded coaxially by a long holed iron cylinderwith relative permeability K_m. The inner radius of the cylinder is b and the external is c. Calculate the total flux of $\vec{B}$ inside a section of lenght l of the cylinder.

Find the current density on the inner and outer surface of the cylinder, also the directions of the currents relative to the current of the wire.

Find the equivalent current density inside the cylinder.

Find $\vec{B}$ at $r > c$ from the wire. How's it affected if the cylinder is taken away?

Solution

The boundary conditions for the Magnetic field are (without free currents)

$$H_\perp^{(1)} = H_\perp^{(2)}, \tag{4.53}$$

$$B_{/\!/}^{(1)} = B_{/\!/}^{(2)}. \tag{4.54}$$

Since the magnetic fiel d generated by a wire is

$$\vec{B} = \frac{\mu_0 I}{2\pi}\frac{1}{r}\hat{\theta} \Rightarrow \vec{B}_m = \frac{\mu I}{2\pi}\frac{1}{r}\hat{\theta}, \tag{4.55}$$

then,

$$\boxed{\Phi_m = \frac{\mu I l}{2\pi}\int_b^c \frac{dr}{r} = K_m \frac{\mu_0 I l}{2\pi}\ln\left(\frac{c}{b}\right).} \tag{4.56}$$

Also, since $\vec{J}_s = \vec{M} \times \hat{n}|_s$, gives the current density on the surface of the conductor. From(4.56) it follows that,

$$\vec{M} = \frac{I}{2\pi r}(K_m - 1)\hat{\theta}, \tag{4.57}$$

thus, on the inner surface,

$$\boxed{\vec{J} = \frac{I}{2\pi b}(K_m - 1)\hat{z},} \tag{4.58}$$

and in the outer surface,

$$\boxed{\vec{J} = -\frac{I}{2\pi c}(K_m - 1)\hat{z}.} \tag{4.59}$$

The internal current is

$$\boxed{\vec{J} = \vec{\nabla} \times \vec{M} = \frac{1}{r}\partial_r(rM_\theta) = 0.} \tag{4.60}$$

Finally, since the currents at $r = b$ and $r = c$ are the same, the magnetic field outside the cylinder is just the sameas without the cylinder,

$$\boxed{\vec{B}(r > c) = \frac{\mu_0 I}{2\pi r}\hat{\theta}.} \tag{4.61}$$

Part II

Quantum Mechanics

Chapter 5

One-dimensional Problems

5.1 Problem 1...

A particle of mass m is confined to a 1-dimensional region $0 \leq x \leq a$ by the potential

$$V(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq a \\ \infty & \text{otherwise} \end{cases} \tag{5.1}$$

At $t = 0$, the wave function is

$$\psi(x, t = 0) = \sqrt{\frac{8}{5a}} \left(1 + \cos\left(\frac{\pi x}{a}\right)\right) \sin\left(\frac{\pi x}{a}\right). \tag{5.2}$$

Calculate

a. $\psi(x, t_0)$ for $t_0 > 0$.

b. The energy of this wave at $t = 0$ and $t = t_0$.

c. What's the probability of finding such a particle in the right half of the well?

Solution

The Schrödinger equation for a particle in an 1-dimensional box is

$$\partial_x \psi(x) + \omega^2 \psi(x) = 0, \tag{5.3}$$

where $\omega^2 = \frac{\sqrt{2mE}}{\hbar^2}$. By imposing the boundary conditions $\psi(0) = \psi(a) = 0$, we get the solution,

$$\psi(x) = A \sin(\omega x), \tag{5.4}$$

with

$$\omega a = n\pi \quad \Rightarrow \quad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \text{ with } n \in \mathbb{N}^*. \tag{5.5}$$

Therefore, (5.2) is nothing but a lineal combination of the solutions with $n = 1$ and $n = 2$

$$\psi(x,0) = \sqrt{\frac{8}{5a}}\sin\left(\frac{\pi x}{a}\right) + \sqrt{\frac{2}{5a}}\sin\left(\frac{2\pi x}{a}\right). \tag{5.6}$$

Next, by applying the evolution operator, $U(t) = e^{-\frac{i}{\hbar}Ht}$, we obtain

$$\psi(x,t) = \sqrt{\frac{8}{5a}}e^{-\frac{i\pi^2\hbar}{2ma^2}t}\sin\left(\frac{\pi x}{a}\right) + \sqrt{\frac{2}{5a}}e^{-\frac{2i\pi^2\hbar}{ma^2}t}\sin\left(\frac{2\pi x}{a}\right). \tag{5.7}$$

Then, the VEV of the energy is the expectation value of the Hamiltonian,

$$\begin{aligned} \langle H\rangle_t &= \int_0^a dx\psi^*(x,t)\left(-\frac{\hbar^2}{2m}\partial_x^2\right)\psi(x,t) \\ &= \frac{4\pi^2\hbar^2}{5ma^3}\int_0^a dx\sin^2\left(\frac{\pi x}{a}\right) + \frac{4\pi^2\hbar^2}{5ma^3}\int_0^a dx\sin^2\left(\frac{2\pi x}{a}\right) \\ &\quad + \sim \int_0^a dx\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right) \\ &= \frac{4\pi^2\hbar^2}{5ma^2}, \end{aligned} \tag{5.8}$$

so that,

$$\int_0^\pi dx sin^2 nx = \frac{\pi}{2}. \tag{5.9}$$

Indeed, since (5.8) is time independent, it follows that

$$\langle E\rangle_0 = \langle E\rangle_t = \frac{4\pi^2\hbar^2}{5ma^2}. \tag{5.10}$$

Finally,

$$\mathcal{P}(x,t)dx = |\psi(x,t)|^2 dx, \tag{5.11}$$

then,

$$\begin{aligned} \mathcal{P}(x \geq a/2, t) &= \int_{a/2}^a dx\psi^*(x,t)\psi(x,t) \\ &= \frac{8}{5a}\int_{a/2}^a dx\sin^2\left(\frac{\pi x}{a}\right) + \frac{2}{5a}\int_{a/2}^a dx\sin^2\left(\frac{2\pi x}{a}\right) \\ &\quad \frac{8}{5a}\int_{a/2}^a dx\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{3\pi^2\hbar^2 t}{2ma^2}\right) \\ &= \frac{1}{2} - \frac{16}{15\pi}\cos\left(\frac{3\pi^2\hbar^2 t}{2ma^2}\right), \end{aligned} \tag{5.12}$$

because

$$\int_{a/2}^{a} dx \sin^2\left(\frac{n\pi x}{a}\right) = \frac{a}{4} \tag{5.13}$$

$$\int_{a/2}^{a} dx \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right) = -\frac{2a}{3\pi}. \tag{5.14}$$

Note that (5.12) is granther than 0 for whatever value of t.

5.2 Problem 2...

A particle of mass m moves on a 1-dimensional potential

$$V(x) = \begin{cases} \lambda\delta\left(x - \frac{L}{2}\right) & \text{if } 0 \leq x \leq L \\ \infty & \text{otherwise} \end{cases} \tag{5.15}$$

a. Find the transcendental equation for the eigenvalues of the energy (In terms of m, L and λ).

b. Find the pair of lowest energies levels and corresponding eigenstates for $Lm/\hbar = 4$ and $\lambda/\hbar = 2$.

Solution

Let's define, like before,

$$\omega^2 = \frac{2mE}{\hbar^2}. \tag{5.16}$$

The solution to the homogeneous (i.e., without the Dirac's delta distribution), is

$$\psi(x) = A\sin(\omega x) + B\cos(\omega X). \tag{5.17}$$

We should consider different solutions in both regions, left- and right-side of the delta. So, by imposing the boundary conditions,

$$\psi(0) = \psi(L) = 0, \tag{5.18}$$

we get,

$$\psi_I(x) = A\sin(\omega x) \tag{5.19}$$

$$\psi_{II}(x) = B\{\sin(\omega x) - \tan(\omega x)\}. \tag{5.20}$$

Then, by continuity $\psi_I(L/2) = \psi_{II}(L/2)$, it follows that

$$A\sin(\omega L/2) = B\{\sin(\omega L/2) - \tan(\omega L/2)\}, \tag{5.21}$$

that can be rewritten as

$$B = -A\cos(\omega L), \tag{5.22}$$

by using the relation

$$\tan(x/2) = \frac{1+\cos(x)}{\sin(x)}. \tag{5.23}$$

Finally, we must use the discontinuity condition on the derivative of the wave function due to the delta distribution,

$$\psi'_{II}(L/2) - \psi'_I(L/2) = \frac{2m\lambda}{\hbar^2} A \sin(\omega L/2), \tag{5.24}$$

which gives us the relation,

$$\cos(\omega L)\left[\cos\left(\frac{\omega L}{2}\right) + \tan(\omega L)\sin\left(\frac{\omega L}{2}\right)\right] + \cos\left(\frac{\omega L}{2}\right) = -\frac{2m\lambda}{\omega\hbar^2}\sin\left(\frac{\omega L}{2}\right). \tag{5.25}$$

This last equation can be reduce, by using the half angle formulae for trigonometric functions, to

$$\tan\left(\frac{\omega L}{2}\right) = -\frac{\omega\hbar^2}{m\lambda}, \tag{5.26}$$

or

$$\tan\left(\frac{Lm}{2\hbar}\sqrt{\frac{2E}{m}}\right) = \sqrt{\frac{2E}{m}}\frac{\hbar}{\lambda}. \tag{5.27}$$

Considering the values $Lm/\hbar = 4$ and $\lambda/\hbar = 2$, we get

$$\tan(x) = -x/4, \tag{5.28}$$

with

$$x = 2\sqrt{\frac{2E}{m}}. \tag{5.29}$$

The lowest eigenvalues for this equation are (x=0 is not allowed because of the uncertainty relations) $x = 2.6$ and $x = 5.3$, or in term of energy, $E_0 \cong .85m$ and $E_1 \cong 3.5m$.

5.3 Problem 3...

A simple model for the states of an electron in a 1-dimensional system is

$$H = \sum_{n=1}^{N} E_0 |n><n| + \sum_{n=1}^{N} W\left\{|n><n+1| + |n+1><n|\right\}, \tag{5.30}$$

where $\{|n>\}_{n=1\ldots N}$ is an orthonormal basis and periodic conditions $|N+j> = |j>$ are assumed. E_0 and W are given parameters.

Calculate the eigenstates and eigenenergies.

Solution

Let's assume there exist a set of eigenvectors $|\lambda_i >= \sum_{n=1}^{N} c_n^{(i)}|n >$ s.t.

$$H|\lambda^i >= \lambda_i|\lambda_i > . \tag{5.31}$$

Then,

$$\begin{aligned} H|\lambda_i > &= \sum_n c_n^{(i)} E_0|n > +W \sum_n c_n \{|n-1 > +|n+1 >\} \\ &= \sum_n \left\{ c_n^{(i)} E_0 + W c_{n-1}^{(i)} + W c_{n+1}^{(i)} \right\} |n > \\ &= \lambda_i \sum_n c_n^{(i)} |n >, \end{aligned} \tag{5.32}$$

thus, from

$$< n|H|\lambda_i >= c_n^{(i)} E_0 + W c_{n-1}^{(i)} + W c_{n+1}^{(i)} = \lambda_i c_n^{(i)}, \tag{5.33}$$

we get the recurrence relation,

$$\lambda_i = E_0 + W \frac{c_{n-1}^{(i)} + c_{n+1}^{(i)}}{c_n^{(i)}}. \tag{5.34}$$

Additionally, the coeffitients must satisfy

$$\sum_n |c_n^{(i)}|^2 = 1 \tag{5.35}$$

$$\sum_n c_n^{(i)} c_n^{(j)} = 0. \tag{5.36}$$

Part III

Thermodynamics and Statistical Mechanics

Chapter 6

Probability

6.1 Problem. Reif 1.1

Solution

In order to get less than 6 point with a set of 3 dice, one should get

Throw	# of permutations
1,1,1	1
1,1,2	3
1,1,3	3
1,1,4	3
1,2,3	6
1,2,2	3
2,2,2	1

Then,

$$P_r = \frac{N_r}{N} = \frac{20}{6^3} = 9.26 * 10^{-2}. \tag{6.1}$$

6.2 Problem. Reif 1.2

Solution

1.

$$P = \frac{6!}{5!1!}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^5 = 0.402 \tag{6.2}$$

2.

$$P = 1 - \left(\frac{5}{6}\right)^6 = 0.667 \tag{6.3}$$

3.
$$P = \frac{6!}{4!2!}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^4 = 0.2 \tag{6.4}$$

6.3 Problem. Reif 1.3

Solution

Each digit, has 5 possible choices for been < 5 and 5 of been ≥ 5, then, $p = q = \frac{1}{2}$.

Thus,
$$P = \frac{10!}{5!5!}\left(\frac{1}{2}\right)^{10} = 0.246. \tag{6.5}$$

6.4 Problem. Reif 1.4

Solution

1.
$$P(N/2) = \frac{N!}{\frac{N}{2}!\frac{N}{2}!}\left(\frac{1}{2}\right)^N. \tag{6.6}$$

2. If N is odd, either n_1 or n_2 is odd and the other is even, so
$$P(m = 0) = 0. \tag{6.7}$$

6.5 Problem. Reif 1.5

Solution

1.
$$P = \left(\frac{5}{6}\right)^N. \tag{6.8}$$

2.
$$P = \left(\frac{5}{6}\right)^{N-1}\frac{1}{6}. \tag{6.9}$$

3.
$$\#\ pull = \frac{1}{P_{shoot}} = 6. \tag{6.10}$$

6.6 Problem. Reif 1.6

Solution

It is known that $\overline{m^a} = \overline{(2n - N)^a}$, then

$$\begin{aligned}
\overline{m} &= 2\overline{n} - N && (6.11)\\
\overline{m^2} &= 4\overline{n^2} - 4N\overline{n} + N^2 && (6.12)\\
\overline{m^3} &= 8\overline{n^3} - 12N\overline{n^2} + 6N^2\overline{n} - N^3 && (6.13)\\
\overline{m^4} &= 16\overline{n^4} - 32N\overline{n^3} + 24N^2\overline{n^2} - 8N^3\overline{n} + N^4. && (6.14)
\end{aligned}$$

Next,

$$\overline{n^a} = \left(p\frac{\partial}{\partial p}\right)^a (p+q)^N, \qquad (6.15)$$

then[1],

$$\begin{aligned}
\left(p\frac{\partial}{\partial p}\right)(p+q)^N &= Np(p+q)^{N-1} && (6.16)\\
\left(p\frac{\partial}{\partial p}\right)^2 (p+q)^N &= Np\left[(p+q)^{N-1} + (N-1)p(p+q)^{N-2}\right] && (6.17)\\
\left(p\frac{\partial}{\partial p}\right)^3 (p+q)^N &= Np\left[(p+q)^{N-1} + 3(N-1)p(p+q)^{N-2}\right.\\
&\quad \left.+(N-1)(N-2)p^2(p+q)^{N-3}\right] && (6.18)\\
\left(p\frac{\partial}{\partial p}\right)^4 (p+q)^N &= Np\left[(p+q)^{N-1} + 7(N-1)p(p+q)^{N-2}\right.\\
&\quad +6(N-1)(N-2)p^2(p+q)^{N-3}\\
&\quad \left.+(N-1)(N-2)(N-3)p^3(p+q)^{N-4}\right] && (6.19)
\end{aligned}$$

Using that $p + q = 1$, and substituting (6.16)-(6.19) into (6.11)-(6.14), one gets

$$\begin{aligned}
\overline{m} &= 0 && (6.20)\\
\overline{m^2} &= N && (6.21)\\
\overline{m^3} &= 0 && (6.22)\\
\overline{m^4} &= 2N^4 + 2N^3 - 3N^2 + 4N && (6.23)
\end{aligned}$$

[1] You must check it by yourself because it does not coinside neither with the Reif results nor the resultofa friend of mine.

6.7 Problem. Reif 1.7

Solution

$$\begin{aligned} W'(n) &= \sum_{i_1=1}^{2} \cdots \sum_{i_N=1}^{2} \omega_{i_1} \cdots \omega_{i_N} \\ &= \sum_{i_1=1}^{2} \omega_{i_1} \cdots \sum_{i_N=1}^{2} \omega_{i_N} \\ &= (\omega_1 + \omega_2) \cdots (\omega_1 + \omega_2) \\ &= (\omega_1 + \omega_2)^N. \end{aligned} \tag{6.24}$$

From the binomial theorem, it follows that the restriction of ω_1 to occurs n times is

$$W(n) = \frac{N!}{n!(N-n)!} \omega_1^n \omega_2^{N-n}. \tag{6.25}$$

Chapter 7

Thermodynamics

7.1 Problem. Huang 1-1

Solution

For an adiabatic process, $\Delta Q = 0$, so, the first law becomes

$$dU = dW = -PdV. \tag{7.1}$$

From the kinetictheory of ideal gases, the internal energy is

$$U = \frac{3}{2}NkT, \tag{7.2}$$

then,

$$dU = \frac{3}{2}NkdT. \tag{7.3}$$

Hence,

$$\frac{3}{2}NkdT = -PdV = -\frac{NkT}{V}dV, \tag{7.4}$$

where the ideal gas equation of states has been used. Integrating (7.4), the result

$$\begin{aligned} \frac{3}{2}\int_{T_0}^{T}\frac{dT}{T} &= -\int_{V_0}^{V}\frac{dV}{V} \\ \frac{V_0}{V} &= \left(\frac{T}{T_0}\right)^{3/2}. \end{aligned} \tag{7.5}$$

Since,

$$V_i = \frac{NkT_i}{P_i}, \tag{7.6}$$

it is possible to change V by T, therefore,

$$\frac{P}{P_0} = \left(\frac{T}{T_0}\right)^{5/2}, \tag{7.7}$$

and also, in the same way,

$$\frac{P}{P_0} = \left(\frac{V_0}{V}\right)^{5/3}. \tag{7.8}$$

Chapter 8

Statistical Mechanics

8.1 Problem G.-C. 2.2

Solution

Since

$$\Omega = \Omega_1 \Omega_2, \tag{8.1}$$

it follows that

$$\Omega_2 \frac{df(\Omega)}{d\Omega} = \frac{d\Omega_1}{d\Omega_1}, \tag{8.2}$$

$$\Omega_1 \frac{df(\Omega)}{d\Omega} = \frac{d\Omega_2}{d\Omega_2}. \tag{8.3}$$

The chain rule gives,

$$\frac{d}{d\ln\Omega} f = \frac{d\Omega}{d\ln\Omega} \frac{df}{d\Omega}, \tag{8.4}$$

thus, (8.2) times Ω_1 gives,

$$\Omega \frac{df(\Omega)}{d\Omega} = \Omega_1 \frac{df(\Omega_1)}{d\Omega_1} \Rightarrow \frac{df(\Omega)}{d\ln\Omega} = \frac{df(\Omega_1)}{d\ln\Omega_1}, \tag{8.5}$$

and similarly,

$$\frac{df(\Omega)}{d\ln\Omega} = \frac{df(\Omega_2)}{d\ln\Omega_2}. \tag{8.6}$$

Furthermore, the Ω_2 derivative of (8.5) and Ω_1 derivative of (8.6) vanish, so

$$\frac{d}{d\Omega_2} \frac{df(\Omega)}{d\ln\Omega} = 0 \tag{8.7}$$

$$\frac{d}{d\Omega_2} \frac{df(\Omega)}{d\ln\Omega} = 0, \tag{8.8}$$

then,

$$f(\Omega) = k \ln \Omega + \Omega_0, \tag{8.9}$$

with k a constant.

By the initial condition,

$$f(\Omega_1) + f(\Omega_2) = f(\Omega),$$

if the integration constant Ω_0 is not zero, it adds. So, the integration constant must be zero.

8.2 Problem G.-C. 2.5

The energy levels allow for a bi-dimensional harmonic oscillator are $\epsilon_i = (n+1)h\nu$ and their degenerancy is $\omega_i = (n+1)$. Show that the partition function for those oscillators is the square of the partition function for the one dimensional oscillator.

Solution

The available data is $\epsilon_i = (n+1)h\nu$ and $\omega_i = n+1$. From the definition,

$$\begin{aligned} Z_2 &= \sum_{n=0}^{\infty}(n+1)e^{-(n+1)h\nu\beta} \\ &= -\frac{\partial}{\partial(h\nu\beta)}\sum_{n=0}^{\infty}e^{-(n+1)h\nu\beta} \\ &= -\frac{\partial}{\partial(h\nu\beta)}\frac{e^{-h\nu\beta}}{1-e^{-h\nu\beta}} \\ &= -\frac{\partial}{\partial(h\nu\beta)}\frac{1}{e^{h\nu\beta}-1} \\ &= \frac{e^{h\nu\beta}}{(e^{h\nu\beta}-1)^2}. \end{aligned} \tag{8.10}$$

Since the partition function for the one-dimensional harmonic oscillator is

$$Z_1 = \frac{e^{\frac{1}{2}h\nu\beta}}{(e^{h\nu\beta}-1)}, \tag{8.11}$$

one conclude that

$$\boxed{Z_2 = (Z_1)^2.} \tag{8.12}$$

8.3 Problem G.-C. 2.6

Obtain the eq. (2.29) of the book.

Solution

From

$$U = \frac{3}{2}Nh\nu + 3N\frac{h\nu}{e^{-\frac{h\nu}{kT}} - 1}, \tag{8.13}$$

it follows that,

$$\begin{aligned} C_v &= \left(\frac{\partial U}{\partial T}\right)_v \\ &= 3Nh\nu\frac{\partial}{\partial T}\frac{1}{e^{-\frac{h\nu}{kT}} - 1} \\ &= 3N\frac{h^2\nu^2}{kT^2}\frac{e^{\frac{h\nu}{kT}}}{\left(e^{\frac{h\nu}{kT}} - 1\right)^2} \\ &= 3Nk\left(\frac{h\nu}{kT}\right)^2\frac{e^{\frac{h\nu}{kT}}}{\left(e^{\frac{h\nu}{kT}} - 1\right)^2}. \end{aligned} \tag{8.14}$$

Finally,

$$\boxed{C_v = 3Nk\left(\frac{h\nu}{kT}\right)^2\frac{e^{\frac{h\nu}{kT}}}{\left(e^{\frac{h\nu}{kT}} - 1\right)^2}} \tag{8.15}$$

8.4 Problem G.-C. 2.8

Repit the calculus of $\langle\mu_z\rangle$ when $J = \frac{1}{2}$. Compare the results with the eqs. (2.37)–(2.39) in the textbook.

Solution

One have that

$$Z_{\frac{1}{2}} = \sum_{m=-1/2}^{1/2} e^{g\mu_0 Hm\beta}, \tag{8.16}$$

and

$$\langle\mu_z\rangle = \sum_{m=-1/2}^{1/2} e^{g\mu_0 Hm\beta} = \frac{\partial}{\partial H\beta}\ln\sum_{m=-1/2}^{1/2} e^{g\mu_0 Hm\beta}. \tag{8.17}$$

Now,

$$\sum_{m=-1/2}^{1/2} e^{g\mu_0 Hm\beta} = 2\cosh\frac{g\mu_0 H\beta}{2}, \tag{8.18}$$

then,

$$\frac{\partial}{\partial H\beta}\ln 2\cosh\left(\frac{g\mu_0 H\beta}{2}\right) = \frac{1}{2}g\mu_0\tanh\left(\frac{g\mu_0 H\beta}{2}\right), \tag{8.19}$$

thus,

$$\boxed{\boxed{\langle\mu_z\rangle = \frac{1}{2}g\mu_0\tanh\left(\frac{g\mu_0 H\beta}{2}\right).}} \tag{8.20}$$

On the other hand,

$$\langle\mu_z\rangle = g\mu_0 J B_J(\eta), \tag{8.21}$$

with $\eta = g\mu_0 H\beta$ and

$$B_J(\eta) = \frac{1}{J}\left[(J+1/2)\coth\left(J+\frac{1}{2}\right)\eta - \frac{1}{2}\coth\frac{\eta}{2}\right]. \tag{8.22}$$

For $J = 1/2$, it follows that

$$B_{1/2}(\eta) = 2\left[\coth\eta - \frac{1}{2}\coth\frac{\eta}{2}\right]. \tag{8.23}$$

By using the identity,

$$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}, \tag{8.24}$$

the last equation can be written as,

$$B_{1/2}(\eta) = 2\left[\frac{1+\tanh^2\eta/2}{2\tanh\eta/2} - \frac{1}{2}\coth\frac{\eta}{2}\right] = \tanh\eta/2, \tag{8.25}$$

and finally,

$$\boxed{\boxed{\langle\mu_z\rangle = \frac{1}{2}g\mu_0\tanh\left(\frac{g\mu_0 H\beta}{2}\right).}} \tag{8.26}$$

8.5 Problem G.-C. 2.10

Show that a solid obeying Einstein model has

$$S = -3Nk\frac{\partial}{\partial T}\left[T\ln\left(1-e^{-\frac{h\nu}{kT}}\right)\right]. \tag{8.27}$$

Discuss physically the limits of low and high frequencies. What rôle plays the zero point energy?

Solution

Since,

$$Z_0 = \sum_{n=0}^{\infty} e^{-(n+1/2)h\nu\beta} = \frac{e^{-\frac{1}{2}h\nu\beta}}{1 - e^{-h\nu\beta}}, \tag{8.28}$$

it follows that

$$F = -NkT \ln Z_0 = \frac{3}{2} Nh\nu + 3NkT \ln\left(1 - e^{-h\nu\beta}\right). \tag{8.29}$$

Finally from the Maxwell relation,one get

$$\boxed{S = -\left(\frac{\partial F}{\partial T}\right)_V = -3Nk\frac{\partial}{\partial T}\left[T \ln\left(1 - e^{-\frac{h\nu}{kT}}\right)\right].} \tag{8.30}$$

8.6 Problem G.-C. 3.1

Deduce the eq. (3.5) of the text.

Solution

Since

$$\begin{aligned} U &= N\frac{\sum_{i=0}^{\infty} \epsilon_i e^{-\epsilon_i\beta}}{\sum_{j=0}^{\infty} e^{-\epsilon_j\beta}} \\ &= -\frac{N}{\sum_{j=0}^{\infty} e^{-\epsilon_j\beta}} \frac{\partial}{\partial\beta} \sum_{i=0}^{\infty} e^{-\epsilon_i\beta} \\ &= -N\frac{\partial}{\partial\beta} \ln\left(\sum_{i=0}^{\infty} e^{-\epsilon_i\beta}\right), \end{aligned} \tag{8.31}$$

Now, by calling

$$Z_0 = \sum_{i=0}^{\infty} e^{-\epsilon_i\beta}, \tag{8.32}$$

and since the chain rule gives

$$\frac{\partial}{\partial\beta} = \frac{\partial T}{\partial\beta}\frac{\partial}{\partial T} \Rightarrow \frac{\partial}{\partial\beta} = -kT^2\frac{\partial}{\partial T}, \tag{8.33}$$

finally one get,

$$\boxed{\langle E\rangle = NkT^2\frac{\partial}{\partial T} \ln Z_0.} \tag{8.34}$$

8.7 Problem G.-C. 3.2

Obtain the eq. (3.8) of the text, from the definition of entropy.

Solution

By the definition of the entropy,

$$S = -\left(\frac{\partial F}{\partial T}\right)_V . \tag{8.35}$$

Since

$$F = -NkT\left\{\ln V + \frac{3}{2}\ln T + \ln\left(\frac{2\pi mk}{h^2}\right)^{3/2}\right\}, \tag{8.36}$$

then,

$$S = Nk\left[\left\{\ln V + \frac{3}{2}\ln T + \ln\left(\frac{2\pi mk}{h^2}\right)^{3/2}\right\} + \frac{3}{2}\right]. \tag{8.37}$$

By substituting $R = N_a k$ and $s = S/n$,

$$\boxed{\frac{s}{R} = \ln V + \frac{3}{2}\ln T + \left(\frac{3}{2} + \ln\left(\frac{2\pi mk}{h^2}\right)^{3/2}\right).} \tag{8.38}$$

8.8 Problem G.-C. 3.3

Use the eqs. (3.17)-(3.20) of the textbook, to obtain the eqs. (3.21) and (3.22) therein.

Solution

Beginning from the corrected distribution

$$W_{n_1,n_2} = \prod_{i,j} \frac{1}{n_i^{(1)}!n_j^{(2)}!}, \tag{8.39}$$

the numberof states is given by

$$\Omega = \sum_{n^{(1)},n^{(2)}}' \prod_{i,j} \frac{1}{n_i^{(1)}!n_j^{(2)}!}, \tag{8.40}$$

with the constrints

$$\sum_i n_i^{(1)} = N_1, \tag{8.41}$$

$$\sum_i n_i^{(2)} = N_2, \tag{8.42}$$

$$\sum_i n_i^{(1)}\epsilon_i^{(1)} + \sum_j n_j^{(2)}\epsilon_j^{(2)} = E. \tag{8.43}$$

Calling

$$t(n^{(1)}, n^{(2)}) = \sum_{n^{(1)},n^{(2)}}{}' \prod_{i,j} \frac{1}{n_i^{(1)}!n_j^{(2)}!}, \tag{8.44}$$

and using

$$\delta t(n^{(1)}, n^{(2)}) = \sum_i \frac{\partial t}{\partial n_i^{(1)}}\delta n_i^{(1)} + \sum_j \frac{\partial t}{\partial n_j^{(2)}}\delta n_j^{(2)}, \tag{8.45}$$

and additionally,

$$\sum_i \delta n_i^{(1)} = \delta N_1 = 0, \tag{8.46}$$

$$\sum_i \delta n_i^{(2)} = \delta N_2, \tag{8.47}$$

$$\sum_i \delta n_i^{(1)}\epsilon_i^{(1)} + \sum_j \delta n_j^{(2)}\epsilon_j^{(2)} = \delta E = 0. \tag{8.48}$$

Thus,

$$\begin{aligned}\delta t(n^{(1)}, n^{(2)}) &= \sum_i \left(\frac{\partial t}{\partial n_i^{(1)}} + \alpha_1 + \beta_1\epsilon_i^{(1)}\right)\delta n_i^{(1)} \\ &\quad + \sum_j \left(\frac{\partial t}{\partial n_j^{(2)}} + \alpha_2 + \beta_2\epsilon_j^{(2)}\right)\delta n_j^{(2)},\end{aligned} \tag{8.49}$$

with α's and β's Lagrange multipliers.

Now,

$$\begin{aligned}\ln t(n^{(1)}, n^{(2)}) &= -\sum_i \ln(n_i^{(1)}!) - \sum_j \ln(n_j^{(2)}!) \\ &= -\sum_i n_i^{(1)} \ln(n_i^{(1)}) + N_1 - \sum_j n_j^{(2)} \ln(n_j^{(2)}) + N_2,\end{aligned} \tag{8.50}$$

since,

$$\frac{\partial}{\partial n_a^{(\alpha)}} \ln t(n^{(1)}, n^{(2)}) = -\ln n_a^{(\alpha)} - 1, \tag{8.51}$$

then, as each term of the sum on (8.49) must vanish independently, it follows that

$$\ln n_a^{(\alpha)} = \alpha_{(\alpha)} + \beta_{(\alpha)}\epsilon_a^{(\alpha)} \quad \Rightarrow \quad n_a^{(\alpha)*} = e^{\alpha_{(\alpha)}} e^{\beta_{(\alpha)}\epsilon_a^{(\alpha)}}. \tag{8.52}$$

Finally, from the restrictions,

$$\sum_a n_a^{(\alpha)*} = \sum_a e^{\alpha_{(\alpha)}} e^{\beta_{(\alpha)}\epsilon_a^{(\alpha)}} = N_{(\alpha)}, \tag{8.53}$$

one gets

$$e^{\alpha_{(\alpha)}} = \frac{N_{(\alpha)}}{\sum_a e^{\beta_{(\alpha)}\epsilon_a^{(\alpha)}}}, \tag{8.54}$$

by definition,

$$Z_{(\alpha)} = \sum_a e^{\beta_{(\alpha)}\epsilon_a^{(\alpha)}}, \tag{8.55}$$

thus,

$$\boxed{n_i^{(1)*} = \frac{N_1}{Z_1} e^{\beta_1 \epsilon_i^{(1)}},} \tag{8.56}$$

and

$$\boxed{n_j^{(2)*} = \frac{N_2}{Z_2} e^{\beta_2 \epsilon_j^{(2)}}.} \tag{8.57}$$

8.9 Problem G.-C. 3.4

Use the result of the previous problem to deduce the eqs. (3.23a) and (3.23b) in the text.

Solution

$$\begin{aligned}
S &= k \ln t(n^{(1)}, n^{(2)})_{max} \\
&= -k \sum_i n_i^{(1)*} \ln(n_i^{(1)*}) + kN_1 - k \sum_j n_j^{(2)*} \ln(n_j^{(2)*}) + kN_2 \\
&= -k \sum_i n_i^{(1)*} \ln(n_i^{(1)*}) + kN_1 - k \sum_j n_j^{(2)*} \ln(n_j^{(2)*}) + kN_2 \\
&= -k\beta E - kN_1(\ln N_1 - 1 - \ln Z_1) - kN_2(\ln N_2 - 1 - \ln Z_2) \\
&= k \ln \frac{Z_1^{N_1}}{N_1!} + k \ln \frac{Z_2^{N_2}}{N_2!} + -\frac{E}{T}.
\end{aligned} \tag{8.58}$$

Thus,

$$\boxed{F = -kT \ln\left(\frac{Z_1^{N_1}}{N_1!}\frac{Z_2^{N_2}}{N_2!}\right).} \tag{8.59}$$

8.10 Problem G.-C. 3.5

The energy of the one-dimensional harmonic oscillator is

$$\epsilon(x,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{8.60}$$

Calculate the clasical partition function and the termal propertiesof a system of $3N$ lineal oscillators. Compare these results with the one of the second chapter of the textbook.

Solution

Two different approaches for solving this problem will be given, the first is by using the definition of the classical partition function and the second by the micro-canonical partition function.

Classical Partition Function

By definition, the classical partition function for a single harmonic oscillator is

$$Z = \frac{1}{\hbar}\int_{\mathbb{R}} dx \int_{\mathbb{R}} dp e^{-\left(\frac{p^2}{2m}+\frac{1}{2}m\omega^2 x^2\right)\beta}, \tag{8.61}$$

since,

$$\int_{\mathbb{R}} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \tag{8.62}$$

it follows that (8.61) yields,

$$\boxed{Z = \frac{kT}{\hbar\omega}.} \tag{8.63}$$

Therefore,

$$F = -kT\ln Z = -kT\ln\frac{kT}{\hbar\omega}, \tag{8.64}$$

so, using Maxwell'srelations, one get

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = 0, \tag{8.65}$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = k\left[1+\ln\left(\frac{kT}{\hbar\omega}\right)\right], \tag{8.66}$$

$$U = F + TS = kT, \tag{8.67}$$

thus,

$$\boxed{S(E,V) = k\left[1+\ln\left(\frac{E}{\hbar\omega}\right)\right].} \tag{8.68}$$

Repiting the above precess for $3N$ harmonic oscillators,

$$\boxed{Z = \left(\frac{kT}{\hbar\omega}\right)^N,} \tag{8.69}$$

where the $N!$ coming from the Gibbs factor is not consider. Hence,

$$\begin{aligned} F &= -3NkT \ln\left(\frac{kT}{\hbar\omega}\right), &(8.70)\\ P &= 0, &(8.71)\\ S &= 3Nk\left[1 + \ln\left(\frac{kT}{\hbar\omega}\right)\right], &(8.72)\\ U &= 3NkT, &(8.73)\end{aligned}$$

thus,

$$\boxed{S(E, V, N) = 3Nk\left[1 + \ln\left(\frac{E}{3N\hbar\omega}\right)\right].} \tag{8.74}$$

Micro-canonical Partition Function

8.11 Problem G.-C. 3.15

Solution

A relativistic ideal gas is composed by relativistic free particles, so its Hamiltonian is given by

$$H = \sum_{i=1}^{N} mc^2 \left\{\left[1 + \left(\frac{\vec{p}}{mc}\right)^2\right]^{1/2} - 1\right\}, \tag{8.75}$$

then, for a single particle its partition function is

$$\begin{aligned} Z(T, V, 1) &= \frac{1}{h^3}\int d^3q \int d^3p e^{-\beta mc^2\left\{\left[1+\left(\frac{\vec{p}}{mc}\right)^2\right]^{1/2}-1\right\}}\\ &= \frac{V}{h^3}e^{\beta mc^2}\int d^3p e^{-\beta mc^2\left\{\left[1+\left(\frac{\vec{p}}{mc}\right)^2\right]^{1/2}\right\}}\\ &= \frac{4\pi V}{h^3}e^{\beta mc^2}\int_0^\infty dp p^2 e^{-\beta mc^2\left\{\left[1+\left(\frac{p}{mc}\right)^2\right]^{1/2}\right\}}. &(8.76)\end{aligned}$$

In order to perform the integral one can substitute $\frac{p}{mc} = \sinh x$, therefore,

$$Z(T, V, 1) = \frac{4\pi V}{h^3}(mc)^3 e^{\beta mc^2}\int_0^\infty dx \cosh x \sinh^2 x e^{-\beta mc^2 \cosh x}, \tag{8.77}$$

calling $u = \beta mc^2$,

$$Z(T, V, 1) = \frac{4\pi V}{h^3}(mc)^3 e^u \int_0^\infty dx \cosh x \sinh^2 x e^{-u\cosh x}. \tag{8.78}$$

Using the identity $\sinh x \cosh x = \frac{1}{2}\sinh 2x$,

$$Z(T, V, 1) = 4\pi V\left(\frac{mc}{h}\right)^3 \frac{1}{2}e^u \int_0^\infty dx \sinh 2x \sinh x e^{-u\cosh x}. \tag{8.79}$$

The above integral is solved by

$$\int_0^\infty dx \sinh x \sinh \gamma x e^{-u\cosh x} = \frac{\gamma}{u}K_2(u). \tag{8.80}$$

The nonrelativistic limit is $u = \beta mc^2 \to \infty$, i.e., $mc^2 >> kT$. Using that

$$K_2(z) \cong \sqrt{\frac{\pi}{2z}}e^{-z},$$

as long as $z \to \infty$, so

$$Z(T, V, 1) \to 4\pi V\left(\frac{mc}{h}\right)^3\left(\frac{1}{\beta mc^2}\right)^{3/2}\sqrt{\frac{\pi}{2}} = V\left(\frac{2\pi mkT}{h^2}\right)^{3/2}, \tag{8.81}$$

which agrees with the ideal gas.

The high temperature limit is $kT >> mc^2$, for which $K_2(u) \cong \frac{2}{u}$, and so

$$Z(T, V, 1) \to 8\pi V\left(\frac{kT}{hc}\right)^3, \tag{8.82}$$

this last is the partition function of an ultra-relativistic ideal gas.

Finally, the partition function of the ideal gas of N particles is

$$Z_n = \frac{1}{N!}(Z_1)^N = \frac{1}{N!}\left[4\pi V\left(\frac{mc}{h}\right)^3 e^{\beta mc^2}\frac{K_2(\beta mc^2)}{\beta mc^2}\right]^N. \tag{8.83}$$

From this,

$$\begin{aligned} F &= -kT \ln Z_N \\ &= -NkT - Nmc^2 - NkT \ln\left[4\pi V\left(\frac{mc}{h}\right)^3 e^{\beta mc^2}\frac{K_2(\beta mc^2)}{\beta mc^2}\right], \end{aligned} \tag{8.84}$$

and

$$\boxed{P = -\left(\frac{\partial F}{\partial V}\right) = \frac{NkT}{V},} \tag{8.85}$$

$$\boxed{\mu = \frac{\partial F}{\partial N} = -kT \ln\left[4\pi V\left(\frac{mc}{h}\right)^3 e^{\beta mc^2}\frac{K_2(\beta mc^2)}{\beta mc^2}\right] - kT - mc^2,} \tag{8.86}$$

8.12 Problem G.-C. 4.3

Solution

Consider the Maxwell-Boltzmann distribution,

$$f(\vec{v})d^3v = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m}{2kT}\vec{v}^2} d^3v. \tag{8.87}$$

Since,

$$\int_0^\infty e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}, \tag{8.88}$$

the total contribution to v_x is the integral over y and z,

$$f(v_x)dv = \left(\frac{m}{2\pi kT}\right)^{1/2} e^{-\frac{m}{2kT}v_x^2} dv_x. \tag{8.89}$$

Then, the probability of finding a particle with celerity on x between 0 and v_{x_0}, is

$$Pr(0 \leq v_x \leq v_{x_0}) = \left(\frac{m}{2\pi kT}\right)^{1/2} \int_0^{v_{x_0}} e^{-\frac{m}{2kT}v_x^2} dv_x, \tag{8.90}$$

by defining $v_m = \sqrt{\frac{2kT}{m}}$ and $x = \frac{v_x}{v_m}$, one get

$$\begin{aligned} n &= NPr(0 \leq v_x \leq v_{x_0}) \\ &= N\left(\frac{m}{2\pi kT}\right)^{1/2} \int_0^{v_{x_0}} e^{-\frac{m}{2kT}v_x^2} dv_x \\ &= N\frac{1}{\sqrt{\pi}} \int_0^{x_0} dx e^{-x^2} \\ &= \frac{N}{2} erf(x_0). \end{aligned} \tag{8.91}$$

Thus,

$$\boxed{n = \frac{N}{2} erf(x_0).} \tag{8.92}$$

8.13 Problem G.-C. 4.4

Solution

Consider the Maxwell-Boltzmann distribution,

$$f(\vec{v})d^3v = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m}{2kT}\vec{v}^2} d^3v. \tag{8.93}$$

This time the interest turns to the total celerity, so one can write the Maxwell-Boltzmann distribution in spherical coordinates, where $d^3v = dvv^2 d\Omega$, and so

$$Pr(0 \leq v \leq v_0) = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \int_0^{v_0} dvv^2 e^{-\frac{m}{2kT}v^2}. \tag{8.94}$$

by defining $v_m = \sqrt{\frac{2kT}{m}}$ and $x = \frac{v_x}{v_m}$, one get

$$\begin{aligned}
Pr(0 \leq v \leq v_0) &= \frac{4}{\sqrt{\pi}} \int_0^{x_0} dxx^2 e^{-ax^2}\bigg|_{a=1} \\
&= \frac{4}{\sqrt{\pi}} \left(-\frac{\partial}{\partial a}\right) \int_0^{x_0} dxe^{-ax^2}\bigg|_{a=1} \\
&= -2\frac{\partial}{\partial a}\frac{1}{\sqrt{a}} erf(\sqrt{a}x_0)\bigg|_{a=1} \\
&= erf(x_0) - \frac{2}{\sqrt{\pi}} x_0 e^{-x_0^2}.
\end{aligned} \tag{8.95}$$

Thus,

$$\boxed{n = N\left[erf(x_0) - \frac{2}{\sqrt{\pi}} x_0 e^{-x_0^2}\right].} \tag{8.96}$$

8.14 Problem G.-C. 4.5

The traslacional energy of a molecul is $\epsilon = \frac{1}{2}mv^2$. Write the Maxwell distribution function in terms of ϵ

Solution

Consider the Maxwell-Boltzmann distribution,

$$f(\vec{v})d^3v = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m}{2kT}\vec{v}^2} d^3v. \tag{8.97}$$

In terms of celerity,

$$f(v)dv = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m}{2kT}v^2} v^2 dv. \tag{8.98}$$

By changing

$$\epsilon = \frac{mv^2}{2}, \quad v = \sqrt{\frac{2\epsilon}{m}}, \quad vdv = \frac{d\epsilon}{m}, \tag{8.99}$$

then,

$$\boxed{f(\epsilon)d\epsilon = 2\pi N(\pi kT)^{-3/2} \sqrt{\epsilon} e^{-\frac{\epsilon}{kT}} d\epsilon.} \tag{8.100}$$

8.15 Problem G.-C. 4.8

For a gas in equilibrium, calculate $\langle \vec{v} \rangle$, $\left\langle v_x^2 \right\rangle$, $\left\langle v_x^3 \right\rangle$ and $\left\langle (v_x + bv_y)^2 \right\rangle$, where b is a constant.

Solution

Since the set up is confined to a volume V, it's invariant under parity symmetry, so the mean value of the velocity vanish,

$$\boxed{\boxed{\langle \vec{v} \rangle = 0.}} \tag{8.101}$$

On the other hand, the average energy by d.o.f. is $\frac{1}{2}kT$. Hence, $E = \frac{1}{2}mv^2$, it follows that,

$$\boxed{\boxed{\left\langle v_x^2 \right\rangle = \frac{N}{m}kT.}} \tag{8.102}$$

By the same symmetry argument as the first case, one get

$$\boxed{\boxed{\left\langle v_x^3 \right\rangle = 0,}} \tag{8.103}$$

as well as the higher odd momenta.

Also,

$$\begin{aligned} \left\langle (v_x + bv_y)^2 \right\rangle &= \left\langle v_x^2 \right\rangle + 2b \left\langle v_x v_y \right\rangle + b^2 \left\langle v_y^2 \right\rangle \\ &= \left(1 + b^2\right) \frac{NkT}{m}. \end{aligned} \tag{8.104}$$

Physically, the higher distribution momenta are irrelevant, because there are not fundamental quantities related with those.

8.16 Problem G.-C. 5.1

Solution

The canonical partition function for an ideal gas is

$$Z_1 = V \left(\frac{2\pi mkT}{h^2} \right)^{3/2}, \tag{8.105}$$

and

$$Z_n = \frac{1}{N!} Z_1^N. \tag{8.106}$$

Since

$$e^{\beta\mu}\sum_i g_i e^{-\beta\epsilon_i} = N, \tag{8.107}$$

then

$$A = \frac{N}{\sum_i g_i e^{-\beta\epsilon_i}} = \frac{N}{Z_N}, \tag{8.108}$$

so, by taking logarithm and $\beta = \dfrac{1}{kT}$,

$$\boxed{\frac{\mu}{kT} = \ln\left(\frac{N}{V}\right) + \frac{3}{2}\ln\left(\frac{h^2}{2\pi mkT}\right).} \tag{8.109}$$

8.17 Problem G.-C. 5.2

Solution

The degeneration parameter is

$$\xi = \frac{V}{N}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}, \tag{8.110}$$

so for an isometric process,

$$T \leq \frac{V}{N}\frac{h^2}{2\pi mk}. \tag{8.111}$$

Moreover, taking

$$V = 1cm^3 \tag{8.112}$$
$$N = 1mol \tag{8.113}$$
$$k = 1.38 * 10^{-16} erg.K^{-1} \tag{8.114}$$
$$h = 6.626 * 10^{-27} erg.s, \tag{8.115}$$

one gets,

Element	Molecular Mass	$T_c(K)$
H_2	2	108.6
D_2	4	54.33
He	4	54.33
Ar	40	5.44
Cl_2	70	3.07

8.18 Problem G.-C. 5.3

Solution

For Helium, $T_c = 54.33K$, so $T = 300K \gg T_c$, which implies $\xi \gg 1$. In what follows the quantites $T = 300K$, $V = 1cm^3$ and $N = 1mol$ will be used.

$$\Gamma(\epsilon) = \frac{1}{8}\frac{4\pi}{3}\left(\frac{8m\epsilon a^2}{h^2}\right)^{3/2} = \frac{\pi V}{6}\left(\frac{8m\epsilon}{h^2}\right)^{3/2}. \tag{8.116}$$

The number of levels

$$\langle\Gamma(\epsilon)\rangle = \frac{\pi V}{6}\left(\frac{8m}{h^2}\right)^{3/2}\left\langle\epsilon^{3/2}\right\rangle. \tag{8.117}$$

Now,

$$\begin{aligned}\left\langle\epsilon^{3/2}\right\rangle &= \frac{N}{Z}\int_0^\infty \rho(\epsilon)\epsilon^{3/2}e^{-\beta\epsilon}d\epsilon \\ &= 2\pi N\left(\frac{1}{2\pi kT}\right)^{3/2}\int_0^\infty d\epsilon\ \epsilon^2 e^{-\beta\epsilon} \\ &= 4\pi N\left(\frac{kT}{2\pi}\right)^{3/2}, \end{aligned}\tag{8.118}$$

then

$$\langle\Gamma(\epsilon)\rangle = N\left[\frac{8V}{3\pi}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\right]. \tag{8.119}$$

Finally since $\frac{m}{N_a} = 4gr/mol$,

$$\boxed{\frac{\langle\Gamma(\epsilon)\rangle}{N} = 6.63 * 10^{24}.} \tag{8.120}$$

And

$$\boxed{N = \frac{PV}{kT} = 2.4 * 10^3,} \tag{8.121}$$

Helium molecules.

8.19 Problem G.-C. 5.4

Solution

Since g_i indicates the number of cells, and n_i the number of particles, if $g_i >> n_i$,paticles can be distribute without a ‘strong’ interaction among them. So they do not note the quantum characteristics of symmetric or anti-symmetric wave function.

8.20 Problem G.-C. 5.6

Solution

As in section 8.18,

$$\xi = \frac{V}{N}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}. \tag{8.122}$$

Since $\rho = 10^7 gr/mol$, $T = 10^7 K$ and $\frac{m_{He}}{N_a} = 4gr/mol$, a direct computation gives

$$\boxed{\boxed{\xi = 32.}} \tag{8.123}$$

8.21 Problem G.-C. 9.1

Solution

Since the partition function is

$$Z = \sum_i g_i e^{-\beta\epsilon_i}, \tag{8.124}$$

in order to past to the classical the differences between energies must satisfy the condition

$$\frac{\Delta\epsilon}{\epsilon_i} \sim 0. \tag{8.125}$$

In doing this, the energies change to

$$\epsilon_i \longrightarrow \epsilon(\vec{r}, \vec{p}) = \frac{\vec{p}^2}{2m} + \varphi(\vec{r}), \tag{8.126}$$

and similarly the sum turns into an integral,

$$\sum_i g_i \longrightarrow \frac{1}{h^3}\int d\Gamma. \tag{8.127}$$

So, the classical partition function is

$$Z = \frac{1}{h^3}\int d\Gamma e^{-\beta\epsilon(\vec{r},\vec{p})}, \tag{8.128}$$

and finally, the fraction of molecules having momentum between $[\vec{p}, \vec{p} + d\vec{p}]$ and position $[\vec{x}, \vec{x} + d\vec{x}]$ is given by

$$\boxed{\boxed{n(\vec{r}, \vec{p})d\Gamma = \frac{Ne^{-\beta\left(\frac{\vec{p}^2}{2m}+\varphi(\vec{r})\right)}d\Gamma}{\int e^{-\beta\left(\frac{\vec{p}^2}{2m}+\varphi(\vec{r})\right)}d\Gamma}.}} \tag{8.129}$$

It's nothing but the Maxwell-Boltzmann distribution.

8.22 Problem G.-C. 9.2

Solution

For a grand canonical ensemble,

$$P_j(N) = \frac{1}{\Xi} e^{-\beta E_j + \alpha N}, \tag{8.130}$$

so,

$$\begin{aligned} \langle E \rangle &= \sum_{N_0}^{\infty} \sum_j E_j P_j(N) \\ &= \frac{1}{\Xi} \sum_{N=0}^{\infty} \sum_j E_j e^{-\beta E_j + \alpha N} \\ &= -\frac{\partial}{\partial \beta} \ln \Xi, \end{aligned} \tag{8.131}$$

then

$$\boxed{\langle E \rangle = -\frac{\partial}{\partial \beta} \ln \Xi.} \tag{8.132}$$

Also,

$$\begin{aligned} \langle N \rangle &= \sum_{N_0}^{\infty} \sum_j N P_j(N) \\ &= \frac{1}{\Xi} \sum_{N=0}^{\infty} N e^{\alpha N} \sum_j e^{-\beta E_j} \\ &= -\frac{\partial}{\partial \alpha} \ln \Xi, \end{aligned} \tag{8.133}$$

then

$$\boxed{\langle N \rangle = -\frac{\partial}{\partial \alpha} \ln \Xi.} \tag{8.134}$$

Finally, one can use

$$TS = U + \overline{N}\mu + \overline{P}V, \tag{8.135}$$

and

$$\overline{P}V = kT \ln \Xi = -E_{Nj} + \mu N - kT \ln P_j(N). \tag{8.136}$$

Substituting both result above, yields

$$TS = \left[\sum_{N=0}^{\infty} \sum_j (E_{Nj} - \mu N) P_j(N) \right] - E_{Nj} + \mu N - kT \ln P_j(N). \tag{8.137}$$

After average the last expresion, and dividing by T,

$$\boxed{S = -k \sum_{N=0}^{\infty} \sum_j P_j(N) \ln P_j(N).} \tag{8.138}$$

8.23 Problem G.-C. 9.13

Solution

Since

$$F = -kT \ln \Xi, \tag{8.139}$$

and by Maxwell relations,

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,\mu}, \tag{8.140}$$

it follows that

$$\boxed{S = k \ln \Xi kT \left(\frac{\partial}{\partial T} \ln \Xi\right)_{V,\mu}.} \tag{8.141}$$

For an open system,

$$\Xi = \sum_{N=0}^{\infty} z^N Z_N, \tag{8.142}$$

so,

$$\langle E \rangle = \frac{1}{\Xi} \sum_{N=0}^{\infty} \sum_j E_j e^{-\beta E_j + \alpha N} = -\frac{\partial}{\partial \beta} \ln \Xi. \tag{8.143}$$

$$\boxed{\langle E \rangle == -\frac{\partial}{\partial \beta} \ln \Xi.} \tag{8.144}$$

8.24 Problem G.-C. 9.14

Solution

Since

$$F = -kT \ln Z, \tag{8.145}$$

and by Maxwell relations,

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N}, \tag{8.146}$$

it follows that

$$\boxed{P = kT\left(\frac{\partial}{\partial V}\ln Z\right)_{T,N}.} \tag{8.147}$$

8.25 Problem H.6.1

Show that the three definitions of entropy given by

$$\begin{aligned} S &= k\ln\Gamma(E) && (8.148)\\ &= k\ln\Sigma(E) && (8.149)\\ &= k\ln\omega(E), && (8.150)\end{aligned}$$

are equivalent. In the above definitions,

$$\begin{aligned} \Sigma(E) &= \int_{H\leq E} d^{3N}p d^{3N}q, && (8.151)\\ \Gamma(E) &= \int_{E\geq H\leq E+\Delta} d^{3N}p d^{3N}q = \Sigma(E+\Delta)-\Sigma(E), && (8.152)\\ \omega(E) &= \frac{\partial}{\partial E}\Sigma(E). && (8.153)\end{aligned}$$

Solution

One can start from the definition

$$\Sigma(E) = \int_{H\leq E} d^{3N}p d^{3N}q, \tag{8.154}$$

which can be written as

$$\Sigma(E) = \int_0^\infty \Theta(E-H) d^{3N}p d^{3N}q. \tag{8.155}$$

Now, for Hamiltonians with only second order dependency on p and q, after certain change of variable, the remain integral is nothing but the integral of the posible states bounded by a “hyper"-half of a $3N$-dimensional sphere of radius $\sqrt{E^*}$, which is the considered energy, E, scaledby factorscoming from the change of variables.

For example, for a single free particle,

$$E^* = \frac{8mV^{2/3}}{h^2}E. \tag{8.156}$$

Since the volume of a $3N$-dimensional sphere is

$$\Sigma(E) = \int_{H\leq E} d^{3N}p d^{3N}q, \tag{8.157}$$

nonetheless, the interesting part for the problem is

$$\frac{1}{2^{3N}}\Sigma(E) = \frac{1}{2^{3N}}\int_{H\leq E} d^{3N}p d^{3N}q, \tag{8.158}$$

because of the boundaries of the set up. Then, formally

$$\Sigma(E) \sim \frac{1}{2^{3N}}\frac{E^{3N/2}}{\left(\frac{3N}{2}\right)!}. \tag{8.159}$$

Now,

$$\omega(E) = \frac{\partial\Sigma}{\partial E}\Delta \sim \frac{3N}{2}\frac{\Delta}{E}\Sigma(E), \tag{8.160}$$

thus,

$$\ln\omega(E) \sim \ln\Sigma(E) + \ln\frac{3N}{2} + \ln\frac{Delta}{E}, \tag{8.161}$$

but

$$\lim_{N\to\infty}\frac{\ln N}{N} = 0 \quad \frac{\delta}{E} << E \Rightarrow \ln\frac{\Delta}{E} = o(0), \tag{8.162}$$

so,

$$\ln\omega(E) \sim \ln\Sigma(E). \tag{8.163}$$

Also, by (8.162) it follows that

$$\ln\Gamma(E) = \ln\omega(E) + \ln\Delta \sim \ln\omega(E) \sim \ln\Sigma(E). \tag{8.164}$$

Finally,

$$\boxed{\ln\Gamma(E) \sim \ln\omega(E) \sim \ln\Sigma(E).} \tag{8.165}$$

8.26 Problem H.6.2

Let the “uniform" ensemble of energy E be defined as the ensemble of all systems of the given type with energy less than E. The equivalence between the definitions of entropy on the above problem means that we should obtain the same thermodynamic functions from the “uniform" ensemble of energy E as fro the microcanonical ensemble of energy E. In particular, the internal energy is E in both ensembles. Explain why this seeminly paradoxical result is true.

Solution

After proving the equivalency of the three diferent definitions of the entropy, it follows that in fact the “uniform" ensamble is equivalent to the micro-canonical ensamble. The reason underlying this situation is that the rate at which the number of states grows with energy is so abrupt that the main contribution is given by the “immediate neighborhood" of E, as was said before,

$$\Sigma(E) \sim E^{3N/2}, \tag{8.166}$$

with large N, i.e., the neighborhood of E makes the overwhelminly dominant contribution to this number.

Since one is finally concerned only in the logarithm of this number, even the “width" of the neighborhood is inconsequential.

8.27 Problem H. 6.4

Using the correct entropy formula (6.62), work out the entropy of mixing for the case of different gases and for the case of identical gases, thus showing expliciply that there is no Gibbs paradox.

Solution

The correct expresion for theentropy is

$$S_i = N_i k \ln \frac{V_i}{N_i} + \frac{3}{2} N_i k \left\{ \frac{5}{3} + \ln\left(\frac{2\pi m_i kT}{h^2} \right) \right\}, \tag{8.167}$$

So,

$$S_t = Nk \ln \frac{V}{N} + \frac{3}{2} Nk \left\{ \frac{5}{3} + \ln\left(\frac{2\pi mkT}{h^2} \right) \right\}, \tag{8.168}$$

where $N = N_1 + N_2$, $V = V_1 + V_2$ and since they are composed of the same gas, $m_1 = m_2 = m$.

Thus,

$$\begin{aligned} \Delta S &= S_t - S_1 - S_2 \\ &= Nk \ln \frac{V}{N} - N_1 k \ln \frac{V_1}{N_1} - N_2 k \ln \frac{V_2}{N_2} \\ &= (N_1 + N_2) k \ln \frac{V_1 + V_2}{N_1 + N_2} - N_1 k \ln \frac{V_1}{N_1} - N_2 k \ln \frac{V_2}{N_2}, \end{aligned} \tag{8.169}$$

moreover, as the initial conditions are the same, the densities of the three systems are the same,

$$\frac{V_1 + V_2}{N_1 + N_2} = \frac{V_1}{N_1} = \frac{V_2}{N_2}, \tag{8.170}$$

so, finally,

$$\boxed{\Delta S = 0.} \tag{8.171}$$

8.28 Problem H. 7.1

- Obtain the preasure of a classical ideal gas as a function of N, T and V, by calculating the partituon function.
- Obtain the same by calculating the grand partition function.

Solution

8.28.1 Canonical Ensamble

Since

$$Z_N(V,T) = \frac{1}{N!} Z_1^N, \tag{8.172}$$

it follows that

$$\begin{aligned} Z_N(V,T) &= \frac{1}{N!h^{3N}} \int d\Gamma\, e^{-\frac{\beta}{2m}\sum_i p_i^2} \\ &= \frac{V^n}{N!h^{3N}} \left[4\pi \int dp\ p^2 e^{-\frac{\beta}{2m}\sum_i p^2} \right]^N \\ &= \frac{1}{N!} \left[\frac{V}{h^3} (2\pi mkT)^{3/2} \right]^N . \end{aligned} \tag{8.173}$$

Then,

$$F = NkT \left[\ln \left\{ \frac{N}{V} \left(\frac{h^2}{2\pi mkT} \right)^{3/2} \right\} - 1 \right], \tag{8.174}$$

thus,

$$\boxed{\mu = \left(\frac{\partial F}{\partial N} \right)_{V,T} = kT \ln \left\{ \frac{N}{V} \left(\frac{h^2}{2\pi mkT} \right)^{3/2} \right\},} \tag{8.175}$$

$$\boxed{P = -\left(\frac{\partial F}{\partial V} \right)_{N,T} = \frac{NkT}{V},} \tag{8.176}$$

$$\boxed{S = -\left(\frac{\partial F}{\partial T} \right)_{N,V} = Nk \left[\ln \left\{ \frac{V}{N} \left(\frac{2\pi mkT}{h^2} \right)^{3/2} \right\} + \frac{5}{2} \right]} \tag{8.177}$$

and

$$\boxed{U = F + ST = \frac{3}{2} NkT.} \tag{8.178}$$

8.28.2 Grand Canonical Ensable

For the grand canonical ensamble, the partition function is

$$\mathcal{Q}(z, V, T) = \sum_{N=0}^{\infty} z^N Z_N(V, T), \tag{8.179}$$

where Z_N is the canonical partition function. Moreover,

$$Z_N = \frac{1}{N!}[Z_1]^N, \tag{8.180}$$

therefore,

$$\mathcal{Q} = \sum_{N=0}^{\infty} \frac{1}{N!}[zZ_1]^N = e^{zZ_1} = e^{zVf} = e^q. \tag{8.181}$$

Now,

$$P = kT\frac{\partial}{\partial V}\ln\mathcal{Q} = zkTf, \tag{8.182}$$

also,

$$N = z\frac{\partial}{\partial z}q = zVf. \tag{8.183}$$

Thus, combining (8.182) and (8.183) one get

$$\boxed{P = \frac{NkT}{V}.} \tag{8.184}$$

By using the equations

$$U = -\frac{\partial}{\partial\beta}q, \tag{8.185}$$

$$F = -kT\ln\frac{\mathcal{Q}}{z^N}, \tag{8.186}$$

$$S = \frac{U-F}{T} = kT\left(\frac{\partial q}{\partial T}\right)_{z,V} - Nk\ln z + kq, \tag{8.187}$$

one finally gets

$$\boxed{U = \frac{3}{2}NkT,} \tag{8.188}$$

$$\boxed{F = -NkT + NkT\ln\left[\frac{N}{V}\left(\frac{h^2}{2m\pi kT}\right)^{3/2}\right],} \tag{8.189}$$

and

$$\boxed{S = Nk\left\{\frac{5}{2} + \ln\left[\frac{V}{N}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\right]\right\}.} \tag{8.190}$$

8.29 Problem H. 7.2

Solution

8.30 Problem H. 7.5

Solution

Since

$$f(\vec{\mu}) = \frac{Ne^{-\beta\vec{\mu}\cdot\vec{H}}}{\int d^3\mu e^{-\beta\vec{\mu}\cdot\vec{H}}} = -\frac{Ne^{-\beta\mu H\cos\theta}}{4\pi\frac{\mu H}{kT}\sinh\left(\frac{\mu H}{kT}\right)}, \tag{8.191}$$

and so

$$\langle\mu_z\rangle = \frac{1}{N}\int f(\vec{\mu})\mu_z d^3\mu = \frac{\theta\cosh\theta - \sinh\theta}{\mu\theta\sinh\theta}. \tag{8.192}$$

From it,

$$M = N\langle\mu_z\rangle = \frac{NkT}{H}\left[\theta\coth\theta - 1\right] = N\mu\left[\coth\theta\frac{1}{\theta}\right], \tag{8.193}$$

so,

$$\boxed{M = N\mu\left[\coth\theta\frac{1}{\theta}\right].} \tag{8.194}$$

The asymptotic behavior are

$\theta \gg 1$	$M \sim \frac{NkT}{H}(\theta - 1)$
$\theta \ll 1$	$M \sim \frac{NkT}{H}\left[\frac{1}{3}\theta^3\right]$

By definition,

$$\chi = \frac{1}{N}, \tag{8.195}$$

and

$$\left(\frac{\partial M}{\partial H}\right)_{T,N} = N\mu\left\{-\frac{\mu}{kT}csch^2\theta + \frac{kT}{\mu H^2}\right\}, \tag{8.196}$$

then

$$\boxed{\chi = \frac{\mu^2}{kT}\left(\frac{1}{\theta^2} - csch^2\theta\right).} \tag{8.197}$$

Curie's law for $T \gg 1$ (or $\theta \ll 1$), yields

$$\chi \longrightarrow \frac{1}{3}\frac{\mu}{H}\left(\frac{\mu H}{kT}\right) = \frac{\mu^2}{3k}\frac{1}{T}, \tag{8.198}$$

thus,

$$\boxed{\chi \propto T^{-1},} \tag{8.199}$$

with a porportional constant $\frac{\mu^2}{3k}$.

8.31 Problem H. 8.2

Derivethe equations of state (8.64) and (8.71) of the book.

Solution

Since

$$\begin{aligned}
\mathcal{Q}(z, T, V) &= \sum_{N=0}^{\infty} z^N Z_N(V, T) \\
&= \sum_{N=0}^{\infty} \sum_{\{n_p\}}' z^N e^{-\beta \sum_p \epsilon_p n_p} \\
&= \sum_{N=0}^{\infty} \sum_{\{n_p\}}' \prod_{\vec{p}} \left(ze^{-\beta\epsilon_p}\right)^{n_p} \\
&= \prod_{\vec{p}} \left[\sum_{\{n_p\}} \left(ze^{-\beta\epsilon_p}\right)^{n_p}\right].
\end{aligned} \tag{8.200}$$

For Bose-Einstein statistics, $n_p = 0, 1, 2, ...$, so,

$$\sum_{\{n_p\}} \left(ze^{-\beta\epsilon_p}\right)^{n_p} = \frac{1}{1 - ze^{-\beta\epsilon_p}}. \tag{8.201}$$

For Fermi-Dirac statistics, $n_p = 0, 1$, so,

$$\sum_{\{n_p\}} \left(ze^{-\beta\epsilon_p}\right)^{n_p} = 1 + ze^{-\beta\epsilon_p}. \tag{8.202}$$

In general,

$$\mathcal{Q} = \prod_{\vec{p}} \left(1 \mp ze^{-\beta\epsilon_p}\right)^{\mp 1}, \tag{8.203}$$

with the upper index rules BE. statistics and the lower index FD's one.

The eqs. of state are

$$\frac{PV}{kT} = \ln \mathcal{Q} = \mp \sum_{\vec{p}} \ln\left(1 \mp ze^{-\beta\epsilon_p}\right). \tag{8.204}$$

Now,

$$N = z\frac{\partial}{\partial z} \ln \mathcal{Q} = \sum_{\vec{p}} \frac{ze^{-\beta\epsilon_p}}{1 \mp ze^{-\beta\epsilon_p}}, \tag{8.205}$$

also,

$$\left\langle n_p \right\rangle = -\frac{1}{\beta}\frac{\partial}{\partial\epsilon_p} \ln \mathcal{Q} = \frac{ze^{-\beta\epsilon_p}}{1 \mp ze^{-\beta\epsilon_p}}. \tag{8.206}$$

If one quantize on a box and after that takes the limit of $V \to \infty$, the corresponding sums converts on integrals, via

$$\sum_{\vec{p}} \longrightarrow \frac{V}{h^3} \int d^3 p. \tag{8.207}$$

Hence, (8.204) can be written like

$$\begin{aligned} \frac{PV}{kT} &= \mp\frac{V}{h^3} \int d^3 p \ln\left(1 \mp ze^{-\beta\epsilon_p}\right) \\ &= \mp\frac{V}{h^3} \int d^3 p \ln\left(1 \mp ze^{-\frac{\beta}{2m}\vec{p}^2}\right) \\ &= \mp\frac{4\pi V}{h^3} \int_0^\infty dp\ p^2 \ln\left(1 \mp ze^{-\frac{\beta}{2m}p^2}\right). \end{aligned} \tag{8.208}$$

Remember that

$$\frac{1}{v} = \frac{N}{V}. \tag{8.209}$$

8.31.1 Ideal Fermi-Dirac Gas

From (8.208),

$$\boxed{\frac{P}{kT} = \frac{4\pi}{h^3} \int_0^\infty dp\ p^2 \ln\left(1 + ze^{-\frac{\beta}{2m}p^2}\right),} \tag{8.210}$$

and from (8.205)

$$N = \frac{4\pi V}{h^3}\int_0^\infty dp\; p^2 \frac{ze^{-\frac{\beta}{2m}p^2}}{1+ze^{-\frac{\beta}{2m}p^2}}, \tag{8.211}$$

or in terms of v,

$$\boxed{\frac{1}{v} = \frac{4\pi}{h^3}\int_0^\infty dp\; p^2 \frac{ze^{-\frac{\beta}{2m}p^2}}{1+ze^{-\frac{\beta}{2m}p^2}}.} \tag{8.212}$$

8.31.2 Ideal Bose-Einstein Gas

The difference of this gases is that the contribution of $p \to \infty$ could be as important as the rest of the momenta, so it must be treated separately.

From (8.208),

$$\boxed{\frac{P}{kT} = -\frac{4\pi}{h^3}\int_0^\infty dp\; p^2 \ln\left(1 - ze^{-\frac{\beta}{2m}p^2}\right) - \frac{1}{V}\ln(1-z),} \tag{8.213}$$

and from (8.205)

$$N = \frac{4\pi V}{h^3}\int_0^\infty dp\; p^2 \frac{1}{z^{-1}e^{\frac{\beta}{2m}p^2}-1} + \frac{z}{1-z}, \tag{8.214}$$

or in terms of v,

$$\boxed{\frac{1}{v} = \frac{4\pi}{h^3}\int_0^\infty dp\; p^2 \frac{1}{z^{-1}e^{\frac{\beta}{2m}p^2}-1} + \frac{1}{V}\frac{z}{1-z}.} \tag{8.215}$$

8.32 Problem H. 8.3

Prove (7.14) in quantum statistical mechanics.

Solution

In quantum mechanics, its known that

$$\langle O \rangle = \frac{Tr(Oe^{-\beta H})}{Tr(e^{-\beta H})}. \tag{8.216}$$

So, for the Hamiltonian, one have

$$\langle H \rangle = \frac{Tr(He^{-\beta H})}{Tr(e^{-\beta H})}, \tag{8.217}$$

which can be written as

$$\langle H \rangle = \frac{\partial}{\partial \beta} \ln Tr(e^{-\beta H}). \tag{8.218}$$

By deriving (8.218), one gets

$$\frac{\partial}{\partial \beta} \langle H \rangle = \frac{Tr(-H^2 e^{-\beta H})}{Tr(e^{-\beta H})} - \frac{Tr(He^{-\beta H})}{Tr(e^{-\beta H})} Tr(-He^{-\beta H}), \tag{8.219}$$

which, by the definiton (8.216) is nothing but

$$\boxed{-\frac{\partial}{\partial \beta} \langle H \rangle = \left\langle H^2 \right\rangle - \langle H \rangle^2 .} \tag{8.220}$$

8.33 Problem H. 8.4

Verify (8.49) for Fermi and Bose statistics, i.e., the fluctuations of cell occupations are small.

Solution

Since,

$$\langle n_k \rangle = -\frac{1}{\beta} \frac{\partial}{\partial \epsilon_k} \ln Q, \tag{8.221}$$

Differentiating this with respect to ϵ_k leads to

$$\left\langle n_k^2 \right\rangle - \langle n_k \rangle^2 = -\frac{1}{\beta} \frac{\partial}{\partial \epsilon_k} \langle n_k \rangle , \tag{8.222}$$

from which one can deduce

$$\left\langle n_k^2 \right\rangle - \langle n_k \rangle^2 = \langle n_k \rangle \pm \langle n_k \rangle^2 , \tag{8.223}$$

with the plus sign for Bose-Einstein statistics and the minus sign for Fermi-Dirac statistics. The fluctuations are not necessarily small. Note, however, that (8.223) refers to the fluctuations of the occupation of individual states, and not the cell occupations.

Note,

$$\left\langle n_k n_p \right\rangle - \langle n_k \rangle \left\langle n_p \right\rangle = -\frac{1}{\beta} \frac{\partial}{\partial \epsilon_k} \left\langle n_p \right\rangle . \quad (p \neq k) \tag{8.224}$$

The RHS. vanish 'cause it depends only on ϵ_p. Thus one gets

$$\left\langle n_k n_p \right\rangle = \langle n_k \rangle \left\langle n_p \right\rangle . \quad (p \neq k) \tag{8.225}$$

In the infinite volume limit the spectrum of states becomes a continuum. The physically interesting question concerns the fluctuations in the occupations of a group of states, or a cell.

Let

$$n^{(i)} = \sum_k n_k^{(i)}, \tag{8.226}$$

where $n_k^{(i)}$ is the k-th state in the (i)-th cell. One is interested in

$$\left\langle \left(n^{(i)}\right)^2 \right\rangle - \left\langle n^{(i)} \right\rangle^2 = \left\langle \left(\sum_k n_k^{(i)} \right)^2 \right\rangle - \left\langle \sum_k n_k^{(i)} \right\rangle^2 . \tag{8.227}$$

By using (8.225), it is easy shown that the RHS. is equal to $\sum_k \left(\left\langle n_k^2 \right\rangle - \langle n_k \rangle^2 \right)$. Hence, using (8.223) one obtains

$$\left\langle \left(n^{(i)}\right)^2 \right\rangle - \left\langle n^{(i)} \right\rangle^2 = \left\langle n^{(i)} \right\rangle \pm \sum_k \left\langle n_k^{(i)} \right\rangle^2 . \tag{8.228}$$

In the infinite volume limit, the k sum is replaced by an integral over a region in k space. No matter how small this region is, the integral is proportional to the volume V of the system.

Thus ended the proof,

$$\boxed{\left\langle \left(n^{(i)}\right)^2 \right\rangle - \left\langle n^{(i)} \right\rangle^2 << \left\langle n^{(i)} \right\rangle^2 .} \tag{8.229}$$

Chapter 9

Special Topics on Statistical Mechanics

9.1 Problem H. 3.1

Solution

In SI units, $\hbar \sim 10^{-34} J.s$, $m \sim 10^{-26} Kg$, $N \sim 10^{23}$, $V \sim 10^{-2} m^3$, $T \sim 300K$ and $k \sim 10^{-23} J/K$.

So, at normal conditions, the parameter

$$\frac{\hbar}{\sqrt{2mkT}}\left(\frac{N}{V}\right)^{1/3} \sim \frac{10^{-34}}{10^{-25}} 10^{8} \sim 10^{-1} \ll 1. \tag{9.1}$$

Therefore, for light gases, whoso mass is $m \sim 10^{-26} Kg$ in their molecular form, the approximation is valid. Moreover, for heavy gases, the approximation becomes even better, because the dependence is inverse to the square root of the mass.

$$\boxed{\frac{\hbar}{\sqrt{2mkT}}\left(\frac{N}{V}\right)^{1/3} \ll 1.} \tag{9.2}$$

9.2 Problem H. 3.2

Solution

At atomic level (not nuclear), the only two elementary interactions are either electromagnetic or gravitational. So, for atomic distances, $m \sim 10^{-26} Kg$, $e \sim 10^{-19} C$, $\frac{1}{4\pi\epsilon_0} \sim 10^9$, $G_N \sim 10^{-11}$ and $r \sim 10^{-10} m$, then

$$F_g \sim \frac{10^{-11} 10^{-52}}{10^{-20}} N = 10^{-43} N, \tag{9.3}$$

$$F_{em} \sim \frac{10^{9} 10^{-38}}{10^{-20}} N = 10^{-9} N. \tag{9.4}$$

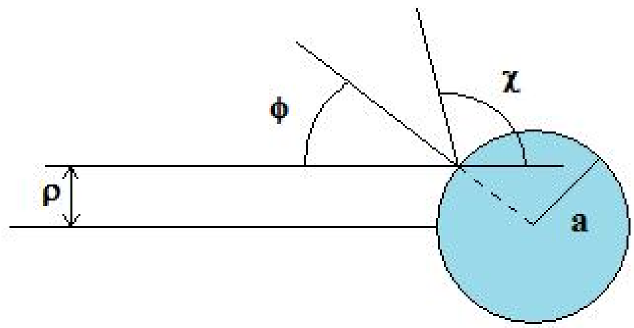


Figure 9.1: Classical Scattering

Then, the only real contribution comes from the electromagnetic interaction.

$$\boxed{F_g \ll F_{em}.} \tag{9.5}$$

9.3 Problem H. 3.3

Solution

Classically

One must relate the impact parameter ρ with the angle of scattering ξ. For a hard sphere,

$$\rho = a \sin \varphi_0 = a \sin \frac{\pi - \xi}{2} = a \cos \frac{\xi}{2}. \tag{9.6}$$

Since,

$$d\sigma = 2\pi\rho(\xi) \left| \frac{d\rho}{d\xi} \right| d\xi, \quad d\Omega = 2\pi \sin \xi d\xi, \tag{9.7}$$

it follows that the differential cross section is given by,

$$\frac{d\sigma}{d\Omega} = \frac{\rho(\xi)}{\sin \xi} \left| \frac{d\rho}{d\xi} \right|, \tag{9.8}$$

therefore,

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{a^2}{4}.} \tag{9.9}$$

and

$$\boxed{\sigma_{cl} = \pi a^2.} \tag{9.10}$$

Note that it coincides with the 2-dimensional area of the sphere, it means, the size of the sphere the other particle can see.

Quantum

A hard sphere is a scattering potential described by

$$V(r) = \begin{cases} \infty; r < R \\ 0; r > R \end{cases} . \tag{9.11}$$

Thus,

$$A_l(r)|_{r=R} = 0 \Longrightarrow j_l(kR)\cos\delta_l = n_l(kR)\sin\delta_l, \tag{9.12}$$

or

$$\tan\delta_l = \frac{j_l(kR)}{n_l(kR)}. \tag{9.13}$$

Consider just $l = 0$, which represents an S-wave scattering, so

$$\tan\delta_0 = \frac{\sin(kR)/kR}{-\cos(kR)/kR} = -\tan kR, \tag{9.14}$$

or

$$\delta_0 = -kR. \tag{9.15}$$

The radial wave function (with the $e^{\imath\delta}$ omitted),

$$A_{l=0}(r) \propto \frac{\sin kr}{kr}\cos\delta_0 + \frac{\cos(kr)}{kr}\sin\delta_0 = \frac{1}{kr}\sin(kr+\delta_0). \tag{9.16}$$

Low energy limit

For $kR \ll 1$, the asymptotic expansions of the spherical functions are

$$j_l(kr) \simeq \frac{(kr)^l}{(2l+1)!!}, \tag{9.17}$$

$$n_l(kr) \simeq -\frac{(2l-1)!!}{(kr)^{l+1}}. \tag{9.18}$$

Then,

$$\tan\delta_l \simeq -\frac{(kR)^{2l+1}}{(2l+1)[(2l-1)!!]^2}, \tag{9.19}$$

and finally,

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\sin^2\delta_0}{k^2} \simeq R^2,} \tag{9.20}$$

and

$$\boxed{\sigma_{nr} = 4\pi R^2.} \tag{9.21}$$

Ultra-relativistic limit

In this limit, $l_{max} \simeq kR$. Then,

$$\sigma_{ur} \simeq \frac{4\pi}{k^2} \sum_{l=0}^{l \simeq kR} (2l+1) \sin^2 \delta_l, \tag{9.22}$$

but

$$in^2 \delta_l = \frac{\tan^2 \delta_l}{1 + \tan^2 \delta_l} = \frac{j_l^2(kR)}{j_l^2(kR) + n_l^2(kR)} \approx \sin^2 \left(kR - \frac{\pi l}{2} \right), \tag{9.23}$$

because,

$$j_l(kR) \sim \frac{1}{kR} in \left(kR - \frac{l\pi}{2} \right)$$

and

$$n_l(kR) \sim -\frac{1}{kR} \cos \left(kR - \frac{l\pi}{2} \right).$$

Finally,

$$\boxed{\sigma_{ur} = \frac{4\pi}{k^2} (kR)^2 \frac{1}{2} = 2\pi R^2.} \tag{9.24}$$

9.4 Problem H. 4.1

Describe an experimental method for the verification of the Maxwell-Boltzmann distribution.

Solution

A plausible experiment is the mean distribution of velocities of a gas expelled from a volume through a small hole.

The number of atoms per unit volume moving in the normal direction to the wall is

$$dn = n f(v) v^2 dv \sin \theta d\theta d\varphi, \tag{9.25}$$

where n is the number density and f is the speed distribution.

So, for The total atomic flow rate, R, through the hole is given by

$$R = 2\pi n A \int_0^\infty f(v) v^3 dv \int_0^{\pi/2} \sin \theta \cos \theta d\theta = \pi A n \int_0^\infty f(v) v^3 dv, \tag{9.26}$$

or

$$R = \frac{An}{4} \langle v \rangle . \tag{9.27}$$

This is highly dependent on $f(v)$, so one can check the Maxwell-Boltzmann distribution.

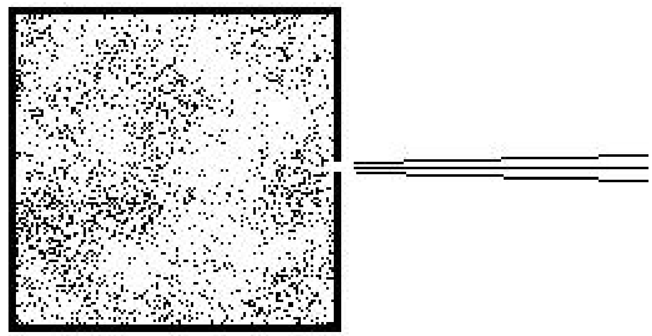


Figure 9.2: Gas enclosed on a box is escaping very slowly, so the process is almost in equilibrium.

NOTE: This is indeed the way neutrons (or other neutral particles) are accelerated, because there is not possibility of applying electric fields.

9.5 Problem H. 4.5

Estimate the probability that a 7 cents airmail stamp (mass=0.1 g) resting on a desk top at room temperature (300 K) will spontaneously fly up to a height of $10^{-8} cm$ above the desktop.

Solution

So, one can use the canonical ensemble, i.e.,

$$P_r = \frac{e^{-\beta\epsilon_r}}{\sum_s e^{-\beta\epsilon_s}}. \tag{9.28}$$

The energy involved is just the gravitational potential energy, mgh. In CGS units, $k = 1.38 * 10^{-16} erg/K$, the gravitational acceleration is $g = 10^3 cm/s^2$, so an estimation is

$$P_r \approx e^{-\frac{mgh}{kT}} \sim e^{-\frac{10^{-1}10^{3}10^{-8}}{10^{-16}10^{2}}} = e^{-10^8}. \tag{9.29}$$

In this thick approximation, the exponent is so huge that changing the base from e to 10 is allowed, thus

$$\boxed{P_r \sim 10^{-10^8}.} \tag{9.30}$$

This is barely non zero.

9.6 Problem H. 4.6

A room of volume $3 \times 3 \times 3$ cubic meters is under standard conditions

1. Estimate the probability that at any instant of time a $1cc$ volume anywhere within this room becomes totally devoid of air because of spontaneously statistical fluctuations.
2. Estimate the same for $1\dot{A}^3$ volume.

Solution

Once more, one can use the canonical ensemble, but since p and T are given, one can use the ideal gas equation of states and rewrite the probability in terms of V,

$$P_v \sim e^{-N\frac{v}{V}}. \tag{9.31}$$

In the room the number of particles is of order $N \simeq 10^{26}$, the volumes are $v_a = 1cm^3$, $v_b = 1\dot{A}^3 = (10^{-8}cm)^3 = 10^{-24}cm^3$ and $V = 27 * 10^6 cm^3$, then

$$\boxed{P_a \sim e^{-\frac{10^{26}}{10^6}} = e^{-10^{20}},} \tag{9.32}$$

whilst

$$\boxed{P_b \sim e^{-\frac{10^{26}10^{-24}}{10^6}} = e^{-10^{-8}}.} \tag{9.33}$$

From (9.32) one conclude that the probability is barely different than zero, but (9.33) is relatively 1.

9.7 Problem H. 4.7

Suppose the situation referred to in Problem 9.6 first part has occurred. Describe qualitatively the behavior of the distribution function thereafter. Estimate the time it takes for the situation to occur again, under the assumption that molecular collisions are such that time sequence of the state of the system is a random sequence of states.

Solution

After the state in which a cubic centimeter has been devoid, the gas will suffer a free expansion and the vacuum will be occupied.

Since the average time of collisions is of order $\Delta t \sim 10^{-11}s$, then the time in which the situation will be repeated is given by

$$\boxed{t = \frac{\Delta t}{P_a} \sim 10^{-11}s * e^{10^{20}} \sim 10^{10^{20}}s.} \tag{9.34}$$

NOTE: The larger estimated age of the universe is 20 billion year ($t_u \sim 10^{20}s$, but $t \gg t_u$. This means that if the situation is reached, the probability of seeing it again could be considered zero.

9.8 Problem H. 4.9

Let

$$H = \int d^3v f(\vec{v}, t) \ln f(\vec{v}, t), \tag{9.35}$$

where $f(\vec{v}, t)$ is arbitrary except for the conditions

$$\int d^3v f(\vec{v}, t) = n \quad \text{and} \quad \int d^3v \frac{1}{2} m v^2 f(\vec{v}, t) = \epsilon. \tag{9.36}$$

Show that H is minimum when f is the Maxwell-Boltzmann distribution.

Solution

In order to extremize the functional H constrained to (9.36), one can use Lagrange multipliers. Then,

$$H_T = \int d^3v \left[f \ln f + \alpha f + \frac{\beta m v^2}{2} f + cont. \right], \tag{9.37}$$

whose variation yields

$$(\delta f) \ln f + f \delta \ln f + \alpha \delta f + \frac{\beta m}{2} \delta(v^2) f + \frac{\beta m v^2}{2} \delta f = 0. \tag{9.38}$$

Since

$$\delta f = \frac{\partial f}{\partial \vec{v}} \cdot \delta \vec{v}, \tag{9.39}$$

then

$$\left[\frac{\partial f}{\partial \vec{v}} \ln f + \frac{\partial f}{\partial \vec{v}} + \alpha \frac{\partial f}{\partial \vec{v}} + \beta m \vec{v} f + \frac{\beta m v^2}{2} \frac{\partial f}{\partial \vec{v}} \right] \cdot \delta \vec{v} = 0. \tag{9.40}$$

But the variation of $\vec{v}$ is arbitrary, and the equation can be written like

$$\vec{\nabla}_v f \left(1 + \alpha + \frac{\beta m v^2}{2} + \ln f \right) = -\beta m \vec{v} f. \tag{9.41}$$

In principle the solution of this equation is known if the term inside the bracket is one. Moreover, the restriction of the bracket is also solution of the complete equation. Thus,

$$\boxed{f(\vec{v}, t) = e^{-\alpha} e^{-\beta \frac{m\vec{v}^2}{2}}.} \tag{9.42}$$

This is nothing but the Maxwell-Boltzmann distribution.

9.9 Problem H. 5.1

Make order-of-magnitude estimates for the mean free path and the collision time for

a.- H_2 molecules in standard conditions (diameter $H_2 = 2.9\dot{A}$)

b.- A proton in a plasma at $T = 3 * 10^5 K$, $n = 10^{15}$ particles/cc, $\sigma = \pi r^2$, where $r = \frac{e^2}{kT}$.

c.- A proton as before but $T = 10^7 K$, where thermonuclear reaction occur.

d.- A proton in the sun's corona, $T = 10^6 K$ and $n = 10^6$ protons/cc.

e.- slow neutrons with $E = 0.5 MeV$ in U^{238} ($\sigma \approx \pi r^2$, $r \approx 10^{-13} cm$)

Solution

The mean-free path is given by

$$\lambda = \frac{1}{4}\sqrt{\frac{\pi}{2}}\frac{1}{n\sigma}, \tag{9.43}$$

and the collision time is

$$\tau = \frac{1}{4}\sqrt{\frac{\pi}{2}}\frac{1}{n\sigma\bar{v}}. \tag{9.44}$$

Then, the task is finding the values of σs, ns and $\bar{v}$s.

In CGS units,

$$\frac{e^2}{kT} = \frac{1.5 * 10^{-7}}{T} cm, \tag{9.45}$$

Then,

$$\begin{aligned} r_a &\sim 10^{-8} cm, \\ r_b &\sim 10^{-12} cm, \\ r_c &\sim 10^{-14} cm, \\ r_d &\sim 10^{-12} cm, \\ r_e &\sim 10^{-13} cm. \end{aligned} \tag{9.46}$$

So, σs are given by (πr^2)

$$\begin{aligned} \sigma_a &\sim 10^{-15} cm^2, \\ \sigma_b &\sim 10^{-23} cm^2, \\ \sigma_c &\sim 10^{-27} cm^2, \\ \sigma_d &\sim 10^{-23} cm^2, \\ \sigma_e &\sim 10^{-25} cm^2. \end{aligned} \tag{9.47}$$

Since $\bar{v} = \sqrt{\dfrac{2kT}{m}} = 4.19 * 10^7 \left(\dfrac{m_e}{m_i}\right)^{1/2} T^{1/2} cm/s$, it follows that,

$$\begin{aligned} \bar{v}_a &\sim 10^7 cm/s, \\ \bar{v}_b &\sim 10^8 cm/s, \\ \bar{v}_c &\sim 10^9 cm/s, \\ \bar{v}_d &\sim 10^8 cm/s, \\ \bar{v}_e &\sim 10^9 cm/s. \end{aligned} \tag{9.48}$$

Thus,

$$\begin{aligned} \lambda_a &\sim 10^{-16} cm, \\ \lambda_b &\sim 10^8 cm, \\ \lambda_c &\sim 10^{12} cm, \\ \lambda_d &\sim 10^{19} cm, \\ \lambda_e &\sim 10^{-18} cm, \end{aligned} \tag{9.49}$$

and

$$\begin{aligned} \tau_a &\sim 10^{-23} s, \\ \tau_b &\sim 10^0 s, \\ \tau_c &\sim 10^3 s, \\ \tau_d &\sim 10^{11} s, \\ \tau_e &\sim 10^{-27} s, \end{aligned} \tag{9.50}$$

9.10 Problem H. 5.4

Show that the velocity of sound in a real substance is to a good approximation given by $c_s = 1/\sqrt{\rho\kappa_s}$, where ρ is the mass density and $\kappa_s$is the adiabatic compressibility, by the following steps:

(a) Show that in a sound wave the density oscillates adiabatically if $K \ll c_s \lambda \rho C_V$, where

$c_s =$	velocity of sound
$\lambda =$	wave length
$\rho =$	mass density
$C_V =$	specific heat
$K =$	Coefficient of thermal conductivity.

(b) Show by numerical examples, that the criterion stated in (a) is well satisfied in most practical situations.

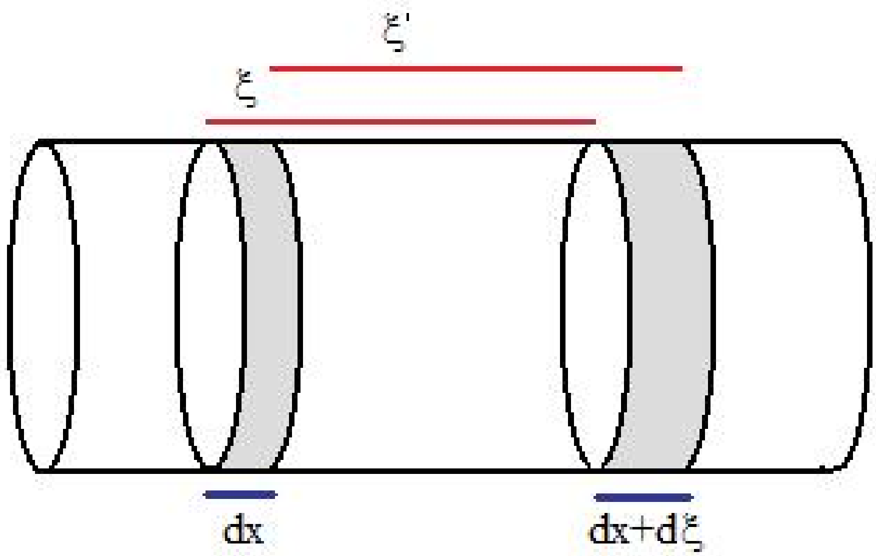


Figure 9.3: Propagation of a sound wave through a tube

Solution

Sound waves cover the following three characteristics

1. Gas moves and varies its density.
2. variation of density correspond to variations of pressure.
3. Differences of pressure generate the movement.

Consider a column of gas in a tube. Let p_0 and ρ_0 be the pressure and density of the gas in its equilibrium. If the pressure varies the elemental volume, Adx, moves. It is displaced a distance ξ, but due to the change of volume the new volume is $A(dx + d\xi)$.

Mass conservation requires that,

$$\rho A(dx + d\xi) = \rho_0 A dx \quad \text{or} \quad \rho = \frac{\rho_0}{1 + \frac{\partial \xi}{\partial x}}. \tag{9.51}$$

In general $\dfrac{\partial \xi}{\partial x}$ is small, so one can use the expansion of $\dfrac{1}{1 + \frac{\partial \xi}{\partial x}}$. Also, $p = f(\rho)$, then up to first order,

$$\rho - \rho_0 = -\rho_0 \left(\frac{\partial \xi}{\partial x} \right) \tag{9.52}$$

and

$$p = p_0 + (\rho - \rho_0)\left(\frac{dp}{d\rho}\right)_0 + \cdots = p_0 - \rho_0\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{dp}{d\rho}\right)_0. \tag{9.53}$$

Further, the equation of motion requires the acceleration $\dfrac{\partial^2 \xi}{\partial t^2}$,so a little algebra yields

$$\frac{\partial^2 \xi}{\partial t^2} = \left(\frac{dp}{d\rho}\right)_0 \frac{\partial^2 \xi}{\partial x^2}, \tag{9.54}$$

then

$$c_s^2 = \left(\frac{dp}{d\rho}\right)_0. \tag{9.55}$$

Now, the coefficient of thermal conductivity is defined as the flux energy through unit area divided by the gravient of temperature, so

$$K = \frac{1}{A}\frac{dE}{dt}\frac{dz}{dT} = \frac{1}{A}\frac{dE}{dT}\frac{dz}{dt} \simeq C_V c_s \rho \lambda, \tag{9.56}$$

then for an adiabatic process, one must requires

$$K \ll C_V \lambda \rho c_s, \tag{9.57}$$

because the transference of heat should be zero. In this case,

$$pV^\gamma = cte, \tag{9.58}$$

and, for an ideal gas,

$$\kappa_S = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_S = -\frac{1}{V}\left(\frac{1}{\frac{\partial p}{\partial V}}\right)_S = \frac{1}{\gamma p}. \tag{9.59}$$

Moreover, the adiabatic relation of p and ρ is $p = cte\rho^\gamma$, thus

$$c_s^2 = \frac{\gamma p}{\rho} = \frac{1}{\kappa_S}, \tag{9.60}$$

finally,

$$\boxed{c_s = \sqrt{\frac{1}{\rho \kappa_S}}.} \tag{9.61}$$

Since $C_V \sim 1.4$, $c_s \sim 3 * 10^4 cm/s$, $\lambda \in [1.5, 1500]$ and $\rho \geq 1$, then in all cases showed in the table, $K \ll C_V c_s \rho \lambda$ is satisfied.

Gas	Temperature (^oC)				
	-100	-50	0	20	100
Air (dry)	3.9	4.9	5.76	6.1	7.4
O_2	3.9	4.9	5.8	6.2	7.6
He	24.6	29.6	34.3	36.1	40.8
H_2	21.8	35.0	41.9	44.5	54.2
CO_2			3.4		

Table 9.1: Thermal conductivity $(cal/cm^2 - s)/(C/cm))$

9.11 Problem H. 9.2

(a) Find hte equations of state for an ideal Bose gas and an ideal Fermi gas in the limit of high temperatures. Include the first correction due to quantum effects.

(b) Estimate, for each of the following ideal gases, the temperature below which quantum effects would become important: H_2, He, N_2.

Solution

For the weak degenerated case, for a Bose gas $\mu(< 0)$ is noot near to zero and for the Fermi gas $\mu < 0$. One can calculate

$$pV = kT \ln \Xi, \quad N = kT \frac{\partial \ln \Xi}{\partial \mu} = \prod_i \left(1 \mp e^{\beta(\mu-\epsilon)}\right)^{\mp 1} .. \tag{9.62}$$

Here on the upper sign is for Bose gas and the lower one is for Fermi gas.

By expanding in terms of $e^{\beta(\mu-\epsilon)}$:

$$pV = \mp kT \sum_i \ln\left(1 \mp e^{\beta(\mu-\epsilon)}\right) = kT \sum_{n=1}^{\infty} (\pm 1)^{n-1} \frac{\lambda^n}{n} C_n, \tag{9.63}$$

$$N = \sum_{n=1}^{\infty} (\pm 1)^{n-1} \lambda^n C_n, \tag{9.64}$$

with

$$C_n = \sum_i e^{-n\beta\epsilon_i} = 2\pi g V \left(\frac{2m}{h^2}\right)^{3/2} \int_0^\infty d\epsilon \frac{\epsilon^{1/2}}{e^{n\beta\epsilon}} = \frac{gV}{\lambda_T^3 n^{3/2}}, \tag{9.65}$$

where g is the weight for the internal degrees of freedom and $\lambda_T = h/\sqrt{2\pi mkT}$. From (9.64) and (9.65),one obtains

$$N = \frac{gV}{\lambda_T^3} \sum_{n=1}^{\infty} (\pm 1)^{n-1} \frac{\lambda^n}{n^{3/2}}. \tag{9.66}$$

Assuming the power series in $x = N\lambda_T^3/gV$ for $\lambda = a_1 x + a_2 x^2 + a_3 x^3 + \cdots$, one can determine the coefficients successively from (9.66),

$$a_1 = 1, \quad a_2 = \mp\frac{1}{2^{3/2}}, \quad a_3 = \frac{1}{4} - \left(\frac{1}{3}\right)^{3/2}, \quad \cdots \tag{9.67}$$

Substituting this values of λ into (9.63), one get

$$\boxed{\frac{pv}{kT} = 1 - \frac{1}{4\sqrt{2}}\left(\frac{\lambda_T^3}{v}\right) + \left(\frac{1}{8} - \frac{2}{9\sqrt{3}}\right)\left(\frac{\lambda_T^3}{v}\right)^2 - \cdots.} \tag{9.68}$$

9.12 Problem H. 9.3

Pair correlation function: The pair correlation function $D(\vec{r}_1, \vec{r}_2)$ of a system of particles is defined as the probability of simultaneously finding a particle in the volume $d\vec{r}_1$ about $\vec{r}_1$ and a particle in the volume $d\vec{r}_2$ about $\vec{r}_2$.

Calculate $D(\vec{r}_1, \vec{r}_2)$ for an ideal Bose gas and an ideal Fermi gas in the limit of High temperatures. Include quantum corrections only to the lowest approximation.

Solution

Classically one has,

$$D(\vec{r}_1, \vec{r}_2) = \frac{N(N-1)\int d^{3N}p d^3 r_3 \cdots d^3 r_N e^{-\beta H(p,r)}}{\int d^{3N}p d^{3N} r e^{-\beta H(p,r)}}. \tag{9.69}$$

In order to include the lowest quantum corrections, one can replace the free Hamiltonian by

$$H(p,r) = \sum_i \frac{p_i^2}{2m} + \sum_{i<j} \tilde{v}_{ij}. \tag{9.70}$$

Assume that the density the gas is almost zero. The limit $N \to \infty$, $V \to \infty$ should be so taken that $N/V \to 0$. Then,

$$\begin{aligned} D(\vec{r}_1, \vec{r}_2) &= \frac{N(N-1)V^{N-2}\left[1 \pm {f_{12}}^2 \pm \frac{N(N-1)}{2V}\int d^3 f^2(r)\right]}{1 \pm \frac{N(N-1)}{2V}\int d^3 f^2(r)} \\ &\approx \frac{1}{v^2}\left[1 \pm \exp\left(-\frac{2\pi}{\lambda^2}\left|\vec{r}_1 - \vec{r}_2\right|^2\right)\right]. \end{aligned} \tag{9.71}$$

This result continues to hold for finite v with $\lambda^3/v \ll 1$, although the derivation did not justify such a conclusion.

9.13 Problem H. 9.5

Consider the grand partition function

$$Q(z,V) = (1+z)^V(1+z^{\alpha V}), \tag{9.72}$$

where α is a positive constant.

(a) Write down the equation of state in the parametric for, eliminate z graphically, and show that there is a first-order phase transition. Find the specific volumes of the two phases.

(b) Find the roots of $Q(z,V) = 0$ in the complex z plane, at fixed V. Show that as $V \to \infty$the root converge toward the real axis at $z = 1$.

(c) Find the equation of state in the "gas" phase. Show that a continuation of this equation beyond the phase transition density fails to show any sign of the transition. This will demonstrate that the order of the operations $z\left(\frac{\partial}{\partial z}\right)$ and $V \to \infty$ can be interchanged only within a single phase region.

Solution

The parametric equation of state are

$$\frac{p}{kT} = \ln(1+z) + \frac{1}{V}\ln(1+z^{\alpha V}), \tag{9.73}$$

$$\frac{1}{v} = \frac{z}{1+z} + \alpha\frac{z^{\alpha V}}{1+z^{\alpha V}}. \tag{9.74}$$

The roots of Q are $z = -1$ which does not belong to the physical system, and $z^{\alpha V} = -1$. This last, so that z is unimodular, can be written as

$$z = e^{\imath\theta} \Rightarrow \quad \imath\theta\alpha V = \pm\pi, \tag{9.75}$$

or

$$\imath\theta = \pm\frac{\pi}{\alpha V}. \tag{9.76}$$

It is clear that in the limit $V \to \infty$, in order to preserve the relation, $\theta \to 0$, thus

$$\boxed{\boxed{z \to 1.}} \tag{9.77}$$

9.14 Problem H. 11.1

Give the numerical estimates for the Fermi energy of

(a) e^- in a typical metal,

(b) nucleons in a heavy nucleus,

(c) He^3 atoms in liquid He^3 (atomic volume = $46.2\dot{A}^3/atom$).

Treat all of the mentioned particles as free particles.

Solution

Since

$$\epsilon_F = \frac{h^2}{2m}\left(\frac{3n}{4\pi(2\sigma+1)}\right)^{2/3}, \tag{9.78}$$

then, for electrons in a metal, whose $n = 5.86 * 10^{28} elect/m^3$, it follows that ($\sigma = 1/2$)

$$\boxed{\boxed{\epsilon_F = 9 * 10^{-19} J \simeq 5.6 eV.}} \tag{9.79}$$

For a nucleon (with energy$E = .5MeV$) on a heavy nucleus, $n \simeq 10^{30} nucleon/cm^3$, $m_n \sim 10^{-30} g$, then

$$\boxed{\boxed{\epsilon_F \sim 10^{-3} erg \sim 10^{-10} J \sim 1GeV.}} \tag{9.80}$$

For He^3, $n \approx 10^{22} atoms/cm^3$, thus

$$\boxed{\boxed{\epsilon_F \sim 10^{-10} erg \sim 10^{-17} J \sim 100 eV.}} \tag{9.81}$$

9.15 Problem H. 11.2

Show that for the ideal gas of N particles the Helmholtz energy at low temperatures is given by

$$\frac{A}{N} = \frac{3}{5}\epsilon_F\left[1 - \frac{5\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2 + \cdots\right]. \tag{9.82}$$

Solution

It was shown that at low temperatures the energy is given by

$$U = \frac{3}{5}\epsilon_F \left[1 + \frac{5\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2 + \cdots\right] = \frac{3}{5}\epsilon_F N. \tag{9.83}$$

Moreover,

$$F = N\mu - pV = N\mu - \frac{2}{3}U = \frac{3}{5}N\epsilon_F \left[1 - \frac{5\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2 + \cdots\right], \tag{9.84}$$

thus,

$$\boxed{\frac{A}{N} = \frac{3}{5}\epsilon_F \left[1 - \frac{5\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2 + \cdots\right].} \tag{9.85}$$

9.16 Problem H. 12.4

Show that the equation of state of the ideal Bose gas in the gas phase can be written in the form of the Virial expansion

$$\frac{pv}{kT} = 1 - \frac{1}{4\sqrt{2}}\left(\frac{\lambda_T^3}{v}\right) + \left(\frac{1}{8} - \frac{2}{9\sqrt{3}}\right)\left(\frac{\lambda_T^3}{v}\right)^2 - \cdots \tag{9.86}$$

Solution

See (9.68)

9.17 Problem H. 14.4

Consider a square lattice of Ising spins in any dimension, with energy given by

$$E_i\{s_i\} = -\epsilon \sum_{<ij>} s_i s_j - H \sum_{i=1}^{N} s_i. \tag{9.87}$$

Show that in the absence of an external field ($H = 0$), the free energy at a given temperature is the same for the ferromagnetic case ($\epsilon > 0$) and the anti-ferromagnetic case ($\epsilon < 0$).

Solution

Consider any lattice site designate it as a site of sub-lattice A. Designate it nearest neighbors as sites belonging to sub-lattice B. The sub-lattices are completely defined by the rule that the nearest neighbors of any site in sub-lattice B are sites of sub-lattice A, and vice versa. In a square lattice, the nearest neighbors are situated on lattice spacing away, along mutually orthogonal directions. Therefore both A and B are square lattices. Denote the spin variables in sub-lattice A by s_{A_i}, and those in sub-lattice B by s_{B_i}. Only spins in different sub-lattices interact. Thus, the partition function can be written in form

$$Q = \sum_{\{s_A\}} \sum_{\{s_B\}} e^{J \sum_{<ij>} s_{A_i} s_{B_j}}, \tag{9.88}$$

where $J = \beta\epsilon$. Since $\{s_A\}$ is being summed over, the above is invariant under $s_{A_i} \to -s_{A_i}$. Therefore it is invariant under $\epsilon \to -\epsilon$.

NOTE: the proof fails for a triangular lattice.

9.18 Problem H. 14.5

Duality of 2-dimensional Ising model.

(a) Let $Q(N,\beta)$ denote the partition function for a 2-dimensional Ising model of N sites on a square lattice at temperature $kT = \frac{1}{\beta}$ with no external field. Show that in the limit $N \to \infty$

$$\frac{\ln Q(N,\beta)}{N} = \frac{\ln Q(N,\beta^*)}{N} - \sinh(2\beta^*), \tag{9.89}$$

where $\beta^* = -\frac{1}{2}\ln\tanh\beta$, or

$$\sinh 2\beta \sinh 2\beta^* = 1. \tag{9.90}$$

Note that β^* is a decreasing function of β. Thus, the high temperature properties of the system are explicitly related to its low temperature properties.

(b) One know through Peierls argument that the system exhibits spontaneous magnetization. Assuming that the critical temperature T_c is unique, one conclude $\beta_c = \beta_c^*$. Show

$$\beta_c = \frac{1}{2}\ln\left(1 + \sqrt{2}\right). \tag{9.91}$$

This is how the transition temperature of the 2-dimensional Ising model was obtained before Onsager's explicit solution.

Solution

$$
\begin{aligned}
Q &= \sum_s \exp\left(J \sum_{<ij>} s_i s_j\right) \\
&= \sum_s \prod_{<ij>} \exp\left(J s_i s_j\right) \\
&= \sum_s \prod_{<ij>} \left(\cosh(J) + s_i s_j \sinh(J)\right) \\
&= \sum_s \prod_{<ij>} \sum_{k=0}^{1} c_k \left(s_i s_j\right)^k
\end{aligned}
\tag{9.92}
$$

where $J = \beta\epsilon$, $c_0 = \cosh J$ and $c_1 = \sinh J$. Where the identity

$$
\exp(ax) = \cosh x + a \sinh x \tag{9.93}
$$

with $a^2 = 1$, has been used.

Each $< ij >$ corresponds to a link b between the two sites. The last expression associates with each link an integer $k = 0, 1$. Denote the set of k's by $\{k\} = \{k_1, k_2, \cdots\}$, where k_b refers to the b-th link. Note that

$$
\prod_{<ij>} \sum_{k=0}^{1} c_k \left(s_i s_j\right)^k = \sum_k \left(c_{k_1} c_{k_2} \cdots\right) \prod_{<ij>} (s_i s_j)^{k_{ij}}, \tag{9.94}
$$

where $k_{ij} \equiv k_b$, with b referring to the link that joints $< ij >$. To derive this, consider $\prod_{<ij>}(c_0 + c_1 s_i s_j)$, which is a product of F factors (F = number of links). Expand the product in to a sum of terms, each made up of F factors, obtained by choosing either the c_0 or c_1 term.

$$
\prod_{<ij>} (s_i s_j)^{k_{ij}} = \prod_i \left(\prod_j{}' (s_i)^{k_{ij}}\right) = \prod_i (s_i)^{n_i}, \tag{9.95}
$$

$n_i \equiv \sum_{b \supset i} k_0$, and the prime on the product on j means product over sites j that are nearest neighbors of i. The sum $\sum_{b \supset i}$ denotes a sum over all links b that meet at the site i. To derive the first equality, note that both sides represent different ways of writing the same product.

$$
\begin{aligned}
Q &= \sum_{\{s\}} \sum_{\{k\}} (c_{k_1} c_{k_2} \ldots) \prod_i (s_i)^{n_i} \\
&= \sum_{\{k\}} (c_{k_1} c_{k_2} \ldots) \sum_{\{s\}} \left[(s_1)^{n_1} (s_2)^{n_2} \ldots\right] \\
&= \sum_{\{k\}} (c_{k_1} c_{k_2} \ldots) \prod_i \sum_{s=-1}^{1} s^{n_i} \\
&= \sum_{\{k\}} (c_{k_1} c_{k_2} \ldots) \prod_i \left[1 + (-1)^{n_i}\right].
\end{aligned}
\tag{9.96}
$$

Note $[1 + (-1)^{n_i}] = 0$ for n_i odd, 2 for n_i even. Since $\{n_i\}$ depends on $\{k_b\}$, only that subset of $\{k_b\}$ corresponding to all n_i even will contribute:

$$Q = 2^N \sum_{\{k\}}' (c_{k_1} c_{k_2} ...) , \tag{9.97}$$

where the prime on the sum denotes the constraint $\sum_{b \supset i} k_b = 0$ (*mod* 2), for all i.

That is, associated with each link b and integer $k_b = 0, 1$, such that $k_1 + ... + k_4 = 0$ (*mod* 2) whenever the links 1, 2, 3, 4 all meet at one site.

One solve the constraint through a geometrical construction. Define the "dual lattice" as the lattice whose sites are located at centers of each square of the original lattice. In the accompanying sketch, the dual lattice sites are marked with an x, and the links of the dual lattice are shown dotted. each original site i is contained in a square of the dual lattice, which is called a "plaquette". Attach to each site of the dual lattice a dual spin variable, σ_i, which can assume only the values ± 1. If 1, 2, 3, 4 are dual sites at successive corners of a plaquette (say in clock-wise order), then

$$\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_1 = 0 \quad (mod\ 4) \tag{9.98}$$

to see this, note that the left side can be written in the form $(\sigma_1 + \sigma_3)(\sigma_2 + \sigma_4) = 0, \pm 4$. Noe each link b of the original lattice cuts a unique link b^* of the dual lattice. The constraint is solved by taking

$$k_b = \frac{1}{2}(1 - \sigma_1\sigma_2) \tag{9.99}$$

where 1, 2 mark the dual sites at the ends of b^*. This is so because, first, $k_b = 0, 1$ as required, and second whenever 4 links meet at the same site, they cut the sides of the plaquette containing the site. One know in passing that an equally acceptable definition of k_b can be obtained by changing the $- \to +$. This shows that in this case the ferromagnetic model and the anti-ferromagnetic model have the same free energy, in agreement with the general theorem stated in the last problem.

Since there is a one-to-one correspondence between links on the original and on the dual lattice, ona can associated with a dual link b^* the integer $k_{b^*} \equiv k_b$. For large N, the number of dual sites is also N. One can associate each dual site with 2 dual links (say, the once pointing north and east). There are $2N$ integers k_{b^*}, of which N are independent (because they satisfy N constraints). Thus, the partition function is obtained by replacing the constraint sum $\sum'_{\{k\}}$ by the un-constraint sum $\frac{1}{2}\sum_{\{\sigma\}}$:

$$Q = 2^{N-1} \sum_{\{\sigma\}} (c_{k_1} c_{k_2} ...) \tag{9.100}$$

where the c's are to be re expressed in terms of the σ 's.

$$\begin{aligned} c_k &= k \sinh J + (1-k)\cosh J \\ &= \frac{1}{2}(1 - \sigma_1\sigma_2)\sinh J + \frac{1}{2}(1 + \sigma_1\sigma_2)\cosh J \\ &= \frac{1}{2}e^J\left(1 + \sigma_1\sigma_2 e^{-2J}\right) \\ &= \frac{1}{2}e^J(1 + \sigma_1\sigma_2 \tanh J^*) \end{aligned} \tag{9.101}$$

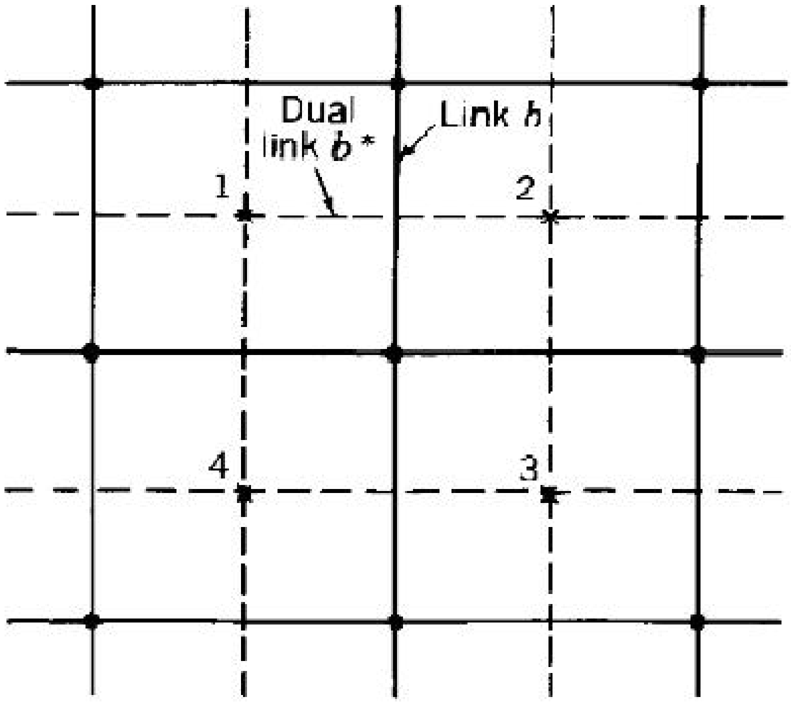


Figure 9.4: Dual lattice

where $J^* \tanh J^* \equiv e^{-2J}$ is defined, with a view to re expressing the above expression as an exponential:

$$\begin{aligned} c_k &= \frac{e^J}{2\cosh J^*}(\cosh J^* + \sigma_1\sigma_2 \sinh J^*) \\ &= [2\sinh(2J)]^{-1/2} \exp(J^*\sigma_1\sigma_2)\,. \end{aligned} \tag{9.102}$$

Thus, finally

$$\boxed{Q = \frac{1}{2}(\sinh 2J^*)^{-N} \sum_{\{\sigma\}} \exp\left(J^* \sum_{<ij>} \sigma_i\sigma_j\right).} \tag{9.103}$$

9.19 Problem G.-C. 6.1

Show that the number of particles in a Fermi gas with celerity between $[v_z, v_z{+}dv_x]$ (or its equivalent in momentum) is

$$n(p_z)dp_z = \left(\frac{2\pi m(2\sigma+1)V}{\beta h^3}\right) \ln\left[\beta\left(\mu - \frac{p_z^2}{2m}\right) - 1\right]. \tag{9.104}$$

Find the limit of this expression when $T \to 0$.

Solution

For a Fermi gas,

$$n(\vec{p}) = \frac{(2\sigma+1)V}{h^3}\frac{1}{e^{-\beta\left(\frac{p^2}{2m}-\mu\right)}+1}, \tag{9.105}$$

then in cylindric coordinates,

$$n(\vec{p})d^3p = \frac{(2\sigma+1)V}{h^3}\frac{p_r dp_r p_z d\theta}{e^{-\beta\left(\frac{p_r^2+p_z^2}{2m}-\mu\right)}+1}. \tag{9.106}$$

Next, one should integrate the angular and radial components of the momentum. By using the integral identity

$$\int \frac{dx}{a+be^{mx}} = \frac{1}{am}\left[mx - \ln(a+be^{mx})\right], \tag{9.107}$$

one gets

$$\boxed{n(p_z)dp_z = \frac{4\pi(2\sigma+1)mV}{\beta h^3}\ln\left[e^{\beta\left(\frac{p_z^2}{2m}-\mu\right)}+1\right]dp_z.} \tag{9.108}$$

Finally, for $T \to 0 \Rightarrow e^{1/T} \to \infty$, $\therefore \ln\left[e^{\beta\left(\frac{p_z^2}{2m}-\mu\right)}+1\right] \to \frac{p_z^2}{2m}-\mu$. Thus,

$$\boxed{n(p_z)dp_z \to \frac{4\pi(2\sigma+1)mV}{h^3}\left(\frac{p_z^2}{2m}-\mu\right)dp_z.} \tag{9.109}$$

9.20 Problem G.-C. 6.5

Show that $e^x/(1+e^x)^2$ is a symmetric function on x, and that it goes to zero as long as $x \to \pm\infty$.

Solution

Under parity transformation,

$$\frac{e^x}{(1+e^x)^2} \mapsto \frac{e^{-x}}{(1+e^{-x})^2} = \frac{e^{-x}}{e^{-2x}(e^x+1)^2} = \frac{e^x}{(1+e^x)^2}. \tag{9.110}$$

Then,

$$\boxed{f(x) = f(-x).} \tag{9.111}$$

Now,

$$f(\pm\infty) = \frac{e^{\pm\infty}}{(1+e^{\pm\infty})^2} = \begin{cases} 0/1 & ;x \to -\infty \\ \frac{e^{+\infty}}{e^{2\infty}} \to 0 & ;x \to +\infty \end{cases} \tag{9.112}$$

Therefore, this equation goes to zero as $x \to \pm\infty$.

9.21 Problem G.-C. 6.16

The state density of electrons in some sample is given by

$$f(\epsilon) = \begin{cases} D = cte & \text{if} \quad \epsilon > 0 \\ 0 & \text{if} \quad \epsilon < 0 \end{cases}$$

Calculate the Fermi potential ϵ_F, the condition for strong degeneration and prove that in this case $C_V \propto T$.

Solution

At $0K$,the energy levels are occupied by electrons up to $\epsilon = \mu_0$. Hence one has $D\mu_0 = N$, thus,

$$\boxed{\boxed{\mu_0 = \frac{N}{D}.}} \tag{9.113}$$

The chemical potential is determined by the condition

$$N = D \int_0^\infty \frac{d\epsilon}{e^{\beta(\epsilon-\mu)} + 1}. \tag{9.114}$$

The condition that guarantees no degeneracy is $e^{-\beta\mu} \gg 1$. When this is satisfied, one has

$$\frac{N}{D} = \int_0^\infty d\epsilon e^{\beta(\mu-\epsilon)} = \frac{e^{-\beta\mu}}{\beta}, \tag{9.115}$$

thus the condition of no-degeneracy becomes

$$\boxed{\boxed{\frac{N}{DkT} \ll 1.}} \tag{9.116}$$

This means that the total number of electrons is very small compared to the number of electrons that can be accommodated in the energy range of width kT. DkT is the number of states in the interval kT.

For $\beta\mu \gg 1$, one has

$$\begin{aligned} N &= D\left[\int_0^\mu d\epsilon - \int_0^\mu d\epsilon \left\{1 - \frac{1}{e^{\beta(\epsilon-\mu)} + 1}\right\} + \int_\mu^\infty \frac{d\epsilon}{e^{\beta(\epsilon-\mu)} + 1}\right] \\ &= D\left[\mu - \int_0^\mu \frac{d\epsilon}{e^{-\beta(\epsilon-\mu)} + 1} + \int_\mu^\infty \frac{d\epsilon}{e^{beta(\epsilon-\mu)} + 1}\right] \\ &\simeq D\left[\mu - \int_0^\infty \frac{dy}{e^{\beta y} + 1} + \int_0^\infty \frac{dy}{e^{\beta y} + 1}\right] \\ &\simeq D\mu, \end{aligned} \tag{9.117}$$

and similarly,

$$\begin{aligned} E &= \int_0^\infty \frac{\epsilon D d\epsilon}{e^{\beta(\epsilon-\mu)}+1} \\ &= D\left[\int_0^\mu \epsilon d\epsilon - \int_0^\mu \frac{\epsilon d\epsilon}{e^{-\beta(\epsilon_m u)}+1} - \int_\mu^\infty \frac{\epsilon d\epsilon}{e^{\beta(\epsilon_m u)}+1}\right] \\ &\simeq D\left[\frac{1}{2}\mu^2 + 2(kT)^2\int_0^\infty \frac{xdx}{e^x+1}\right] \\ &= \frac{1}{2}D\mu^2 + \frac{1}{6}\pi^2 D(kT)^2. \end{aligned} \tag{9.118}$$

The first term in (9.118) does not depend on the temperature as can be seen from (9.117). Hence one gets, at low temperatures,

$$\boxed{C_V = \frac{\partial E}{\partial T} = \frac{1}{3}\pi^2 D k^2 T.} \tag{9.119}$$

9.22 Problem G.-C. 6.17

Show that if the distribution of velocities of a Fermi gas is written, it becomes the Maxwell-Boltzmann distribution when $kT \gg \epsilon_F$. Estimate the temperature required for this to occur if $n = 10^{24}$ electrons/cc.

Solution

The distribution of velocities of a Fermi gas is given by

$$n(\vec{v}) = \frac{(2\sigma+1)V}{h^3}\frac{1}{e^{\beta\left(\frac{1}{2}mv^2-\epsilon_F\right)}+1}. \tag{9.120}$$

Then, in the limit $kT \gg \epsilon_F$,

$$\begin{aligned} N &= \frac{(2\sigma+1)V}{h^3}4\pi\int \frac{p^2dp}{e^{\beta\left(\frac{1}{2}mv^2-\epsilon_F\right)}+1} \\ &= \frac{(2\sigma+1)V}{h^3}4\pi e^{\epsilon_F}\frac{\sqrt{\pi}}{4}(2mkT)^{3/2}, \end{aligned} \tag{9.121}$$

or

$$e^{\beta\epsilon_F} = \left(\frac{h^2}{2\pi mkT}\right)^{3/2}\frac{N}{(2\sigma+1)V}. \tag{9.122}$$

Therefore,

$$\begin{aligned} n(v)d^3v &\approx \frac{(2\sigma+1)V}{h^3}e^{\beta\epsilon_F}e^{-\beta\frac{mv^2}{2}}d^3v \\ &= \frac{N}{(2\pi mkT)^{3/2}}e^{-\beta\frac{m}{2}v^2}d^3v. \end{aligned} \tag{9.123}$$

Equation (9.123) is the same as the Maxwell-Boltzmann distribution.

9.23 Problem G.-C. 7.15

For massive particles, $pV = \frac{2}{3}U$. In the case of photons as massless particles whose momentum is $p = \frac{h\nu}{c}$, show that $p = \frac{1}{3}u$. Explain the difference between them.

Solution

The Helmholtz free energy $F(= N\mu - pV)$ of the photon gas is equal to $-pV$ because $\mu = 0$.

On the other hand, the grand partition function is

$$pV = kT \ln \Xi = -kT \sum_i \ln\left(1 - e^{-\beta h\nu_i}\right). \tag{9.124}$$

The energy spectrum of the photon gas is given by $E(k) = \hbar ck$, k is the modulo of the wave vector. Then, the summation over i is to be interpreted as that over the wave vector $\vec{k}$, and this can be replaced by an integral. Taking into account the weight 2 which comes from the two different polarizations of the photon, one obtains

$$\begin{aligned} pV &= -kT \int_0^\infty \ln\left(1 - e^{-\beta\hbar cq}\right) \frac{2 * 4\pi V}{(2\pi)^3} q^2 dq \\ &= -\frac{(kT)^4 V}{\pi^2(\hbar c)^3} \int_0^\infty \ln\left(1 - e^{-x}\right) x^2 dx \\ &= \frac{(kT)^4 V}{3\pi^2(\hbar c)^3} \int_0^\infty \ln\left(1 - e^{-x}\right) x^3 dx, \end{aligned} \tag{9.125}$$

after integration by parts.

The integral in (9.125) turns to have the value $\pi^4/15$.

Hence,one has, finally

$$pV = -F = \frac{V\pi^2(kT)^4}{45(\hbar c)^3} = \frac{4\sigma V}{3c} T^4, \tag{9.126}$$

with σ the Stefan-Boltzmann constant. The entropy is given by

$$S = \frac{16\sigma V}{3c} T^3, \tag{9.127}$$

then

$$U(= F + TS) = \frac{4\sigma V}{c} T^4. \tag{9.128}$$

Thus,

$$\boxed{p = \frac{u}{3}.} \tag{9.129}$$

9.24 Problem G.-C. 7.16

Suppose that sun emits electromagnetic waves from its surface ($T \sim 10^6 K$) calculate the pressure exerts by that radiation. How is it compared with the one of a mol of ideal gas enclose on a volume of $1cm^3$?

Solution

In SI units, $\sigma = 5.67 * 10^{-8} Jm^{-2}K^{-4}s^{-1}$, $R = 8.315J/K * mol$ and $c = 3 * 10^8 m/s$.

From (9.129), it follows that

$$p_s = \frac{u}{3} = \frac{4\sigma}{3c}T^4 = 2.52 * 10^8 J/m^3. \tag{9.130}$$

Similarly, from the ideal gas equation of states, it follows that

$$p_{(i)} = \frac{8.315J/s * 300K}{10^{-6}M^3} = 2.5 * 10^9 J/m^3. \tag{9.131}$$

Then,

$$\boxed{\boxed{p_{(i)} \sim 10 p_s.}} \tag{9.132}$$

9.25 Problem G.-C. 7.20

Show that an ideal boson gas on 2-D does not condensate.

Solution

The energy levels of free particles in a 2-D region on $L_x \times L_y$ are given by

$$\epsilon(k_x, k_y) = \frac{\hbar^2}{2m}(k_x^2 + k_y^2), \tag{9.133}$$

with $k_i = \frac{2\pi n_i}{L_i}$ and $n_i \in \mathbb{Z}$.

One can use the formula which is established generally in the Bose-Einstein statistics, i.e.,

$$N = \sum_i \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}, \tag{9.134}$$

where N is the total number of particles, μ is the chemical potential and summation is over all possible energy levels. Assuming L_i sufficiently large, one can replace this summations by integration in the (k_x, k_y)-space,

$$N = \frac{L_x L_y}{(2\pi)^2} 2\pi \int_0^\infty \frac{kdk}{e^{\beta\left(\frac{\hbar^2 k^2}{2m} - \mu\right)} - 1} = 2\pi L_x L_y \frac{m}{h^2} \int_0^\infty \frac{d\epsilon}{e^{\beta(\epsilon - \mu)} - 1}, \tag{9.135}$$

or

$$N = L_x L_y \frac{2\pi mkT}{h^2} \sum_{l=1}^{\infty} \frac{1}{l} e^{\frac{l\mu}{kT}}. \tag{9.136}$$

From (9.136), one knows that always is possible to get a value of μwhich is not of order $1/N$.Hence one can conclude that there are no levels which are occupied by a number of molecules of order N, so that Bose-Einstein condensation does not occur.

9.26 Problem G.-C. 7.21

Photons obey Bose-Einstein statistics, so, one would assure that a photon gas condense. Is it correct? If this is correct, calculate T_c.

Solution

Electromagnetic waves confined in a container can be regarded as superpositions of normal mode of oscillations.

Let ν_i be the frequency of the i-th normal mode and n_i be the quantum of that mode which can be treated as a quantized harmonic oscillator. Then,

$$E(n_0, n_1, \cdots) = \sum_i n_i h\nu_i, \tag{9.137}$$

is the energy of the electromagnetic wave which is in the quantum state specified by $(n_0, n_1, \cdots)$.

The zero point of energy is omitted by the adjustment of energy. By (9.137), one can consider n_i as the number of photons with energy $h\nu_i$. Hence the canonical ensemble partition function for this photon gas is given by

$$Z(T,V) = \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} \cdots e^{-\beta E(n_0,n_1\ldots)} = \prod_i \left(1 - e^{-\beta h\nu_i}\right)^{-1}, \tag{9.138}$$

and the average value of n_j is given by

$$\overline{n}_j = \frac{1}{e^{-\beta h\nu_j} - 1}. \tag{9.139}$$

One can interpret (9.138) as the grand partition function and (9.139) as the distribution function for an ideal gas with $\mu = 0$ respectively.

Thus the condition $\mu \sim kT$ is fulfilled just for $T \to 0$ which is the critical temperature of the photon gas.

9.27 Problem G.-C. 10.1

Obtain the equations $m\ddot{\eta}_r = -m\omega_r^2 \eta_r$ and $H = \sum_r \left(\frac{p_r^2}{2m} + \frac{m\omega_r^2}{2}\eta_r^2\right)$.

Solution

For a solid, the equations of motion are given by a set of coupled harmonic oscillators,

$$m\omega^2\ddot{\xi}_j = -\sum_i \Phi_{ij}\xi_i. \tag{9.140}$$

Nonetheless, one one can introduce a set of orthonormal basis vectors, a_j, s.t. $\xi_j = \sum_r a_j^{(r)}\eta_r$ are the principal vectors of the potential Φ, e.i., $\sum_i \Phi_{ij}a_j^{(r)} = m\omega_r^2\delta_{ij}a_j^{(r)}$.

Then, (9.140) yields,

$$\sum_r a_j^{(r)}\left(m\ddot{\eta}_r + m\omega_r^2\eta_r\right) = 0 \qquad \forall a_j^{(r)}, \tag{9.141}$$

thus,

$$\boxed{m\ddot{\eta}_r = -m\omega_r^2\eta_r.} \tag{9.142}$$

Similarly, the Hamiltonian

$$H = \sum_{i=1}^{3N}\frac{p_i^2}{2m} + \frac{1}{2}\sum_{i,j=1}^{3N}\Phi_{ij}\xi_i\xi_j, \tag{9.143}$$

becomes[1],

$$\begin{aligned}
\sum_i \frac{p_i^2}{2m} &= \frac{m^2}{2m}\sum_i\sum_{rr'} a_i^{(r)}a_i^{(r')}\dot{\eta}_r\dot{\eta}_{r'} \\
&= \frac{1}{2m}\sum_{rr'}\delta_{rr'}m\dot{\eta}_r m\dot{\eta}_{r'} \\
&= \frac{1}{2m}\sum_r p_r^2,
\end{aligned} \tag{9.144}$$

and

$$\begin{aligned}
\frac{1}{2}\sum_{ij}\Phi_{ij}\xi_i\xi_j &= \frac{1}{2}\sum_{ij}\sum_{rr'}\Phi_{ij}a_i^{(r)}a_j^{(r')}\eta_r\eta_{r'} \\
&= \frac{1}{2}\sum_{ij}\sum_{rr'}m\omega_r^2\delta_{ij}a_i^{(r)}a_j^{(r')}\eta_r\eta_{r'} \\
&= \frac{1}{2}\sum_{i}\sum_{rr'}m\omega_r^2 a_i^{(r)}a_i^{(r')}\eta_r\eta_{r'} \\
&= \frac{1}{2}\sum_r m\omega_r^2\eta_r^2.
\end{aligned} \tag{9.145}$$

[1] Since $p_i = m\dot{\xi}_i = m\sum_r a_i^{(r)}\dot{\eta}_r$.

From (9.144) and (9.145) it follows that

$$\boxed{H = \sum_r \left(\frac{p_r^2}{2m} + \frac{1}{2} m\omega_r^2 \eta_r^2 \right).} \tag{9.146}$$

9.28 Problem G.-C. 10.2

a.- The energy levels of a 3D isotropic harmonic oscillator are given by $\epsilon_n = (n + 3/2)h\nu$. Show that the n-th level has degeneration $\frac{1}{2}(n+1)(n+2)$.

b.- Calculate the partition function for N oscillators and relate the result with the Einstein model of solids.

Solution

In a 3D harmonic oscillator, $\epsilon_n = (n + 3/2)\hbar\omega$, where $n = n_x + n_y + n_z$, so the degeneration is given by the number of ways one can array n object in three different boxes.

The number of ways of distributing M white balls among N labeled boxes is equivalent to the permutation of $N-1$ black balls in a row together with the white balls. Therefore, the combinatorial factor is

$$g_M = \frac{(M+N-1)!}{M!(N-1)!}. \tag{9.147}$$

In the problem, $M = n$ and $N = 3$, so

$$\boxed{g_n = \frac{1}{2}(n+1)(n+2).} \tag{9.148}$$

The partition function is given by

$$\begin{aligned}
Z_1 &= \frac{1}{2}\sum_n (n+1)(n+2)e^{-\beta(n+3/2)h\nu} \\
&= \frac{1}{2}e^{\beta\frac{1}{2}h\nu}\sum_n (n+1)(n+2)e^{-\beta(n+2)h\nu} \\
&= \frac{1}{2}e^{\beta\frac{1}{2}h\nu}\left(-\frac{\partial}{\partial(\beta h\nu)}\right)\sum_n (n+1)e^{-\beta(n+2)h\nu} \\
&= \frac{1}{2}e^{\beta\frac{1}{2}h\nu}\left(-\frac{\partial}{\partial(\beta h\nu)}\right)e^{-\beta h\nu}\sum_n (n+1)e^{-\beta(n+1)h\nu} \\
&= \frac{1}{2}e^{\beta\frac{1}{2}h\nu}\left(-\frac{\partial}{\partial(\beta h\nu)}\right)e^{-\beta h\nu}\left(-\frac{\partial}{\partial(\beta h\nu)}\right)\sum_n e^{-\beta(n+1)h\nu} \\
&= \frac{1}{2}e^{\beta\frac{1}{2}h\nu}\left(-\frac{\partial}{\partial(\beta h\nu)}\right)e^{-\beta h\nu}\left(-\frac{\partial}{\partial(\beta h\nu)}\right)\frac{1}{e^{\beta h\nu}-1} \\
&= \frac{1}{2}e^{\beta\frac{1}{2}h\nu}\left(-\frac{\partial}{\partial(\beta h\nu)}\right)\frac{1}{(e^{\beta h\nu}-1)^2} \\
&= \frac{e^{\beta\frac{3}{2}h\nu}}{(e^{\beta h\nu}-1)^3},
\end{aligned} \tag{9.149}$$

or

$$\boxed{Z_N = \frac{e^{\beta\frac{3N}{2}h\nu}}{(e^{\beta h\nu}-1)^{3N}}.} \tag{9.150}$$

This yield the same results as the Einstein's theory of crystals.

9.29 Problem G.-C. 10.3

a.- Use directly the definition of Debye's function to showing

$$D(x) = \begin{cases} 1 - \frac{3x}{8} + \frac{x^2}{20} + O(x^3) & ;x = \frac{\theta_D}{T \ll 1} \\ \frac{\pi^4}{5x^3} - 3e^{-x}(1 + O(1/x)); & x \gg 1 \end{cases} \tag{9.151}$$

b.- Use the above result to showing that

$$\frac{C_V}{Nk} = \begin{cases} 3\left[1 - \frac{1}{20}x^2 + \cdots\right] & ;x \ll 1 \\ \frac{12}{5}\pi^4\frac{1}{x^3}\left[1 - \frac{15}{4\pi}x^3e^{-x} + \cdots\right] & ;x \gg 1 \end{cases} \tag{9.152}$$

Solution

Since,

$$D(x) = \frac{3}{x^3}\int_0^x \frac{x^3 dx}{e^x - 1}, \tag{9.153}$$

it follows that for $x \ll 1$, $e^x - 1 \simeq x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, then

$$\begin{aligned} \int_0^x dx \frac{x^2}{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots} &\approx \int_0^x dx x^2 \left(1 - \left(\frac{x}{2} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots\right) + \left(\frac{x}{2} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots\right)^2 + \cdots\right) \\ &\approx \int_0^x dx x^2 - \frac{1}{2}\int_0^x x^3 + \frac{1}{12}\int_0^x dx x^4 + \cdots \\ &= \frac{x^3}{3} - \frac{x^4}{8} + \frac{x^5}{60} + \cdots . \end{aligned} \tag{9.154}$$

Therefore,

$$\boxed{D(x) \approx 1 - \frac{3x}{8} + \frac{x^2}{20},} \quad \text{for } x \ll 1. \tag{9.155}$$

Similarly,for $x \gg 1$ one get

$$\begin{aligned} \int_0^x dx \frac{x^3}{e^x - 1} &= \int_0^\infty dx \frac{x^3}{e^x - 1} - \int_x^\infty dx \frac{x^3}{e^x - 1} \\ &= \frac{\pi^4}{15} - \int_x^\infty dx \frac{x^3 e^{-x}}{1 - e^{-x}} \\ &= \frac{\pi^4}{15} - \int_x^\infty dx x^3 e^{-x}\left(1 + e^{-x} + e^{-2x} + \cdots\right) \\ &\approx \frac{\pi^4}{15} - x^3 e^{-x} + \cdots . \end{aligned} \tag{9.156}$$

Thus,

$$\boxed{D(x) = \frac{\pi^4}{5x^3} - 3e^{-x} + O\left(\frac{1}{x}\right).} \tag{9.157}$$

Moreover,

$$U = U_0^{(D)} + 3NkTD(x), \tag{9.158}$$

then, for $x \ll 1$ yields,

$$\boxed{C_V = 3NkT - \frac{3}{20}Nk\left(\frac{\theta_D}{T}\right)^2 + \cdots = 3Nk\left(1 - \frac{x^2}{20} + cdots\right).} \tag{9.159}$$

And for $x \gg 1$,

$$C_V \approx \frac{12}{5} NkT\pi^4 \left(\frac{T}{\theta_D}\right) - 9Nke^{-\frac{\theta_D}{T}} + \cdots , \tag{9.160}$$

thus,

$$\frac{C_V}{Nk} = \frac{12}{5}\pi^4 \left(\frac{1}{x^3}\right)\left[1 - \frac{15}{4\pi^4}x^3 e^{-x} + \cdots\right]. \tag{9.161}$$

9.30 Problem G.-C. 12.1

The measurement of polarization of an ideal gas is taken a $300K$ using an electric field of $2000V/m$. Is it allowed to use $\mathcal{L}(x) = \frac{1}{3}x$? Why? Assume $\mu_e = 2.64$ Debyes.

Solution

Langevin's function is definedby

$$\mathcal{L}(x) = \coth x - \frac{1}{x} = \begin{cases} \frac{x}{3} + \frac{x^3}{45} & ; \quad x < 1 \\ 1 & ; \quad x \gg 1 \end{cases} \tag{9.162}$$

with $x = \mu_e E/kT$.

In order to approximate Langevin's function just like $x/3$, $x \ll 1$.

In the given set up, the condition $x = 1$ requires

$$E_c \simeq \frac{1.38 * 10^{-16} * 300}{3 * 10^{-18}} = 1.38 * 10^4 ues \simeq 4 * 10^6 V/m, \tag{9.163}$$

since $E \ll E_c$, then is valid taking the linear approximation.

9.31 Problem G.-C. 12.2

How would be modified the equation

$$\frac{\kappa - 1}{4\pi n} = \alpha + \frac{\mu_e^2}{3kT},$$

if the first non-linear term of Langevin's function is include? Under what conditions are usefull this corrections?

Solution

Since

$$\langle \mu_e \rangle = \mu_e \mathcal{L}(x), \tag{9.164}$$

then,

$$\langle \mu_e \rangle = \frac{\mu_e^2 E}{3kT}\left(1 + \frac{x^2}{15}\right). \tag{9.165}$$

It follows that

$$P = \frac{n}{3}\frac{\mu_e^2 E}{kT}\left(1 + \frac{x^2}{15}\right). \tag{9.166}$$

If one includes the polarization of the molecules, hence,

$$\chi = n\left[\alpha + \frac{\mu_e^2}{3kT}\left(1 + \frac{x^2}{15}\right)\right]. \tag{9.167}$$

Finally the electric displacement definition, yields

$$\boxed{\boxed{\frac{\kappa - 1}{4\pi n} = \alpha + \frac{\mu_e^2}{3kT}\left(1 + \frac{\mu_e^2 E^2}{15(kT)^2}\right).}} \tag{9.168}$$

This effect is not worthing unless one is interested in low temperatures and/or huge electric fields (order of MV/m).

9.32 Problem G.-C. 12.11

Use the equation

$$C_{\mathcal{M}} = -\frac{2nJ}{g\mu_B} V\sigma\mathcal{M}\left[\frac{\partial}{\partial T} B_\sigma\left(b\frac{\mathcal{M}}{T}\right)\right]_V, \tag{9.169}$$

and the definition of the Brillouin to showing that $C_{\mathcal{M}} = 0$ when $T \to 0$.

Solution

By definition of the Brillouin function,

$$B_\sigma(B\mathcal{M}/T) = (\sigma + 1/2)\coth\left[(\sigma + 1/2)\frac{B\mathcal{M}}{T}\right] - \frac{1}{2}\coth\left(\frac{B\mathcal{M}}{T}\right). \tag{9.170}$$

Since,

$$\frac{\partial}{\partial T}\coth(1/T) \propto \frac{1 - \coth^2(1/t)}{T^2}, \tag{9.171}$$

when $T \to \infty$, $\coth^2(1/t) \to 1$ exponentially, whilst the denominator explode potentially. Then,

$$\boxed{C_{\mathcal{M}} \to 0.} \tag{9.172}$$

as $T \to 0$.

9.33 Problem G.-C. 15.1

Show that the minimum work required to producing a fluctuation in entropy ΔS is precisely the numerator of the argument of

$$\Omega = C \exp - \frac{\Delta E - T\Delta S_I + p\Delta V - \mu \Delta N}{kT}. \tag{9.173}$$

Solution

Using the thermodynamical relations for the total entropy $S = S_I + S_{II}$,the fluctuation is given by

$$T\Delta S = \Delta E - T\Delta S_I + p\Delta V - \mu\Delta N, \tag{9.174}$$

then, one can write

$$\Omega = C \exp - \frac{\Delta S}{k}. \tag{9.175}$$

Equation (9.175) is valid for the compose system $I + II$. Let the total system be transferred from the states A^* into the state $A^* + A'$, when the subsystem I is transferred from the state α^* to the state $\alpha^* + \alpha'$, external work is being done.

Then, since whenever work is done the quasi-static (or reversible) way, it becomes minimum, one obtains

$$W_{min}(\alpha^*, \alpha') = \int_{A^*}^{A^*+A'} (dU - TdS), \tag{9.176}$$

where U and S denote the energy and entropy of the composite system $I + II$, and T is the temperature at each intermediate stage of the process.

Since α is the change in a sufficiently small subsystem I one regards A' as a very small deviation of the total system $I + II$, and therefore approximate T by the value T^* at the state A^*.

The composite system is supposes to be isolated, so the internal energy must be equal at both states. In this case,

$$W_{min}(\alpha^*, \alpha') = T^* \Delta S. \tag{9.177}$$

Finally

$$\boxed{\Omega(\alpha^*, \alpha') = C \exp\left(-\frac{W_{min}(\alpha^*, \alpha')}{kT^*}\right).} \tag{9.178}$$

9.34 Problem G.-C. 15.8

Calculate the fluctuations of the square energy for Rayleigh-Jeans, Wien and Planck.

Solution

The fluctuation of the energy square is given by

$$\left\langle E^2(\nu, T)\right\rangle = hTV_0 d\nu \left(\frac{\partial u(\nu, T)}{\partial T}\right)_{V_0}. \tag{9.179}$$

Also, Rayleigh-Jeans energy density is

$$u(\nu, T) = \frac{8\pi\nu^2}{c^3}kT, \Longrightarrow \quad \frac{\partial u}{\partial T} = \frac{8\pi\nu^2}{c^3}k. \tag{9.180}$$

For Wien's energy density,

$$u(\nu, T) = \alpha\nu^3 e^{-\beta\nu/T}, \Longrightarrow \quad \frac{\partial u}{\partial T} = \frac{\alpha\beta\nu^4}{T^2}e^{-\beta\nu/T}. \tag{9.181}$$

Finally for Planck's energy density

$$u(\nu, T) = \frac{8\pi\nu^3 h}{c^3}\frac{1}{e^{\frac{h\nu}{kT}}-1}, \Longrightarrow \quad \frac{\partial u(\nu, T)}{\partial T} = \frac{8\pi\nu^4 h^2}{c^3 kT^2}\frac{e^{\frac{h\nu}{kT}}}{(e^{\frac{h\nu}{kT}}-1)^2}. \tag{9.182}$$

If one uses the identity,

$$\frac{e^{\frac{h\nu}{kT}}}{(e^{\frac{h\nu}{kT}}-1)^2} = \frac{1}{e^{\frac{h\nu}{kT}}-1} + \frac{1}{(e^{\frac{h\nu}{kT}}-1)^2}, \tag{9.183}$$

it follows directly that,

$$\boxed{\left\langle E^2(\nu, T)\right\rangle = \begin{cases} u^2(\nu, T)\frac{c^3}{8\pi\nu^2}V_0 d\nu & \text{R.J.} \\ \beta k\nu V_0 u(\nu, T) d\nu & \text{W.} \\ \left[h\nu u(\nu, T) + \frac{c^3}{8\pi\nu^2}u^2(\nu, T)\right]V_0 d\nu & \text{Planck} \end{cases}} \tag{9.184}$$

Part IV

Relativity

Chapter 10

Relativity

10.1 Problem 1

Write down the line elements for

- A sphere of radius a, using the angles (θ, ϕ) as spherical coordinates.
- A cylinder whose transverse section is a circle of radius a, using (phi, z) as the coordinates for the cylinder. Is this cylinder plane or curved?
- A hyperbolic paraboloid, whose parametric equation is

$$\vec{r} = (u + v)\hat{e}_x + (u - v)\hat{e}_y + 2uv\hat{e}_z, \tag{10.1}$$

using the (u, v) parametrisation.

Answer

In order to find the line element for each case, we just substitute the differentials of the line element in terms of the new coordinates.

Spherical coordinates

The change of basis is given by

$$\begin{aligned} x &= a\sin\theta\cos\phi, \\ y &= a\sin\theta\sin\phi, \\ z &= a\cos\theta. \end{aligned} \tag{10.2}$$

then,

$$\begin{aligned}\mathbf{d}x &= a\left(\cos\theta\mathbf{d}\theta\cos\phi - \sin\theta\sin\phi\mathbf{d}\phi\right)\\ \mathbf{d}y &= a\left(\cos\theta\mathbf{d}\theta\sin\phi + \sin\theta\cos\phi\mathbf{d}\phi\right)\\ \mathbf{d}z &= -a\sin\theta\mathbf{d}\theta,\end{aligned} \tag{10.3}$$

and

$$\begin{aligned}\mathbf{d}x^2 &= a^2\left(\cos^2\theta\cos^2\phi\mathbf{d}\theta^2 + \sin^2\theta\sin^2\phi\mathbf{d}\phi^2 - 2\sin\theta\cos\theta\cos\phi\sin\phi\mathbf{d}\theta\mathbf{d}\phi\right)\\ \mathbf{d}y^2 &= a^2\left(\cos^2\theta\sin^2\phi\mathbf{d}\theta^2 + \sin^2\theta\cos^2\phi\mathbf{d}\phi^2 + 2\sin\theta\cos\theta\cos\phi\sin\phi\mathbf{d}\theta\mathbf{d}\phi\right)\\ \mathbf{d}z^2 &= a^2\sin^2\theta\mathbf{d}\theta^2.\end{aligned} \tag{10.4}$$

Thus,

$$\mathbf{d}s^2 = a^2\left(\mathbf{d}\theta^2 + \sin^2\theta\mathbf{d}\phi^2\right). \tag{10.5}$$

Cylindric coordinates

For cylindric coordinates, the change of basis is given by

$$\begin{aligned}x &= a\cos\phi,\\ y &= a\sin\phi,\\ z &= z.\end{aligned} \tag{10.6}$$

Then

$$\begin{aligned}\mathbf{d}x &= -a\sin\phi\mathbf{d}\phi\\ \mathbf{d}y &= a\cos\phi\mathbf{d}\phi\\ \mathbf{d}z &= \mathbf{d}z,\end{aligned} \tag{10.7}$$

and

$$\begin{aligned}\mathbf{d}x^2 &= a^2\sin^2\phi\mathbf{d}\phi^2\\ \mathbf{d}y^2 &= a^2\cos^2\phi\mathbf{d}\phi^2\\ \mathbf{d}z^2 &= \mathbf{d}z^2.\end{aligned} \tag{10.8}$$

Thus

$$\mathbf{d}s^2 = a^2\mathbf{d}\phi + \mathbf{d}z^2. \tag{10.9}$$

We note that a cylinder is a direct product $\mathbb{R}\times S^1$ this embedded is plane in $\mathbb{R}^3$.

Hyperbolic paraboloid

For hyperbolic paraboloid coordinates, the change of basis is given by

$$\begin{aligned} x &= u + v, \\ y &= u - v, \\ z &= 2uv. \end{aligned} \tag{10.10}$$

Then

$$\begin{aligned} \mathbf{d}x &= \mathbf{d}u + \mathbf{d}v \\ \mathbf{d}y &= \mathbf{d}u - \mathbf{d}v \\ \mathbf{d}z &= 2v\mathbf{d}u + 2u\mathbf{d}v, \end{aligned} \tag{10.11}$$

and

$$\begin{aligned} \mathbf{d}x^2 &= \mathbf{d}u^2 + \mathbf{d}v^2 + 2\mathbf{d}u\mathbf{d}v \\ \mathbf{d}y^2 &= \mathbf{d}u^2 + \mathbf{d}v^2 - 2\mathbf{d}u\mathbf{d}v \\ \mathbf{d}z^2 &= 4v^2\mathbf{d}u^2 + 4u^2\mathbf{d}v^2 + 8uv\mathbf{d}u\mathbf{d}v. \end{aligned} \tag{10.12}$$

Thus

$$\mathbf{d}s^2 = (2 + 4v^2)\mathbf{d}u^2 + (2 + 4u^2)\mathbf{d}v^2 + 8uv\mathbf{d}u\mathbf{d}v. \tag{10.13}$$

10.2 Problem 2

Show that ${\delta_\mu}^\nu$ has the expected tensorial character. Prove that ${\delta_\mu}^\nu$ is a constant tensor which has the same components in every coordinate frame. Evaluate ${\delta_\mu}^\mu$ and ${\delta_\mu}^\nu{\delta_\nu}^\mu$.

Answer

Let us consider the change of coordinates $x^\mu \mapsto y^\mu$, therefore, the matrix transformation is $\frac{\partial x^\mu}{\partial y^\alpha}$ for covariant indices, and $\frac{\partial y^\alpha}{\partial x^\mu}$ for contravariant ones.

Hence,

$$\begin{aligned} {\delta_\mu}^\nu &\mapsto \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial y^\beta}{\partial x^\nu}{\delta_\mu}^\nu \\ &= \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial y^\beta}{\partial x^\mu} \\ &= {\delta_\alpha}^\beta, \end{aligned} \tag{10.14}$$

i.e., it has the same componets in whatever frame we consider.

Furthermore, since ${\delta_\mu}^\nu$ represent the unit matrix, then ${\delta_\mu}^\mu = {\delta_\mu}^\nu{\delta_\nu}^\mu = D$, where D is the dimension of the spacetime, it's due to the fact that ${\delta_\mu}^\mu$ is nothing but the trace of the unit matrix.

10.3 Problem 3

Show that whatever covariant (or contravariant) tensor of second rank can be written the sum of a symmetric and an antisymmetric tensor.

Answer

A symmetric tensor satisfy that $T_{ab} = T_{ba}$ and an antisymmetric one satisfy $T_{ab} = -T_{ba}$.

For an arbitrary tensor, we can define its symmetrisation as

$$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba}), \tag{10.15}$$

which satisfy the symmetric condition. Similarly, we can define its antisymmetric part of this tensor as

$$T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba}). \tag{10.16}$$

Adding the last two equations, we get

$$T_{[ab]} + T_{(ab)} = \frac{1}{2}(T_{ab} - T_{ba}) + \frac{1}{2}(T_{ab} + T_{ba}) = T_{ab}. \tag{10.17}$$

i.e., every tensor of second rank can be written as a sum of a symmetric tensor and an antisymmetric tensor.

10.4 Problem 4

Show that the Levì-Civita ϵ^{abcd} is a tensor under Lorentz transformations and a tensorial density of weight -1 under general coordinate transformations. Also show that its components are the same for every reference frame.

Answer

Under a general transformation, the epsilon transforms by

$$\epsilon^{abcd} \mapsto \Lambda^a_m \Lambda^b_n \Lambda^c_p \Lambda^d_q \epsilon^{mnpq}. \tag{10.18}$$

Now, we use the definition of the determinant of a matrix in term of the epsilon tensor, and the symmetries of (10.18), so we write

$$\Lambda^a_m \Lambda^b_n \Lambda^c_p \Lambda^d_q \epsilon^{mnpq} = det(\Lambda)\epsilon^{abcd}. \tag{10.19}$$

If we consider just Lorentz transformation, then $\Lambda \in SO(3,1)$ and so $det(\Lambda) = 1$, thus

$$\epsilon^{abcd} \mapsto \epsilon^{abcd}, \tag{10.20}$$

i.e., epsilon is an invariant tensor under Lorentz transformations.

In case of general coordinates transformations, $\det(\Lambda) = |J|^{-1}$, and so

$$\epsilon^{abcd} \mapsto |J|^{-1}\epsilon^{abcd}, \tag{10.21}$$

this transformation law defines a density of weight -1. Then, we conclude that in fact epsilon is an invariant tensor under Lorentz transformations but a density under general coordinates transformations.

10.5 Problem 5

Consider a plane parameterized with both, a cartesian coordinate frame (x, y) and a polar coordinate frame (r, θ). Calculate:

- The matrices which define the transformation laws co- and contra-variant.
- The natural basis (e'_1, e'_2) and its dual defined through the relation $e'^i \cdot e'_j = \delta^i_j$

Answer

We know the transformation

$$x = r\cos\theta \tag{10.22}$$

$$y = r\sin\theta, \tag{10.23}$$

then, the transformation matrices are

$$\Lambda^{\nu}_{\mu'} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \tag{10.24}$$

$$\Lambda^{\mu'}_{\nu} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}. \tag{10.25}$$

Finally,

$$e_r = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath} \tag{10.26}$$

$$e_\theta = -r\sin\theta\hat{\imath} + r\cos\theta\hat{\jmath} \tag{10.27}$$

$$\omega^r = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath} \tag{10.28}$$

$$\omega^\theta = -\frac{\sin\theta}{r}\hat{\imath} + \frac{\cos\theta}{r}\hat{\jmath}. \tag{10.29}$$

10.6 Problem 6

Among the set of general coordinate transformations consider the subset defined by the linear orthogonal transformations, $x'^i = A^i_j x^j$ with A^i_j constants. Show that for this subset the relation

$$\frac{\partial x'^i}{\partial x^j} = \frac{\partial x^j}{\partial x'^i}, \tag{10.30}$$

is satisfied.

What does this imply?

Answer

Indeed, the relation (10.30) defines the group of orthogonal transformations.

In term of contra- and co-variant components it implies that they transform as inverse of the other.

10.7 Problem 7

Show that any infinitesimal orthogonal transformation can be written in the form $x'^i = x^i + \epsilon^i_j x^j$ with $\epsilon^{ij} = -\epsilon^{ji}$.

Answer

Since orthogonal transformations form a Lie group, they can be written in the form

$$O = e^{\imath\theta^i T_i}.$$

with $\theta^i T_i = \epsilon$. Thus, at first order it yields,

$$O^t O = (1 + \epsilon^t)(1 + \epsilon) = 1, \tag{10.31}$$

therefore,

$$\epsilon^t = -\epsilon. \tag{10.32}$$

10.8 Problem 8

IT WAS ALREADY SOLVE IN THE PREVIOUS HOMEWORK.

10.9 Problem 9

On a space, apparently 3-dimensional, the distance between two points is given by

$$ds^2 = dx^2 + dy^2 + dz^2 - \frac{1}{169}(3dx + 4dy + 12dz)^2.$$

Show that this space is locally $\mathbb{R}^2$. Find a couple of new coordinates s.t. the line element is $ds^2 = d\xi^2 + d\eta^2$.

Answer

In order to prove that this is in fact a 2-dimensional...

10.10 Problem 10

Consider the metric space (M, d), where $M = \{(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}\}$ and d is the distance given by the number of sides of the shortest path between a couple of points.

- Draw a circumference of radius 4.
- Compute the value of π in this geometry. Is it irrational?

Answer

From figure 10.1, we can see that the 'perimeter' of this circumference is 32. Since the value of π is perimeter by diameter,

$$\pi = \frac{perimeter}{diameter} = \frac{32}{8} = 4. \tag{10.33}$$

Thus the value of π in this geometry is not irrational but natural.

10.11 Problem 11

$\mathbb{R}^2$ in polar coordinates has as length $\mathbf{d}s^2 = \mathbf{d}r^2 + r^2\mathbf{d}\theta^2$. Compute in this coordinates the Christoffel symbols.

Answer

Since $\mathbf{d}s^2 = g_{ab}\mathbf{d}x^a \otimes \mathbf{d}x^b$, we have the metric tensor, and we can construct the Lagrangian,

$$L = g_{ab}\dot{x}^a\dot{x}^b. \tag{10.34}$$

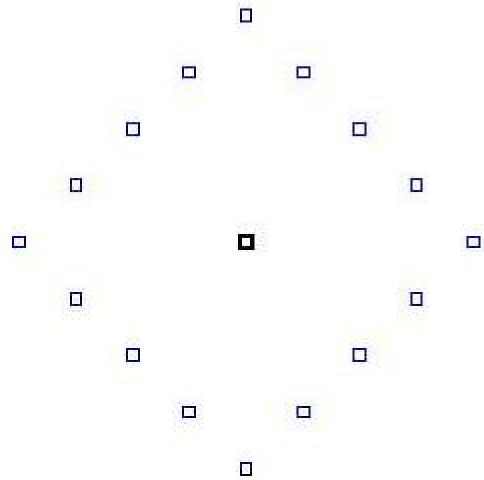


Figure 10.1: A drawn of a circle of radius 4 in a lattice. The black dot represent the centre and the blue ones are the circumference.

So, for our particular case, we have

$$L = \dot{r}^2 + r^2\dot{\theta}^2, \tag{10.35}$$

therfore, the Euler-Lagrange equations are,

$$\begin{aligned} \frac{d}{dt}2r^2\dot{\theta} &= 0 \\ \Rightarrow \quad \ddot{\theta} + \frac{2}{r}\dot{\theta}\dot{r} &= 0. \end{aligned} \tag{10.36}$$

$$\begin{aligned} \frac{d}{dt}(2\dot{r}) - 2r\dot{\theta}^2 &= 0 \\ \Rightarrow \quad \ddot{r} - r\dot{\theta}^2 &= 0. \end{aligned} \tag{10.37}$$

Now, from the geodesic equation, it follows that,

$$\Gamma^r_{\theta\theta} = -r, \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}. \tag{10.38}$$

10.12 Problem 12

How do the Christoffel's symbols transform under a general change of coordinates?

Answer

Let's remind that the coefficients of the connection are defined by

$$\nabla_{e_a} e_b = \Gamma^c_{ab} e_c, \tag{10.39}$$

then, if we consider a different basis $f_m = \frac{x^a}{y^m} e_a$, where x's denote the old coordinates and y's the new ones, the new connection coefficients are

$$\nabla_{f_m} f_n = \Gamma^p_{mn} f_p. \tag{10.40}$$

Furthermore,

$$\begin{aligned}
\nabla_{f_m} f_n &= \nabla_{f_m} \left(\frac{\partial x^b}{\partial y^n} e_b \right) \\
&= \frac{\partial^2 x^c}{\partial y^m \partial y^n} e_c + \frac{\partial x^a}{\partial y^m} \frac{\partial x^b}{\partial y^n} \nabla_{e_a} e_b \\
&= \left(\frac{\partial^2 x^c}{\partial y^m \partial y^n} + \frac{\partial x^a}{\partial y^m} \frac{\partial x^b}{\partial y^n} \Gamma^c_{ab} \right) e_c.
\end{aligned} \tag{10.41}$$

Finally,

$$\Gamma^p_{mn} = \frac{\partial x^a}{\partial y^m} \frac{\partial x^b}{\partial y^n} \frac{\partial y^p}{\partial x^c} \Gamma^c_{ab} + \frac{\partial^2 x^a}{\partial y^m \partial y^n} \frac{\partial y^p}{\partial x^a}. \tag{10.42}$$

10.13 Problem 13

Compute the curvature as the change on a parallel transported vector through a parallelogram in a manifold M.

Answer

Consider a parallelogram $pqrs$ whose coordinates are x^a, $x^a + \epsilon^a$, $x^a + \epsilon^a$ and $x^a + \delta^a$, and let $V^a(p)$ be a vector at the point $p \in M$.

If we start at p, the parallel transport vector at q is

$$V^a(q) = V^a(p) - V^b(p)\Gamma^a_{cb}(p)\epsilon^c. \tag{10.43}$$

Then,

$$\begin{aligned}
V^a(r) &= V^a(q) - V^b(q)\Gamma^a_{cb}(q)\delta^c \\
&= V^a(p) - V^b(p)\Gamma^a_{cb}(p)\epsilon^c - \left(V^b(p) - V^m(p)\Gamma^b_{nm}(p)\epsilon^n \right) \Gamma^a_{cb}(q)\delta^c \\
&\cong V^a(p) - V^b(p)\Gamma^a_{cb}(p)\epsilon^c - \left(V^b(p) - V^m(p)\Gamma^b_{nm}(p)\epsilon^n \right) \\
&\quad * \left(\Gamma^a_{cb}(p) + \partial_k \Gamma^a_{cb}(p)\epsilon^k \right) \delta^c \\
&= V^a(p) - V^b(p)\Gamma^a_{cb}(p)(\delta^c + \epsilon^c) \\
&\quad -V^b(p) \left(\partial_k \Gamma^a_{cb}(p) - \Gamma^l_{kb}(p)\Gamma^a_{cl}(p) \right) \epsilon^k \delta^c.
\end{aligned} \tag{10.44}$$

Similarly, if we reach r through the point s, we get,

$$\begin{aligned} V^a(r') &= V^a(p) - V^b(p)\Gamma^a_{cb}(p)(\delta^c + \epsilon^c) \\ &\quad - V^b(p)\left(\partial_k \Gamma^a_{cb}(p) - \Gamma^l_{kb}(p)\Gamma^a_{cl}(p)\right)\epsilon^c\delta^k. \end{aligned} \tag{10.45}$$

So, both vectors differ by (substract (10.44) from (10.45))

$$\begin{aligned} V^a(r') - V^a(r) &= V^b\left(\partial_c \Gamma^a_{kb}(p) - \partial_k \Gamma^a_{cb}(p)\right. \\ &\quad \left. + \Gamma^l_{kb}(p)\Gamma^a_{cl}(p) - \Gamma^l_{cb}(p)\Gamma^a_{kl}(p)\right)\epsilon^c\delta^k \qquad (10.46) \\ &= V^b R^a{}_{bcd}\epsilon^c\delta^d. \qquad (10.47) \end{aligned}$$

10.14 Problem 14

Show that the covariant derivative $D_a u^b$ transforms like a $\binom{1}{1}$ tensor.

Answer

Under coordinate transformations,

$$\begin{aligned} D_a u^b \mapsto D_\alpha u^\beta &= \partial_\alpha u^\beta + \Gamma^\beta_{\alpha\gamma}u^\gamma \\ &= \Lambda^a_\alpha \partial_a(\Lambda^\beta_b u^b) + \Lambda^a_\alpha \Lambda^\beta_b \Gamma^b_{ac} u^c + \Lambda^a_\alpha(\partial_a \Lambda^n_\gamma)\Lambda^\beta_n \Lambda^\gamma_c u^c. \end{aligned} \tag{10.48}$$

Note that in the last line, the matrices coming from the γ index vanishes.

Now, from the identity

$$\Lambda^a_b \Lambda^b_c = \delta^a_c, \tag{10.49}$$

it follows that

$$(\partial_d \Lambda^a_b)\Lambda^b_c = -\Lambda^a_b(\partial_d \Lambda^b_c), \tag{10.50}$$

then,

$$\begin{aligned} D_\alpha u^\beta &= \Lambda^a_\alpha \partial_a(\Lambda^\beta_b u^b) + \Lambda^a_\alpha \Lambda^\beta_b \Gamma^b_{ac} u^c + \Lambda^a_\alpha(\partial_a \Lambda^n_\gamma)\Lambda^\beta_n \Lambda^\gamma_c u^c \\ &= \Lambda^a_\alpha(\partial_a \Lambda^\beta_b)u^b + \Lambda^a_\alpha \Lambda^\beta_b \partial_a u^b + \Lambda^a_\alpha \Lambda^\beta_b \Gamma^b_{ac} u^c \\ &\quad - \Lambda^a_\alpha(\partial_a \Lambda^\beta_b)u^b \\ &= \Lambda^a_\alpha \Lambda^\beta_b \partial_a u^b + \Lambda^a_\alpha \Lambda^\beta_b \Gamma^b_{ac} u^c \\ &= \Lambda^a_\alpha \Lambda^\beta_b(\delta^b_c \partial_a + \Gamma^b_{ac})u^c \\ &= \Lambda^a_\alpha \Lambda^\beta_b D_a u^b. \end{aligned} \tag{10.51}$$

10.15 Problem 15

Show that the Torsion and the Riemann curvature transform like tensors under general changes of coordinates.

Answer

Let's start from the definitions of both, Torsion and Curvature,

$$\begin{aligned} T(X,Y) &= \nabla_X Y - \nabla_Y X - [X,Y] &(10.52)\\ R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. &(10.53)\end{aligned}$$

Let's change $X \to fX$, $Y \to gY$ in (10.52), so

$$\begin{aligned} T(fX,gY) &= f\nabla_X(gY) - g\nabla_Y(fX) - [fX,gY]\\ &= fX[g]Y + fg\nabla_X Y - gY[f]X - (fX[g]Y + fgXY - gY[f]X - gfYX\\ &= fgT(X,Y), &(10.54)\end{aligned}$$

therefore, we can take $X = e_a$, $Y = e_b$ and $f = \frac{\partial x^a}{\partial y^m}$, $g = \frac{\partial x^b}{\partial y^n}$, so

$$T(f_m, f_n) = \frac{\partial x^a}{\partial y^m}\frac{\partial x^b}{\partial y^n}T(e_a,e_b). \tag{10.55}$$

Similarly, for the curvature (10.53),

$$\begin{aligned} R(fX,gY)hZ &= f\nabla_X(g\nabla_Y(h))Z - g\nabla_Y(f\nabla_X(hZ)) - \nabla_{[fX,gY]}hZ\\ &= fX[g]\nabla_Y(hZ) + fg\nabla_X\nabla_Y(hZ) - gY[f]\nabla_X(hZ)\\ &\quad -fg\nabla_Y\nabla_X(hZ) - \nabla_{[fX,gY]}(hZ). &(10.56)\end{aligned}$$

Since

$$\begin{aligned} \nabla_{[fX,gY]}(hZ) &= \nabla_{fXgY}(hZ) - \nabla_{gYfX}(hZ)\\ &= fX[g]\nabla_Y(hZ) - gY[f]\nabla_X(hZ) + fg\nabla_{[X,Y]}(hZ), &(10.57)\end{aligned}$$

then,

$$\begin{aligned} R(fX,gY)hZ &= fX[g]\nabla_Y(hZ) + fg\nabla_X\nabla_Y(hZ) - gY[f]\nabla_X(hZ)\\ &\quad -fg\nabla_Y\nabla_X(hZ) - fX[g]\nabla_Y(hZ)\\ &\quad +gY[f]\nabla_X(hZ) - fg\nabla_{[X,Y]}(hZ)\\ &= fg\nabla_X\nabla_Y(hZ) - fg\nabla_Y\nabla_X(hZ) - fg\nabla_{[X,Y]}(hZ)\\ &= fgX[h]\nabla_Y Z + fgY[h]\nabla_X Z + fgXY[h]Z\\ &\quad +fgh\nabla_X\nabla_Y(Z) - fgY[h]\nabla_X Z - fgX[h]\nabla_Y Z\\ &\quad -fgXY[h]Z - fgh\nabla_Y\nabla_X(Z) - fg[X,Y][h]Z\\ &\quad -fgh\nabla_{[X,Y]}(Z)\\ &= fgh\nabla_X\nabla_Y(Z) - fgh\nabla_Y\nabla_X(Z) - fgh\nabla_{[X,Y]}(Z)\\ &= fghR(X,Y)Z. &(10.58)\end{aligned}$$

Thus,

$$R(f_m,f_n)f_p = \frac{\partial x^a}{\partial y^m}\frac{\partial x^b}{\partial y^n}\frac{\partial x^c}{\partial y^p}R(e_a,e_b)e_c. \tag{10.59}$$

10.16 Problem 16

Prove the first and second Bianchi identities for a general connection. Reduce'em to the Leì-Civita case.

Answer

I'M WORKING ON IT...

10.17 Problem 17

Show that the covariant divergence is

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}V^\mu). \tag{10.60}$$

Answer

By definition,

$$\begin{aligned}
\nabla_\mu V^\mu &= \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda \\
&= \partial_\mu V^\mu + \frac{1}{\sqrt{g}}\partial_\lambda(\sqrt{g})V^\lambda \\
&= \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}V^\mu).
\end{aligned} \tag{10.61}$$

We've used the identity $\Gamma^\lambda_{\lambda\mu} = \partial_\mu \ln(\sqrt{g}) = \frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}$. A rough way of showing this identity is to give a diagonal metric. Then

$$\begin{aligned}
\Gamma^\lambda_{\mu\lambda} &= \frac{1}{2}g^{\mu\mu}g_{\mu\mu,\mu} \\
&= \frac{1}{2}\frac{g_{\mu\mu,\mu}}{g_{\mu\mu}} \\
&= \frac{1}{2}\partial_\mu \ln g \\
&= \partial_\mu \ln\sqrt{g}.
\end{aligned} \tag{10.62}$$

10.18 Problem 18

Given the metric $\mathbf{d}s^2 = \frac{\mathbf{d}x^2+\mathbf{d}y^2}{y^2}$, Compute the connection coefficients and the timelike geodesic curves.

Answer

With the given metric, we can construct a Lagrangian,

$$L = \frac{1}{2}t^{-2}(\dot{x}^2 - \dot{t}^2), \tag{10.63}$$

then, from the Euler-Lagrange equations, we get

$$\begin{aligned} \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{d}{ds}(t^{-2}\dot{x}) \\ &= -2t^{-3}\dot{t}\dot{x} + t^{-2}\ddot{x} = 0. \end{aligned} \tag{10.64}$$

$$\begin{aligned} \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{t}}\right) - \frac{\partial L}{\partial t} &= \frac{d}{ds}\left(-\frac{\dot{t}}{t^2}\right) + 2t^{-3}(\dot{x}^2 - \dot{t}^2) \\ &= -t^{-2}\ddot{t} + t^{-3}\dot{x}^2 = 0. \end{aligned} \tag{10.65}$$

Therefore,

$$\ddot{x} - 2t^{-1}\dot{t}\dot{x} = 0, \tag{10.66}$$

and

$$\ddot{t} - t^{-1}(\dot{x}^2 + \dot{t}^2) = 0. \tag{10.67}$$

The last equations yield

$$\Gamma^x_{xt} = \Gamma^x_{tx} = \frac{1}{t} \quad \text{and} \quad \Gamma^t_{tt} = \Gamma^t_{xx} = -\frac{1}{t}. \tag{10.68}$$

In order to write down the geodesic equations, we should take in account the Lorentian signature of the metric, then, by recall $\dot{x}^a = u^a$, so

$$g_{ab}u^a u^b = \frac{\dot{x}^2 - \dot{t}^2}{t^2} = \kappa, \tag{10.69}$$

with

$$\kappa = \begin{cases} -1 & ; \quad \text{for timelike geodesics} \\ 0 & ; \quad \text{for null geodesics} \\ 1 & ; \quad \text{for spacelike geodesics} \end{cases} \tag{10.70}$$

A part is still missing.

10.19 Problem 19

Consider the Poincarè metric

$$\mathbf{d}s^2 = \frac{\mathbf{d}x^2 + \mathbf{d}y^2}{y^2},$$

defined for $y > 0$.

Determine the geodesics. Compute the Ricci scalar and compare with a 2-sphere.

Answer

With the metric we can construct a Lagrangian,

$$L = \frac{\dot{x}^2 + \dot{y}^2}{2y^2}. \tag{10.71}$$

From the Euler Lagrange equations, we have

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}}\right) \sim \ddot{x} - 2y^{-1}\dot{x}\dot{y} = 0 \tag{10.72}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} \sim \ddot{y} + y^{-1}(\dot{x}^2 - \dot{y}^2) = 0. \tag{10.73}$$

Therefore,

$$\Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}, \quad \text{and} \quad -\Gamma^y_{yy} = \Gamma^y_{xx} = \frac{1}{y}. \tag{10.74}$$

Thus,

$$\partial_y \Gamma^x_{xy} = \partial \Gamma^x_{yx} = y^{-2} \tag{10.75}$$

$$\partial_y \Gamma^y_{xx} = -y^{-2} \tag{10.76}$$

$$\partial_y \Gamma^y_{yy} = y^{-2}. \tag{10.77}$$

We must remind that in 1-dimension, the Riemann tensor has only one independent component, the component$xyxy$. Then,

$$\begin{aligned} R^x{}_{yxy} &= \partial_y \Gamma^x_{xy} - \cancel{\partial_x \Gamma^x_{yy}} + \Gamma^m_{xy}\Gamma^x_{ym} - \Gamma^m_{yy}\Gamma^x_{xm} \\ &= \frac{1}{y^2}. \end{aligned} \tag{10.78}$$

Now,

$$R_{xyxy} = g_{xx} R^x_{yxy} = \frac{1}{y^4}. \tag{10.79}$$

Since in 2-dimensions

$$R_{abcd} = \frac{R}{2}(g_{ac}g_{bd} - g_{ad}g_{bc}), \tag{10.80}$$

it follows that for the Poincarè metric

$$R = 2, \tag{10.81}$$

which is the same Ricci scalar than for a 2-sphere of radius one.

10.20 Problem 20

Prove that if a geodesic is time-like some of this points is so everywhere.

Answer

Note in equation (10.69) that the LHS in fact depend on the point, but in not the case for the RHD. Thus if in a point the geodesic is time-like, it'll be so in any other point. Same for light-like or time-like.

10.21 Problem 21

A Killing vector satisfy $\xi_{a;b} + \xi_{b;a} = 0$. Show that is possible to construct conserved quantities by considering objects $u^a \xi_a$, with u^a the 4-velocity of the particle.

Answer

A quantity is said to be conserved if its derivative vanish, so, we must expect that

$$\nabla_b (u^a \xi_a) = 0. \tag{10.82}$$

Then,

$$\nabla_b (u^a \xi_a) = (\nabla_b u^a) \xi_a + u^a (\nabla_b \xi_a). \tag{10.83}$$

By the Killing vector equation, it follows that the bracket in the last term is antisymmetric under change $a \leftrightarrow b$, so if we contract the whole expression with u^b, we get

$$u^b \nabla_b (u^a \xi_a) = (u^b \nabla_b u^a) \xi_a + u^b u^a (\nabla_b \xi_a) = 0, \tag{10.84}$$

the latter term vanish because of the symmetries, and the former because of the geodesic equation of the particle.

10.22 Problem 22

What is the Riemann for a 1-dimensional manifold? Express the Riemann and Ricci tensors in term of the metric for the 2- and 3-dimensional case.

Answer

Because of the symmetries of the Riemann tensor, it's identically zero for a 1-dimensional manifold. In fact, any 1-dimensional manifold is diffeomorphic to $\mathbb{R}^1$ or S^1 depending if it's compact or not.

A PART IS STILL MISSING.

10.23 Problem 6.10

Show that $\nabla_c\nabla_d X^a_b - \nabla_d\nabla_c X^a_b = R^a{}_{ecd}X^e_b + R^e{}_{bcd}X^a_e$.

Answer

Let's compute the first term.

$$\begin{aligned}
\nabla_c\nabla_d X^a_b &= \nabla_c(\partial_d X^a_b + \Gamma^a_{dm}X^m_b - \Gamma^m_{db}X^a_m) \\
&= \partial_c\partial_d X^a_b - \Gamma^m_{cd}\partial_m X^a_b + \Gamma^a_{cm}\partial_d X^m_b - \Gamma^m_{cb}\partial_d X^a_m \\
&\quad +\partial_c(\Gamma^a_{dm}X^m_b) - \Gamma^n_{cd}\Gamma^a_{nm}X^m_b + \Gamma^a_{cn}\Gamma^n_{dm}X^m_b - \Gamma^n_{cb}\Gamma^a_{dm}X^m_n \\
&\quad -\partial_c(\Gamma^m_{db}X^a_m) + \Gamma^n_{cd}\Gamma^m_{nb}X^a_m - \Gamma^a_{cn}\Gamma^m_{db}X^n_m + \Gamma^n_{cb}\Gamma^m_{dn}X^a_n.
\end{aligned} \tag{10.85}$$

Since the second term is interchanging $c \leftrightarrow d$, thus

$$\begin{aligned}
[\nabla_c, \nabla_d]X^a_b &= -T^m_{cd}(\partial_m X^a_b - \Gamma^n_{mb}X^a_n + \Gamma^a_{mn}X^n_b) \\
&\quad +X^m_b(\partial_c\Gamma^a_{dm} - \partial_d\Gamma^a_{cm} + \Gamma^a_{cn}\Gamma^n_{dm} - \Gamma^a_{dn}\Gamma^n_{cm}) \\
&\quad -X^a_m(\partial_c\Gamma^m_{db} - \partial_d\Gamma^m_{cb} + \Gamma^m_{cn}\Gamma^n_{db} - \Gamma^m_{dn}\Gamma^n_{cb}) \\
&= -T^m_{cd}\nabla_m X^a_b + R^a{}_{mcd}X^m_b + R^m{}_{bcd}X^a_m.
\end{aligned} \tag{10.86}$$

10.24 Problem 6.11

Show that

$$\nabla_X(\nabla_Y Z^a) - \nabla_Y(\nabla_X Z^a) - \nabla_{[X,Y]}Z^a = R^a_{bcd}Z^b X^c Y^d. \tag{10.87}$$

Answer

Since $R(X, Y, Z) = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$ is a tensor, it satisfy that

$$R(X, Y, Z) = X^a Y^b Z^c R(e_a, e_b)e_c. \tag{10.88}$$

Moreover,

$$\begin{aligned}
\left\langle dx^d, R(e_a, e_b)e_c\right\rangle &= \left\langle dx^d, [\nabla_{e_a}, \nabla_{e_b}]e_c\right\rangle \\
&= \partial_a\Gamma^d_{bc} - \partial_b\Gamma^d_{ac} + \Gamma^d_{am}\Gamma^m_{bc} - \Gamma^d_{bm}\Gamma^m_{ac} \\
&= R^d_{cab}.
\end{aligned} \tag{10.89}$$

Thus,

$$\nabla_X(\nabla_Y Z^d) - \nabla_Y(\nabla_X Z^d) - \nabla_{[X,Y]}Z^d = R^d_{cab}Z^c X^a Y^b. \tag{10.90}$$

10.25 Problem 6.14

The line elements or $\mathbb{R}^3$ in cartesian, cylindrical and spherical coordinates are given respectively by

- $ds^2_{(1)} = dx^2 + dy^2 + dz^2$.
- $ds^2_{(2)} = dr^2 + r^2 d\theta^2 + dz^2$.
- $ds^2_{(3)} = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$.

Find g_{ab}, g^{ab} and g.

Answer

Since $ds^2 = g_{ab}dx^a dx^b$, it follows that,

$$g_{(1)ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10.91}$$

$$g_{(2)ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10.92}$$

$$g_{(3)ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{10.93}$$

$$g^{ab}_{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10.94}$$

$$g^{ab}_{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10.95}$$

$$g^{ab}_{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix} \tag{10.96}$$

$$g_{(1)} = 1 \tag{10.97}$$

$$g_{(2)} = r^2 \tag{10.98}$$

$$g_{(3)} = r^4\sin^2\theta. \tag{10.99}$$

10.26 Problem 6.17

Find the geodesic equation for $\mathbb{R}^3$ in cylindrical coordinates.

Answer

From (10.92) we get,

$$\ddot{r} - r\dot{\theta}^2 = 0 \tag{10.100}$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} = 0 \tag{10.101}$$

$$\ddot{z} = 0 \tag{10.102}$$

10.27 Problem 6.18

Consider a space with coordinates $x^a = (x, y, z)$ and line element

$$ds^2 = dx^2 + dy^2 - dz^2. \tag{10.103}$$

Prove that the null geodesics are given by

$$x = lu + l', \quad y = mu + m', \quad z = nu + n', \tag{10.104}$$

where u is a parameter and l, l', m, m', n, n' are arbitrary constants satisfying $l^2 + m^2 - n^2 = 0$.

Answer

Since

$$ds^2 = dx^2 + dy^2 - dz^2, \tag{10.105}$$

we can write a Lagrangian for this line element,

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 - \dot{z}^2). \tag{10.106}$$

Thus, the geodesic equations are,

$$\ddot{x} = 0, \quad \ddot{y} = 0, \quad \ddot{z} = 0. \tag{10.107}$$

Then, the solution of the geodesic equations are,

$$x = lu + l' \tag{10.108}$$

$$y = mu + m' \tag{10.109}$$

$$z = nu + n'. \tag{10.110}$$

Moreover, for null geodesics from (10.69), we know that,

$$g_{ab}\dot{x}^a\dot{x}^b = 0, \tag{10.111}$$

so,

$$l^2 + m^2 - n^2 = 0. \tag{10.112}$$

10.28 Problem 6.19

Prove that $\nabla_c g_{ab} = 0$. Deduce that $\nabla_b X_a = g_{ac}\nabla_b X^c$.

Answer

Since

$$\Gamma^a_{bc} = \frac{1}{2}g^{am}(g_{mc,b} + g_{mb,c} - g_{bc,m}), \tag{10.113}$$

it follows that,

$$\begin{aligned}\nabla_c g_{ab} &= \partial_c g_{ab} - \frac{1}{2}(g_{cb,a} + g_{ab,c} - g_{ac,b}) - \frac{1}{2}(g_{ba,c} + g_{ca,b} - g_{bc,a}) \\ &= 0.\end{aligned} \tag{10.114}$$

I. e., the metric tensor is co-variantly constant, so,

$$\nabla_b X_a = \nabla_b(g_{ac}X^c) = g_{ac}\nabla_b X^c. \tag{10.115}$$

10.29 Problem 6.20

Suppose we have an arbitrary symmetric connection Γ^a_{bc} satisfying $\nabla g = 0$. Deduce that Γ must be the metric connection.

Answer

Since,

$$\begin{aligned}\nabla_a g_{bc} &= \partial_a g_{bc} - \Gamma^m_{ab}g_{mc} - \Gamma^m_{ac}g_{bm} \\ \nabla_b g_{ac} &= \partial_b g_{ac} - \Gamma^m_{ba}g_{mc} - \Gamma^m_{bc}g_{am} \\ -\nabla_c g_{ab} &= -\partial_c g_{ab} + \Gamma^m_{ca}g_{ma} + \Gamma^m_{cb}g_{am}.\end{aligned} \tag{10.116}$$

By adding all eqs. (10.116), we get,

$$0 = (\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}) - 2g_{cm}\Gamma^m_{bc} \tag{10.117}$$

$$\Rightarrow \Gamma^m_{ab} = \frac{1}{2}g^{mc}(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}). \tag{10.118}$$

10.30 Problem 6.25

Show that $G_{ab} = 0$ iff $R_{ab} = 0$.

Answer

Obviously, if $R_{ab} = 0$ then $G_{ab} = 0$. Therefore, let's show the other way.

Since

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R, \tag{10.119}$$

we have that for $a \neq b$, $G_{ab} = 0 \Rightarrow R_{ab} = 0$. Moreover, if $a = b$, the trace of G yields

$$R = \frac{n}{2} R, \tag{10.120}$$

but for $n \neq 2$, (10.120) doesn't hold, therefore,

$$G_{ab} = 0 \quad \text{iff} \quad R_{ab} = 0. \tag{10.121}$$

Part V

Quantum Field Theory

Chapter 11

QFT I 6

11.1 Problem 1

Consider the massless K.G. action,

$$S = \frac{1}{2}\int d^4x\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi, \tag{11.1}$$

and the dilatation transformation,

$$\begin{aligned} x^\mu &\rightarrow x'^\mu = E^\alpha x^\mu, & (11.2)\\ \phi(x) &\rightarrow \phi'(x') = e^{-d_\phi\alpha}\phi(x). & (11.3)\end{aligned}$$

1. Show that this transformation is a global symmetry for an appropiate choice of the parametre d_ϕ. Find the conserved current associated to this symmery, and verify it's conserved when ϕ is a classical solution.

2. Show that if there is a mass term then dilatations are no mero a symmetry. Show also that in contrast, a term $V(\phi) = \lambda\phi^4$ doesn't spoil the symmetry. What should be the dimension of the coupling constant λ?

Answer

Let us find the apropiate value of the constant d_ϕ. In order for dilatations being a symmetry, $s = s'$. Then

$$\begin{aligned} S \rightarrow S' &= \frac{1}{2}\int d^4xe^{4\alpha}\eta^{\mu\nu}e^{(2-2d_\phi)\alpha}\partial_\mu\phi\partial_\nu\phi \\ &= \frac{1}{2}\int d^4x\eta^{\mu\nu}e^{(2-2d_\phi)\alpha}\partial_\mu\phi\partial_\nu\phi \\ &= S \qquad \text{if } d_\phi = 1. \end{aligned} \tag{11.4}$$

Thus,

$$\phi(x) \to \phi'(x') = e^{-\alpha}\phi(x). \tag{11.5}$$

Now, if we restrict ourselves to infinitesimal transformations, we can find the Nöther conserved current by using the formulae derived in class,

$$\begin{aligned} j_a^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\left[\mathscr{A}_a^\nu \partial_\nu \phi_i - \mathscr{F}_{i,a}\right] - \mathscr{A}_a^\mu \mathcal{L} \\ &= \mathscr{A}_{a\nu}\Theta^{\mu\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\mathscr{F}_{i,a}. \end{aligned} \tag{11.6}$$

Then, for our particular sut up, we have,

$$\delta x^\mu = \alpha x^\mu = \epsilon^a \mathscr{A}_a^\mu, \tag{11.7}$$

$$\delta\phi = -\alpha\phi = \epsilon^a \mathscr{F}_{i,a}, \tag{11.8}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi. \tag{11.9}$$

In the above equations, $i = \phi$, $a = 1$, $\epsilon = \alpha$, $\mathscr{A}^\mu = x^\mu$ and $\mathscr{F}_{i,a} = -\phi$.

On the other hand, for the massless KG theory,

$$\Theta^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi. \tag{11.10}$$

$$\begin{aligned} \Longrightarrow \quad \partial_\mu\Theta^{\mu\nu} &= \partial^\nu\phi\Box\phi + \partial^\mu\phi\partial_\mu\partial^\nu\phi - \eta^{\mu\nu}\partial_\mu\partial^\alpha\phi\partial_\alpha\phi \\ &= \partial^\nu\phi\Box\phi + \partial^\mu\phi\partial_\mu\partial^\nu\phi - \partial^\nu\partial^\alpha\phi\partial_\alpha\phi \\ &= \partial^\nu\phi\Box\phi \\ &= 0. \end{aligned} \tag{11.11}$$

In the last line we have used the eqs. of motion.

From (11.10), we calculate the trace,

$$\Theta_\mu^\mu = \partial_\mu\phi\partial^\mu\phi - 2\partial_\mu\phi\partial^\mu\phi = -\partial_\mu\phi\partial^\mu\phi. \tag{11.12}$$

Thus,

$$\begin{aligned} \partial_\mu j^\mu &= \Theta_\mu^\mu + \mathscr{A}_\nu\partial_\mu\Theta^{\mu\nu} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\mathscr{F}\right) \\ &= -\partial^\mu\phi\partial_\mu\phi + \partial_\mu(\phi\partial^\mu\phi) \\ &= \phi\Box\phi \\ &= 0. \end{aligned} \tag{11.13}$$

.

We saw that the massless KG equation is indeed dilatation invariant. If we add a mass term or a $\lambda\phi^4$ potential, those do not affect the invariance of the kinetic term, so we can verify its invariance (or not) individually. Then,

$$S_m \rightarrow S_m{}' \frac{1}{2} \int d^4 x e^{4\alpha} m^2 \phi^2 e^{-2\alpha} \neq S. \tag{11.14}$$

$$S_\lambda \rightarrow S_\lambda{}' = \lambda \int d^4 x e^{4\alpha} \phi^4 e^{-4\alpha} = S_\lambda. \tag{11.15}$$

Thus, the mass term spoils the dilatation symmetry whilst the quartic potential on phi doesn't. Additionally, the λ must be adimensional, so that the ϕ^4 match the dimensions of the volume form.

11.2 Problem 2

Consider th elagrangian of QED, for massless fermions, and the dilatation transformation,

$$x^\mu \rightarrow e^\alpha x^\mu \tag{11.16}$$

$$A_\mu \rightarrow e^{-d_A \alpha} A_\mu \tag{11.17}$$

$$\psi \rightarrow e^{-d_\psi \alpha} \psi. \tag{11.18}$$

1. Find the values of d_A and d_ψ in order for dilatations being a symmetry.

2. Calculate the conserved current for this symmetry and express it in terms os the energy-momentum tensor of the theory. Verify that that its conservation is a consecuence of the fact that the energy-momentum is traceless in the massless theory.

3. Include the mass term for fermions in the lagrangian. Compute the new Nöther corrent and verify it's not conserved, relate its divergence with the trace of the energy-momentum tensor.

Answer

As we've discussed in the above problem, we can check the dilatation invariace term by term. Since the lagrangian of QED is

$$\mathscr{L}_{QED} = \imath\bar{\psi}(\partial\!\!\!/ - m)\psi - \frac{1}{4}F^2 - q\bar{\psi}\gamma^\mu A_\mu \psi, \tag{11.19}$$

let's check first the EM lagrangian.

Each term of this lagrangian contents a couple of derivatis and a couple of A's, therefore,

$$\begin{aligned} S_{EM} \to S'_{EM} &= -\int d^4x\, e^{4\alpha}\frac{1}{4}F^2 e^{-2-2d_A} \\ &= S \qquad \text{if } d_A = 1. \end{aligned} \tag{11.20}$$

$$\begin{aligned} S_D \to S'_D &= \int d^4x e^{4\alpha}\bar{\psi}\gamma^\mu\partial_\mu\psi e^{-\alpha-2\alpha} \\ &= S \qquad \text{if } d_\psi = \frac{3}{2}. \end{aligned} \tag{11.21}$$

$$S_I \to S'_I = -\int e^{4\alpha} q\bar{\phi}\gamma^\mu A_\mu \phi e^{-4\alpha} = S_I. \tag{11.22}$$

Then, the right values of the parameters are,

$$d_A = 1 \qquad \text{and} \qquad d_\psi = \frac{3}{2}. \tag{11.23}$$

Let's calculate the energy-momentum tensor using,

$$\Theta^{\mu\nu} = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\partial^\nu\phi_i - \eta^{\mu\nu}\mathscr{L}. \tag{11.24}$$

Then,

$$\Theta^{\mu\nu} = \imath\bar{\psi}\partial^\mu\gamma^\nu\psi - F^{\mu\lambda}\partial^\nu A_\lambda - \eta_{\mu\nu}\mathscr{L}_{QED}. \tag{11.25}$$

And, for infinitesimal transformation, we have,

$$\delta x^\mu = \alpha x^\mu, \tag{11.26}$$

$$\delta A^\mu = -\alpha A^\mu, \tag{11.27}$$

$$\delta\psi = -\frac{3}{2}\alpha\phi. \tag{11.28}$$

Following the usual method, the conserved current must be,

$$j^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\left[\mathscr{A}^\nu\partial_\nu\phi_i - \mathscr{F}_i\right] - \mathscr{A}^\mu\mathscr{L} \tag{11.29}$$

$$= \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\left[\mathscr{A}_\nu\partial^\nu\phi_i - \mathscr{F}_i\right] - \mathscr{A}_\nu\eta^{\mu\nu}\mathscr{L} \tag{11.30}$$

$$= -\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_i)}\mathscr{F}_i + \mathscr{A}_\nu\Theta^{\mu\nu} \tag{11.31}$$

11.3 Problem 3

Consider the Dirac lagrangians

$$\mathscr{L} = \bar{\psi}(\imath\partial\!\!\!/ - m)\psi \tag{11.32}$$

and

$$\mathscr{L}' = \bar{\psi}\left(\frac{\imath}{2}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu} - m\right)\psi. \tag{11.33}$$

Verify that they are classically equivalents. Compute the energy-momentum tensor to show they're different, but their conserved charges are the same.

Answer

Consider the prime action,

$$\begin{aligned} S' &= \int d^4x\bar{\psi}\left(\frac{\imath}{2}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu} - m\right)\psi \\ &= \int d^4x\left(\frac{\imath}{2}\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - \frac{\imath}{2}\partial_{\mu}\bar{\psi}\gamma^{\mu}\psi - m\bar{\psi}\psi\right) \\ &= \int d^4x\left(\imath\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - \frac{\imath}{2}\partial_{\mu}(\bar{\psi}\gamma^{\mu}\psi) - m\bar{\psi}\psi\right) \\ &= \int d^4x\left(\imath\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi\right) \\ &= S. \end{aligned} \tag{11.34}$$

We have made use of the fact that $\phi \to 0$ as long as $|x| \to \infty$

Hence, classically both lagrangians are equivalent.

Now, let's calculate their energy-momentum tensors. Since,

$$\Theta^{\mu\nu} = \frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\phi_i)}\partial^{\nu}\phi_i - \eta^{\mu\nu}\mathscr{L}, \tag{11.35}$$

then,

$$\begin{aligned} \Theta_1^{\mu\nu} &= \imath\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi - \eta^{\mu\nu}\left(\bar{\psi}(\imath\partial\!\!\!/ - m)\psi\right), \\ &= \imath\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi \end{aligned} \tag{11.36}$$

$$\begin{aligned} \Theta_2^{\mu\nu} &= \frac{\imath}{2}\left(\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi - \partial^{\nu}\bar{\psi}\gamma^{\mu}\psi\right) - \eta^{\mu\nu}\bar{\psi}\left(\frac{\imath}{2}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu} - m\right)\psi \\ &= \frac{\imath}{2}\left(\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi - \partial^{\nu}\bar{\psi}\gamma^{\mu}\psi\right). \end{aligned} \tag{11.37}$$

We have used the equations of motion for both lagrangians,

$$(\imath\partial\!\!\!/ - m)\psi = 0; \qquad \imath\partial_{\mu}\bar{\psi}\gamma^{\mu} + m\bar{\psi} = 0, \tag{11.38}$$

That's nothing but evaluate $\phi \to \phi_{cl}$.

The conserved currents are,

$$P_1^\mu = \imath \int d^3x\, \bar{\psi}\gamma^0\partial^\mu\psi. \tag{11.39}$$

$$\begin{aligned} P_2^\mu &= \int d^3x \frac{\imath}{2}\left(\bar{\psi}\gamma^0\partial^\mu\psi - \partial^\mu\bar{\psi}\gamma^0\psi\right) \\ &= \imath \int d^3x\, \bar{\psi}\gamma^0\partial^\mu\psi + (B.T.) \end{aligned} \tag{11.40}$$

Thus, the conserved charges are the same, as we expected.

Chapter 12

QFT I 7

12.1 Problem 1

- Show that the conserved $U(1)$-charge in the quantum theory of a free complex scalar,

$$Q_{U(1)} = \imath \int d^3x \ : \phi^\dagger \overleftrightarrow{\partial}_0 \phi :, \tag{12.1}$$

  is given by

$$Q_{U(1)} = \int \frac{d^p}{(2\pi)^3} \left(a_p^\dagger a_p - b_p^\dagger b_p \right). \tag{12.2}$$

- Has the state $a_p^\dagger b_q^\dagger |0\rangle$ a defined charge?
- Calculate the commutator $[Q_{U(1)}, H]$, where H is the Hamiltonian of the theory.

Answer

We already know that the modes expansion of the complex scalar field is

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-\imath p\cdot x} + b_p^\dagger e^{\imath p\cdot x} \right\} \Big|_{p^0=E_p} \tag{12.3}$$

and

$$\phi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ b_p e^{-\imath p\cdot x} + a_p^\dagger e^{\imath p\cdot x} \right\} \Big|_{p^0=E_p} . \tag{12.4}$$

Then,

$$\partial_0\phi(x) = -\imath \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} \left\{ a_p e^{-\imath p\cdot x} - b_p^\dagger e^{\imath p\cdot x} \right\} \Big|_{p^0=E_p} , \tag{12.5}$$

and

$$\phi(x) = -\imath \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} \left\{ b_p e^{-\imath p\cdot x} - a_p^\dagger e^{\imath p\cdot x} \right\} \Bigg|_{p^0=E_p} . \tag{12.6}$$

Thus

$$\begin{aligned}
Q_{U(1)} &= \imath \int d^3x : \{\phi^\dagger \partial_0 \phi - (\partial_0 \phi^\dagger)\phi\} : \\
&= : \int d^3x \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{2} \sqrt{\frac{E_q}{E_p}} \left[\left\{ b_p e^{-\imath p\cdot x} + a_p^\dagger e^{\imath p\cdot x} \right\} \left\{ a_q e^{-\imath q\cdot x} - b_q^\dagger e^{\imath q\cdot x} \right\} \right. \\
&\quad \left. - \left\{ b_q e^{-\imath q\cdot x} - a_q^\dagger e^{\imath q\cdot x} \right\} \left\{ a_p e^{-\imath p\cdot x} + b_p^\dagger e^{\imath p\cdot x} \right\} \right] : \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{2} (2\pi)^3 \delta^3(p-q) \\
&\quad e^{\imath(E_p - E_q)t} \sqrt{\frac{E_q}{E_p}} : \left\{ -b_p b_q^\dagger + a_p^\dagger a_q - b_q b_p^\dagger + a_q^\dagger a_p \right\} : \\
&= \int \frac{d^3p}{(2\pi)^3} (a_p^\dagger a_p - b_p^\dagger b_p).
\end{aligned} \tag{12.7}$$

Let us now calculate the charge of the state $a_p^\dagger b_q^\dagger |0\rangle$. Then,

$$\begin{aligned}
Q_{U(1)} a_p^\dagger b_q^\dagger |0\rangle &= \int \frac{d^3k}{(2\pi)^3} (a_k^\dagger a_k - b_k^\dagger b_k) a_p^\dagger b_q^\dagger |0\rangle \\
&= a_p^\dagger b_q^\dagger |0\rangle - a_p^\dagger b_q^\dagger |0\rangle \\
&= 0 a_p^\dagger b_q^\dagger |0\rangle ,
\end{aligned} \tag{12.8}$$

therefore, this state has no charge, as it was expected so that each creator operator contribute with opposite charge.

On the other hand,

$$\begin{aligned}
[Q_{U(1)}, H] &= \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{2\pi)^3} E_{p'} \left[a_p^\dagger a_p - b_p^\dagger b_p, a_{p'}^\dagger a_{p'} + b_{p'}^\dagger b_{p'} \right] \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{2\pi)^3} E_{p'} \left\{ \left[a_p^\dagger a_p, a_{p'}^\dagger a_{p'} \right] - \left[b_p^\dagger b_p, b_{p'}^\dagger b_{p'} \right] \right\} \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{2\pi)^3} E_{p'} \left\{ a_p^\dagger \left[a_p, a_{p'}^\dagger \right] a_{p'} + a_{p'}^\dagger \left[a_p^\dagger, a_{p'} \right] a_p \right. \\
&\quad \left. - b_p^\dagger \left[b_p, b_{p'}^\dagger \right] b_{p'} - b_{p'}^\dagger \left[b_p^\dagger, b_{p'} \right] b_p \right\} \\
&= 0.
\end{aligned} \tag{12.9}$$

12.2 Problem 2

In the quantum theory for a free Dirac field, consider the modes expansion

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \left(a_{p,s} u_s(p) e^{-\imath p\cdot x} + b^\dagger_{p,s} v_s(p) e^{\imath p\cdot x} \right)\Big|_{p^0=E_p} . \tag{12.10}$$

Suppose that the spinors u and v are normalized like

$$\bar{u}_s(p) u_{s'}(p) = N_u \delta_{ss'}, \quad \bar{v}_s(p) v_{s'}(p) = -N_v \delta_{ss'}, \tag{12.11}$$

with N_u and N_v are positive factors (possibly depending on p)

- Find the expression for the creation and annihilation operators in term of the field operators ψ and $\bar{\psi}$.
- Compute the whole set of anti-commutator of the creator and annihilator operators.
- How must the normalization constant of spinors be chosen in order to satisfy the relations

$$\{a_{p,s}, a^\dagger_{p',s'}\} = \{b_{p,s}, b^\dagger_{p',s'}\} = (2\pi)^3 \delta^{(3)}(p - p') \delta_{ss'}? \tag{12.12}$$

Answer

Hence,

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \left(a_{p,s} u_s(p) e^{-\imath p\cdot x} + b^\dagger_{p,s} v_s(p) e^{\imath p\cdot x} \right)\Big|_{p^0=E_p} , \tag{12.13}$$

it follows that

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \left(b_{p,s} \bar{v}_s(p) e^{-\imath p\cdot x} + a^\dagger_{p,s} \bar{u}_s(p) e^{\imath p\cdot x} \right)\Big|_{p^0=E_p} . \tag{12.14}$$

Now, we can take the Fourier transform of ψ and $\bar{\psi}$,

$$\int d^3x e^{\imath \vec{k}\cdot\vec{x}} \psi(x) = \frac{1}{\sqrt{2E_k}} \sum_{s=1,2} \left(a_{k,s} u_s(k) e^{-\imath E_k t} + b^\dagger_{-k,s} v_s(-k) e^{\imath E_k t} \right) \tag{12.15}$$

$$\int d^3x e^{\imath \vec{k}\cdot\vec{x}} \bar{\psi}(x) = \frac{1}{\sqrt{2E_k}} \sum_{s=1,2} \left(b_{k,s} \bar{v}_s(k) e^{-\imath E_k t} + a^\dagger_{-k,s} \bar{u}_s(-k) e^{\imath E_k t} \right). \tag{12.16}$$

By using the normalization conditions, we get,

$$
\begin{aligned}
a_{k,s} &= \frac{\sqrt{2E_k}}{N_u} \int d^3x \, e^{\imath k\cdot x} \bar{u}_s(k)\psi(x) && (12.17)\\
a^\dagger_{k,s} &= \frac{\sqrt{2E_k}}{N_u} \int d^3x \, e^{-\imath k\cdot x} u_s(k)\bar{\psi}(x) && (12.18)\\
b_{k,s} &= -\frac{\sqrt{2E_k}}{N_v} \int d^3x \, e^{\imath k\cdot x} v_s(k)\bar{\psi}(x) && (12.19)\\
a_{k,s} &= -\frac{\sqrt{2E_k}}{N_v} \int d^3x \, e^{-\imath k\cdot x} \bar{v}_s(k)\psi(x). && (12.20)
\end{aligned}
$$

As we can to calculate their anti-commutator relations, we should remind that

$$
\{\psi(t,\vec{x}), \bar{\psi}(t,\vec{y})\} = \delta^{(3)}(x-y), \tag{12.21}
$$

and whatever other anti-commutator vanish.

So,

$$
\begin{aligned}
\{a_{k,s}, a_{k',s'}\} &= 0 && (12.22)\\
\left\{a^\dagger_{k,s}, a^\dagger_{k',s'}\right\} &= 0 && (12.23)\\
\{b_{k,s}, b_{k',s'}\} &= 0 && (12.24)\\
\left\{b^\dagger_{k,s}, b^\dagger_{k',s'}\right\} &= 0 && (12.25)\\
\left\{a_{k,s}, b^\dagger_{k',s'}\right\} &= 0 && (12.26)\\
\left\{b_{k,s}, a^\dagger_{k',s'}\right\} &= 0. && (12.27)
\end{aligned}
$$

Additionally, we know that

$$
\bar{u}_s(p)v_{s'}(p) = \bar{v}_s(p)u_{s'}(p) = 0. \tag{12.28}
$$

Then,

$$
\begin{aligned}
\{a_{k,s}, b_{k',s'}\} &= 0 && (12.29)\\
\left\{a^\dagger_{k,s}, b^\dagger_{k',s'}\right\} &= 0. && (12.30)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\left\{a_{k,s}, a^\dagger_{k',s'}\right\} &= \frac{2\sqrt{EE'}}{N_u^2} \int d^3x d^3y \bar{u}_s(k)u_{s'}(k')e^{\imath(k\cdot x - k'\cdot y)} \{\psi(t,x)\bar{\psi}(t,y)\}\\
&= \frac{2\sqrt{EE'}}{N_u^2} \bar{u}_s(k)u_{s'}(k') \int d^3x e^{\imath(k-k')\cdot x}\\
&= (2\pi)^3 \frac{2\sqrt{EE'}}{N_u^2} \bar{u}_s(k)u_{s'}(k')\delta^{(3)}(k-k')\\
&= (2\pi)^3 \frac{2E}{N_u} \delta_{ss'}\delta^{(3)}(k-k'), && (12.31)
\end{aligned}
$$

and

$$
\begin{aligned}
\left\{b_{k,s}, b^{\dagger}_{k',s'}\right\} &= \frac{2\sqrt{EE'}}{N_v^2}\int d^3x d^3y \bar{v}_s(k) v_{s'}(k') e^{\imath(k\cdot x - k'\cdot y)}\left\{\psi(t,x)\bar{\psi}(t,y)\right\} \\
&= \frac{2\sqrt{EE'}}{N_v^2}\bar{v}_s(k) v_{s'}(k')\int d^3x e^{\imath(k-k')\cdot x} \\
&= (2\pi)^3\frac{2\sqrt{EE'}}{N_v^2}\bar{v}_s(k) v_{s'}(k')\delta^{(3)}(k-k') \\
&= -(2\pi)^3\frac{2E}{N_u}\delta_{ss'}\delta^{(3)}(k-k'). \qquad (12.32)
\end{aligned}
$$

12.3 Problem 3

Show that the quantity $E_p\delta^{(3)}(p-q)$ is Lorentz invariant, therefore the 1-particle states $(2E_p)^{1/2}a_p^{\dagger}|0\rangle$ have an Lorentz invariant normalization.

Answer

First, we note that the rotacional part of the Lorentz group keeps the quantity, we are interested on, invariant, so that the energy remains the same and the determinante of a rotation matrix (for a proper rotation) is 1. Therefore, we can consentrate just on the boost transformation.

Actually, we know how a delta function transform

$$
\delta(f(x) - f(x_0)) = \sum_i \frac{1}{|f'(x_i)|}\delta(x - x_i), \qquad (12.33)
$$

with x_i are the zeroes of the funtion f.

Without loss of generality, we can onsider a boost in the x direction, s.t. $p' = p'_x = \gamma(p + \beta E)$ and $E' = \gamma(E + \beta p)$. Then,

$$
\begin{aligned}
E\delta^{(3)}(p-q) &= E\delta^{(3)}(p'-q')\frac{dp'}{dp} \\
&= E\delta^{(3)}(p'-q')\gamma\left(1 + \beta\frac{dE}{dp}\right) \\
&= E\delta^{(3)}(p'-q')\frac{\gamma}{E}(E + \beta p) \\
&= E\delta^{(3)}(p'-q')\frac{E'}{E} \\
&= E'\delta^{(3)}(p'-q'). \qquad (12.34)
\end{aligned}
$$

Thus, the quantity $E\delta^{(3)}(p-q)$ in Lorentz invariant.

Furthermore, for a one-particle state,$(2E_p)^{1/2}a_p^\dagger\,|0\rangle$, we have

$$\left\langle 0\left|(2E_q)^{1/2}a_q(2E_p)^{1/2}a_p^\dagger\right|0\right\rangle = 2E_p(2\pi)^3\delta^{(3)}(p-q), \tag{12.35}$$

which we have proved to be Lorentz invariant.

12.4 Problem 4

For the electromagnetic field quantized in the radiation gauge, show that

- $$[A^i(t,\vec{x}), E^j(t,\vec{y})] = -\imath\int\frac{d^3k}{(2\pi)^3}e^{\imath\vec{k}\cdot(\vec{x}-\vec{y})}\left(\delta^{ij}-\frac{k^ik^j}{|\vec{k}|^2}\right).$$

- $$H = \int\frac{d^3k}{(2\pi)^3}\sum_{\lambda=1,2}\omega_k a_{k,\lambda}^\dagger a_{k,\lambda},$$

  and

  $$\vec{P} = \int\frac{d^3k}{(2\pi)^3}\sum_{\lambda=1,2}\vec{k}a_{k,\lambda}^\dagger a_{k,\lambda}.$$

- $$S^{ij} = \imath\int\frac{d^3k}{(2\pi)^3}\sum_{\lambda,\lambda'}\left[\epsilon_{\lambda'}^i(k)\epsilon_\lambda^{j*}-\epsilon_\lambda^{i*}(k)\epsilon_{\lambda'}^j\right]a_{k,\lambda}^\dagger a_{k,\lambda'}.$$

Answer

The modes expansion of the vector potential is

$$A^i(x) = \int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\sum_{\lambda=1,2}\left\{\epsilon_\lambda^i(k)a_{k,\lambda}e^{-\imath k\cdot x}+\epsilon_\lambda^{i*}(k)a_{k,\lambda}^\dagger e^{\imath k\cdot x}\right\}\Bigg|_{k^0=\omega_k}. \tag{12.36}$$

Then,

$$E^i(x) = \imath\int\frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_k}{2}}\sum_{\lambda=1,2}\left\{\epsilon_\lambda^i(k)a_{k,\lambda}e^{-\imath k\cdot x}-\epsilon_\lambda^{i*}(k)a_{k,\lambda}^\dagger e^{\imath k\cdot x}\right\}\Bigg|_{k^0=\omega_k}. \tag{12.37}$$

From these equations, it follows that,

$$\begin{aligned}
[A^i(t,\vec{x}), E^j(t,\vec{y})] &= \imath \int \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{2}\sqrt{\frac{\omega_q}{\omega_k}} \\
&\sum_{\lambda,\lambda'}\Big\{-\epsilon^i_\lambda(k)\epsilon^{j*}_{\lambda'}(q)[a_{k,\lambda}, a^\dagger_{q,\lambda'}]e^{-\imath(\omega_k-\omega_q)t}e^{\imath(\vec{k}\cdot\vec{x}-\vec{q}\cdot\vec{y})} \\
&+\epsilon^{i*}_\lambda(k)\epsilon^j_{\lambda'}(q)[a^\dagger_{k,\lambda}, a_{q,\lambda'}]e^{\imath(\omega_k-\omega_q)t}e^{-\imath(\vec{k}\cdot\vec{x}-\vec{q}\cdot\vec{y})}\Big\} \\
&= -\imath\int\frac{d^3k}{(2\pi)^3}\frac{1}{2}\sum_\lambda\left\{\epsilon^i_\lambda(k)\epsilon^{j*}_{\lambda'}(k)+\epsilon^{i*}_\lambda(-k)\epsilon^j_{\lambda'}(-k)\right\}e^{\imath\vec{k}\cdot(\vec{x}-\vec{y})} \\
&= -\imath\int\frac{d^3k}{(2\pi)^3}e^{\imath\vec{k}\cdot(\vec{x}-\vec{y})}\left(\delta^{ij}-\frac{k^ik^j}{|\vec{k}|^2}\right) \\
&= -\imath\delta^{ij}_{tr}(\vec{x}-\vec{y})
\end{aligned} \tag{12.38}$$

Now, we shall compute the quantum Hamiltonian of the system,

$$H = \frac{1}{2}\int d^3x : E^2 + B^2 : . \tag{12.39}$$

Since

$$\vec{E}(x) = \imath\int\frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega}{2}}\sum_\lambda\left\{\vec{\epsilon}_\lambda(k)a_{k,\lambda}e^{-\imath k\cdot x}-\vec{\epsilon}^*_\lambda(k)a^\dagger_{k,\lambda}e^{\imath k\cdot x}\right\}, \tag{12.40}$$

and

$$\vec{B}(x) = \imath\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega}}\sum_\lambda\left\{\vec{k}\times\vec{\epsilon}_\lambda(k)a_{k,\lambda}e^{-\imath k\cdot x}-\vec{k}\times\vec{\epsilon}^*_\lambda(k)a^\dagger_{k,\lambda}e^{\imath k\cdot x}\right\}, \tag{12.41}$$

it follows that

$$\begin{aligned}
:E^2: &= -:\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\sqrt{\omega\omega'}\sum_{\lambda,\lambda'}\left\{\vec{\epsilon}_\lambda(k)a_{k,\lambda}e^{-\imath k\cdot x}-\vec{\epsilon}^*_\lambda(k)a^\dagger_{k,\lambda}e^{\imath k\cdot x}\right\} \\
&\left\{\vec{\epsilon}_{\lambda'}(k')a_{k',\lambda'}e^{-\imath k'\cdot x}-\vec{\epsilon}^*_{\lambda'}(k')a^\dagger_{k',\lambda'}e^{\imath k'\cdot x}\right\}: \\
&= -\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\sqrt{\omega\omega'}\sum_{\lambda,\lambda'}\Big\{\vec{\epsilon}_\lambda(k)\cdot\vec{\epsilon}_{\lambda'}(k')a_{k,\lambda}a_{k',\lambda'}e^{-\imath(k+k')\cdot x} \\
&-\vec{\epsilon}_\lambda(k)\cdot\vec{\epsilon}^*_{\lambda'}(k')a^\dagger_{k',\lambda'}a_{k,\lambda}e^{-\imath(k-k')\cdot x}-\vec{\epsilon}^*_\lambda(k)\cdot\vec{\epsilon}_{\lambda'}(k')a^\dagger_{k,\lambda}a_{k',\lambda'}e^{\imath(k-k')\cdot x} \\
&+\vec{\epsilon}^*_\lambda(k)\cdot\vec{\epsilon}^*_{\lambda'}(k')a^\dagger_{k,\lambda}a^\dagger_{k',\lambda'}e^{\imath(k+k')\cdot x}\Big\},
\end{aligned} \tag{12.42}$$

so,

$$\begin{aligned}
\int d^3x:E^2: &= -\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}|\vec{k}|\sum_{\lambda,\lambda'}\Big\{\vec{\epsilon}_\lambda(k)\cdot\vec{\epsilon}_{\lambda'}(-k)a_{k,\lambda}a_{-k,\lambda'}e^{-2\imath\omega t} \\
&-\vec{\epsilon}_\lambda(k)\cdot\vec{\epsilon}^*_{\lambda'}(k)a^\dagger_{k,\lambda'}a_{k,\lambda}-\vec{\epsilon}^*_\lambda(k)\cdot\vec{\epsilon}_{\lambda'}(k)a^\dagger_{k,\lambda}a_{k,\lambda'} \\
&+\vec{\epsilon}^*_\lambda(k)\cdot\vec{\epsilon}^*_{\lambda'}(-k)a^\dagger_{k,\lambda}a^\dagger_{-k,\lambda'}e^{2\imath\omega t}\Big\}.
\end{aligned} \tag{12.43}$$

And

$$
\begin{aligned}
:B^2: \;=\;& -\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{\omega\omega'}}\sum_{\lambda,\lambda'} :\left\{\vec{k}\times\vec{\epsilon}_\lambda(k)a_{k,\lambda}e^{-\imath k\cdot x} - \vec{k}\times\vec{\epsilon}^{\,*}_\lambda(k)a^\dagger_{k,\lambda}e^{\imath k\cdot x}\right\} \\
& \left\{\vec{k'}\times\vec{\epsilon}_{\lambda'}(k')a_{k',\lambda'}e^{-\imath k'\cdot x} - \vec{k'}\times\vec{\epsilon}^{\,*}_{\lambda'}(k')a^\dagger_{k',\lambda'}e^{\imath k'\cdot x}\right\}: \\
=\;& -\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{\omega\omega'}}\sum_{\lambda,\lambda'}\left\{(\vec{k}\times\vec{\epsilon}_\lambda(k))\cdot(\vec{k'}\times\vec{\epsilon}_{\lambda'}(k'))a_{k,\lambda}a_{k',\lambda'}e^{-\imath(k+k')\cdot x}\right. \\
& -(\vec{k}\times\vec{\epsilon}_\lambda(k))\cdot(\vec{k'}\times\vec{\epsilon}^{\,*}_{\lambda'}(k'))a^\dagger_{k',\lambda'}a_{k,\lambda}e^{-\imath(k-k')\cdot x} \\
& -(\vec{k}\times\vec{\epsilon}^{\,*}_\lambda(k))\cdot(\vec{k'}\times\vec{\epsilon}_{\lambda'}(k'))a^\dagger_{k,\lambda}a_{k',\lambda'}e^{\imath(k-k')\cdot x} \\
& \left.+(\vec{k}\times\vec{\epsilon}^{\,*}_\lambda(k))\cdot(\vec{k'}\times\vec{\epsilon}^{\,*}_{\lambda'}(k'))a^\dagger_{k,\lambda}a^\dagger_{k',\lambda'}e^{\imath(k+k')\cdot x}\right\},
\end{aligned}
\tag{12.44}
$$

so that

$$
\begin{aligned}
\int d^3x :B^2: \;=\;& -\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{|\vec{k}|}\sum_{\lambda,\lambda'}\left\{(\vec{k}\times\vec{\epsilon}_\lambda(k))\cdot(-\vec{k}\times\vec{\epsilon}_{\lambda'}(-k))a_{k,\lambda}a_{-k,\lambda'}e^{-2\imath\omega t}\right. \\
& -(\vec{k}\times\vec{\epsilon}_\lambda(k))\cdot(\vec{k}\times\vec{\epsilon}^{\,*}_{\lambda'}(k))a^\dagger_{k,\lambda'}a_{k,\lambda} \\
& -(\vec{k}\times\vec{\epsilon}^{\,*}_\lambda(k))\cdot(\vec{k}\times\vec{\epsilon}_{\lambda'}(k))a^\dagger_{k,\lambda}a_{k,\lambda'} \\
& \left.+(\vec{k}\times\vec{\epsilon}^{\,*}_\lambda(k))\cdot(-\vec{k}\times\vec{\epsilon}^{\,*}_{\lambda'}(-k))a^\dagger_{k,\lambda}a^\dagger_{-k,\lambda'}e^{2\imath\omega t}\right\}.
\end{aligned}
\tag{12.45}
$$

Now, we shall use the vectorial identity

$$
(\vec{k}\times\vec{\epsilon})\cdot(\vec{k}\times\vec{\epsilon'}) = |\vec{k}|^2\vec{\epsilon}\cdot\vec{\epsilon'} - (\vec{k}\cdot\vec{\epsilon})(\vec{k}\cdot\vec{\epsilon'}),
\tag{12.46}
$$

and the polarization conditions

$$
\vec{k}\cdot\vec{\epsilon} = \vec{k}\cdot\vec{\epsilon'} = 0,
\tag{12.47}
$$

we get finally,

$$\begin{aligned}
H &= -\frac{1}{4}\int \frac{d^3k}{(2\pi)^3}|\vec{k}|\sum_{\lambda,\lambda'}\Big\{\vec{\epsilon}_\lambda(k)\cdot\vec{\epsilon}_{\lambda'}(-k)a_{k,\lambda}a_{-k,\lambda'}e^{-2\imath\omega t} \\
&-\vec{\epsilon}_\lambda(k)\cdot\vec{\epsilon}^*_{\lambda'}(k)a^\dagger_{k,\lambda'}a_{k,\lambda} - \vec{\epsilon}^*_\lambda(k)\cdot\vec{\epsilon}_{\lambda'}(k)a^\dagger_{k,\lambda}a_{k,\lambda'} \\
&+\vec{\epsilon}^*_\lambda(k)\cdot\vec{\epsilon}^*_{\lambda'}(-k)a^\dagger_{k,\lambda}a^\dagger_{-k,\lambda'}e^{2\imath\omega t}\Big\} \\
&-\frac{1}{4}\int \frac{d^3k}{(2\pi)^3}|\vec{k}|\sum_{\lambda,\lambda'}\Big\{-\vec{\epsilon}_\lambda(k)\cdot\vec{\epsilon}_{\lambda'}(-k)a_{k,\lambda}a_{-k,\lambda'}e^{-2\imath\omega t} \\
&-\vec{\epsilon}_\lambda(k)\cdot\vec{\epsilon}^*_{\lambda'}(k)a^\dagger_{k,\lambda'}a_{k,\lambda} \\
&-\vec{\epsilon}^*_\lambda(k)\cdot\vec{\epsilon}_{\lambda'}(k)a^\dagger_{k,\lambda}a_{k,\lambda'} \\
&-\vec{\epsilon}^*_\lambda(k)\cdot\vec{\epsilon}^*_{\lambda'}(-k)a^\dagger_{k,\lambda}a^\dagger_{-k,\lambda'}e^{2\imath\omega t}\Big\} \\
&= \int \frac{d^3k}{(2\pi)^3}|\vec{k}|\sum_{\lambda,\lambda'}\frac{1}{2}\left(\vec{\epsilon}_\lambda(k)\cdot\vec{\epsilon}^*_{\lambda'}(k) + \vec{\epsilon}^*_{\lambda'}(k)\cdot\vec{\epsilon}_\lambda(k)\right)a^\dagger_{k,\lambda}a_{k,\lambda'} \\
&= \int \frac{d^3k}{(2\pi)^3}|\vec{k}|\sum_{\lambda}a^\dagger_{k,\lambda}a_{k,\lambda}. \qquad (12.48)
\end{aligned}$$

We have use

$$\frac{1}{2}\left(\vec{\epsilon}_\lambda(k)\cdot\vec{\epsilon}^*_{\lambda'}(k) + \vec{\epsilon}^*_{\lambda'}(k)\cdot\vec{\epsilon}_\lambda(k)\right) = \delta_{\lambda,\lambda'}. \qquad (12.49)$$

Also, in the same way,

$$\begin{aligned}
\vec{P} &= \int d^3x : \vec{E}\times\vec{B} : \\
&= -\int d^3x\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{2}\frac{1}{\sqrt{EE'}}\sum_{\lambda\lambda'}\Big\{\vec{\epsilon}_\lambda(k)a_{k,\lambda}e^{-\imath k\cdot x} - \vec{\epsilon}^*_\lambda(k)a^\dagger_{k,\lambda}e^{\imath k\cdot x}\Big\} \\
&\times\Big\{\vec{k'}\times\vec{\epsilon}_{\lambda'}(k')a_{k',\lambda'}e^{-\imath k'\cdot x} - \vec{k'}\times\vec{\epsilon}^*_{\lambda'}(k')a^\dagger_{k',\lambda'}e^{\imath k'\cdot x}\Big\} \\
&= -\int d^3x\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{2}\frac{1}{\sqrt{EE'}}\sum_{\lambda\lambda'}\Big\{\vec{\epsilon}_\lambda(k)\times\vec{k'}\times\vec{\epsilon}_{\lambda'}(k')a_{k,\lambda}a_{k',\lambda'}e^{-\imath(k+k')\cdot x} \\
&-\vec{\epsilon}_\lambda(k)\times\vec{k'}\times\vec{\epsilon}^*_{\lambda'}(k')a^\dagger_{k',\lambda'}a_{k,\lambda}e^{-\imath(k-k')\cdot x} - \vec{\epsilon}^*_\lambda(k)\times\vec{k'}\times\vec{\epsilon}_{\lambda'}(k')a^\dagger_{k,\lambda}a_{k',\lambda'}e^{\imath(k-k')\cdot x} \\
&+\vec{\epsilon}^*_\lambda(k)\times\vec{k'}\times\vec{\epsilon}^*_{\lambda'}(k')a^\dagger_{k,\lambda}a^\dagger_{k',\lambda'}e^{\imath(k+k')\cdot x}\Big\} \\
&= \int \frac{d^3k}{(2\pi)^3}\sum_{\lambda\lambda'}\frac{1}{2}\Big\{\vec{k}(\vec{\epsilon}_\lambda(k)\cdot\vec{\epsilon}_{\lambda'}(-k))a_{k,\lambda}a_{-k,\lambda'}e^{-2\imath\omega t} \\
&+\vec{k}(\vec{\epsilon}_\lambda(k)\cdot\vec{\epsilon}^*_{\lambda'}(k))a^\dagger_{k,\lambda'}a_{k,\lambda} + \vec{k}(\vec{\epsilon}^*_\lambda(k)\cdot\vec{\epsilon}_{\lambda'}(k))a^\dagger_{k,\lambda}a_{k,\lambda'} \\
&\vec{k}(\vec{\epsilon}^*_\lambda(k)\cdot\vec{\epsilon}^*_{\lambda'}(-k))a^\dagger_{-k,\lambda'}a_{k,\lambda}e^{2\imath\omega t}\Big\} \\
&= \int \frac{d^3k}{(2\pi)^3}\sum_{\lambda}\vec{k}a^\dagger_{k,\lambda}a_{k,\lambda}, \qquad (12.50)
\end{aligned}$$

where we've used (12.49) and

$$\vec{\epsilon}_\lambda(k) \cdot \vec{\epsilon}^{(*)}_{\lambda'}(-k) = 0. \tag{12.51}$$

Finally, from (12.36) and (12.37), we get

$$\begin{aligned}
S^{ij} &= -\int d^3x \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{1}{2} \sqrt{\frac{\omega'}{\omega}} \sum_{\lambda\lambda'} \\
&\quad \Big\{ \left(\epsilon^i_\lambda(k) \epsilon^j_{\lambda'}(k') - \epsilon^j_\lambda(k) \epsilon^i_{\lambda'}(k') \right) a_{k,\lambda} a_{k',\lambda'} e^{-\imath(k+k')\cdot x} \\
&\quad - \left(\epsilon^i_\lambda(k) \epsilon^{*j}_{\lambda'}(k') - \epsilon^j_\lambda(k) \epsilon^{*i}_{\lambda'}(k') \right) a^\dagger_{k',\lambda'} a_{k,\lambda} e^{-\imath(k-k')\cdot x} \\
&\quad + \left(\epsilon^{*i}_\lambda(k) \epsilon^j_{\lambda'}(k') - \epsilon^{*j}_\lambda(k) \epsilon^i_{\lambda'}(k') \right) a^\dagger_{k,\lambda} a_{k',\lambda'} e^{\imath(k-k')\cdot x} \\
&\quad - \left(\epsilon^{*i}_\lambda(k) \epsilon^{*j}_{\lambda'}(k') - \epsilon^{*j}_\lambda(k) \epsilon^{*i}_{\lambda'}(k') \right) a^\dagger_{k,\lambda} a^\dagger_{k',\lambda'} e^{\imath(k+k')\cdot x} \Big\} \\
&= -\int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \sum_{\lambda,\lambda'} 2 \left\{ \epsilon^j_\lambda(k) \epsilon^{*i}_{\lambda'}(k) - \epsilon^i_\lambda(k) \epsilon^{*j}_{\lambda'}(k) \right\} a^\dagger_{k,\lambda} a_{k,\lambda'} \\
&= -\int \frac{d^3k}{(2\pi)^3} \sum_{\lambda,\lambda'} \left\{ \epsilon^j_\lambda(k) \epsilon^{*i}_{\lambda'}(k) - \epsilon^i_\lambda(k) \epsilon^{*j}_{\lambda'}(k) \right\} a^\dagger_{k,\lambda} a_{k,\lambda'}
\end{aligned} \tag{12.52}$$

Chapter 13

Miscellaneous

13.1 Notation

Through this article we shall use the following notation.

- Minkowski metric: $\eta_{\mu\nu} = diag(+1, -1, -1, -1)$.
- Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{13.1}$$

$$\sigma^2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} \tag{13.2}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13.3}$$

The Pauli matrices satisfy the Clifford algebra $\{\sigma^i, \sigma^j\} = 2\mathbb{1}\delta^{ij}$. Additionally, $\sigma^\mu = (\mathbb{1}, \sigma^i)$, and $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$.

13.2 Gamma matrices in the chiral representation

Consider the Gamma matrices in the chiral representation

$$\Gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{13.4}$$

Let's prove that they satisfy the Clifford algebra.

$$\left\{ \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}, \begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & \end{pmatrix} \right\} = \begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu \end{pmatrix}. \tag{13.5}$$

Now, let's consider the cases $\mu = \nu = 0$, $\mu = 0, \nu = i$ and $\mu = i, \nu = j$.

1. If $\mu = \nu = 0$,
$$\sigma^0\bar{\sigma}^0 + \sigma^0\bar{\sigma}^0 = 2\mathbb{1}, \tag{13.6}$$
and
$$\bar{\sigma}^0\sigma^0 + \bar{\sigma}^0\sigma^0 = 2\mathbb{1}. \tag{13.7}$$

2. If $\mu = 0.\nu = i$, since $\bar{\sigma}^i = -\sigma^i$,
$$\begin{aligned}\sigma^0\bar{\sigma}^i + \sigma^i\bar{\sigma}^0 &= -\sigma^i + \sigma^i \\ &= 0,\end{aligned} \tag{13.8}$$
and
$$\begin{aligned}\bar{\sigma}^0\sigma^i + \bar{\sigma}^i\sigma^0 &= \sigma^i - \sigma^i \\ &= 0.\end{aligned} \tag{13.9}$$

3. If $\mu = i, \nu = j$,
$$\begin{aligned}\sigma^i\bar{\sigma}^j + \sigma^j\bar{\sigma}^i &= -\{\sigma^i, \sigma^j\} \\ &= -2\mathbb{1},\end{aligned} \tag{13.10}$$
and
$$\begin{aligned}\bar{\sigma}^i\sigma^j + \bar{\sigma}^j\sigma^i &= -\{\sigma^i, \sigma^j\} \\ &= -2\mathbb{1}.\end{aligned} \tag{13.11}$$

Finally, from (13.6), (13.7), (13.10) and (13.11), we can write
$$\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = 2\eta^{\mu\nu}\mathbb{1} \tag{13.12}$$
$$\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu = 2\eta^{\mu\nu}\mathbb{1}, \tag{13.13}$$
and
$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}. \tag{13.14}$$

13.3 Scalars from Weyl spinors

Consider the products $\psi_L^\dagger\psi_R$ and $\psi_R^\dagger\psi_L$. We want to prove that they transform like scalars.

Since, under Lorentz transformations
$$\psi_L \mapsto \psi'_L = \Lambda_L\psi_L = \exp\left((-\imath\vec{\theta} - \vec{\eta})\frac{\vec{\sigma}}{2}\right)\psi_L, \tag{13.15}$$
$$\psi_R \mapsto \psi'_R = \Lambda_R\psi_R = \exp\left((-\imath\vec{\theta} + \vec{\eta})\frac{\vec{\sigma}}{2}\right)\psi_R, \tag{13.16}$$

Then

$$\begin{aligned}
\psi_L^\dagger \phi_R \mapsto {\psi'_L}^\dagger \psi'_R &= (\Lambda_L \psi_L)^\dagger \Lambda_R \psi_R \\
&= \psi_L^\dagger \exp\left((\imath\vec{\theta} - \vec{\eta})\frac{\vec{\sigma}}{2}\right) \exp\left((-\imath\vec{\theta} + \vec{\eta})\frac{\vec{\sigma}}{2}\right) \psi_R \\
&= \psi_L^\dagger \psi_R. \qquad (13.17)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\psi_R^\dagger \phi_L \mapsto {\psi'_R}^\dagger \psi'_L &= (\Lambda_R \psi_R)^\dagger \Lambda_L \psi_L \\
&= \psi_R^\dagger \exp\left((\imath\vec{\theta} + \vec{\eta})\frac{\vec{\sigma}}{2}\right) \exp\left((-\imath\vec{\theta} - \vec{\eta})\frac{\vec{\sigma}}{2}\right) \psi_L \\
&= \psi_R^\dagger \psi_L. \qquad (13.18)
\end{aligned}$$

13.4 Determinant of a deformation of the identity

Consider a deformation of the identity matrix, let's say $\mathbb{1} + X$, we want to compute

$$det(\mathbb{1} + X) = ?. \qquad (13.19)$$

Actually, we can do more than one trick for calculating this quantity.
First, we'll use the formulae

$$\begin{aligned}
det(\exp(M)) &= \exp(tr(M)) \qquad (13.20) \\
\exp(M) &= \sum_{n=0}^{\infty} \frac{M^n}{n!} \qquad (13.21) \\
\ln(1 + M) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{M^n}{n}. \qquad (13.22)
\end{aligned}$$

From (13.20), we have

$$\begin{aligned}
det(\mathbb{1} + X) &= \exp(tr(\ln(\mathbb{1} + X))) \\
&= \exp\left(tr\left(X - \frac{X^2}{2} + \frac{X^3}{3} - \cdots\right)\right) \\
&= 1 + tr(X) - \frac{1}{2}tr(X^2) + \frac{1}{2}(tr(X))^2 \\
&\quad + \frac{1}{12}\left(4tr(X^3) - 3tr(X^2)tr(X) + 2(tr(X))^3\right) + \cdots . \qquad (13.23)
\end{aligned}$$

Nonetheless, if the components of $X \ll 1$, we get

$$det(\mathbb{1} + X) \cong 1 + tr(X). \qquad (13.24)$$

Another way of get the solution is by writing

$$det(\mathbb{1}+X)=\epsilon^{i_1\cdots i_n}(\delta_{1i_1}+X_{1i_1})\cdots(\delta_{ni_n}X_{ni_n}), \tag{13.25}$$

from this equation we can see that the expansion of above is never infinite, and we obtain a sum of “generalised" cofactors matrices. So,

$$det(\mathbb{1}+X)=1+tr(X)+\cdots+det(X). \tag{13.26}$$

13.5 Lorentz currents for a real scalar field

In the class we've check that for a Lorentz transformation, a real scalar field has a Nöther conserved current

$$j^{\mu}_{\underline{\rho\sigma}}=\partial^{\mu}\phi(\delta^{\nu}_{\rho}x_{\sigma}-\delta^{\nu}_{\sigma}x_{\rho})\partial_{\nu}\phi-(\delta^{\mu}_{\rho}x_{\sigma}-\delta^{\mu}_{\sigma}x_{\rho})\mathscr{L}, \tag{13.27}$$

now, we want to write that in term of the energy momentum tensor,

$$\Theta^{\mu\nu}=\partial^{\mu}\phi\partial^{\nu}\phi-\eta^{\mu\nu}\mathscr{L}. \tag{13.28}$$

Then

$$\begin{aligned}
j^{\mu\underline{\rho\sigma}} &= \partial^{\mu}\phi(\eta^{\nu\rho}x^{\sigma}-\eta^{\nu\sigma}x^{\rho})\partial_{\nu}\phi-(\eta^{\mu\rho}x^{\sigma}-\eta^{\mu\sigma}x^{\rho})\mathscr{L}\\
&= \partial^{\mu}\phi(\delta^{\rho}_{\nu}x^{\sigma}-\delta^{\sigma}_{\nu}x^{\rho})\partial^{\nu}\phi-(\eta^{\mu\rho}x^{\sigma}-\eta^{\mu\sigma}x^{\rho})\mathscr{L}\\
&= \partial^{\mu}\phi(\delta^{\rho}_{\nu}x^{\sigma}-\delta^{\sigma}_{\nu}x^{\rho})\partial^{\nu}\phi-\eta^{\mu\nu}(\delta^{\rho}_{\nu}x^{\sigma}-\delta^{\sigma}_{\nu}x^{\rho})\mathscr{L}\\
&= (\delta^{\rho}_{\nu}x^{\sigma}-\delta^{\sigma}_{\nu}x^{\rho})(\partial^{\mu}\phi\partial^{\nu}\phi-\eta^{\mu\nu}\mathscr{L})\\
&= (\delta^{\rho}_{\nu}x^{\sigma}-\delta^{\sigma}_{\nu}x^{\rho})\Theta^{\mu\nu}\\
&= x^{\sigma}\Theta^{\mu\rho}-x^{\rho}\Theta^{\mu\sigma}.
\end{aligned} \tag{13.29}$$

13.6 Conservation of the KG inner product

Let's define the Klein-Gordon inner product as

$$\langle\phi_1|\phi_2\rangle=\frac{\imath}{2}\int\mathbf{d}^3x\,\phi_1\overleftrightarrow{\partial}\phi_2. \tag{13.30}$$

Then

$$\begin{aligned}
\partial_0\langle\phi_1|\phi_2\rangle &= \frac{\imath}{2}\int\mathbf{d}^3x\left(\cancel{\partial_0\phi_1\partial_0\phi_2}+\phi_1\partial_0^2\phi_2-\partial_0^2\phi_1\phi_2-\cancel{\partial_0\phi_1\partial_0\phi_2}\right)\\
&= \frac{\imath}{2}\int\mathbf{d}^3x\left(\cancel{m^2\phi_1\phi_2}-\cancel{m^2\phi_1\phi_2}+\phi_1\nabla^2\phi_2-\nabla^2\phi_1\phi_2\right)\\
&= \frac{\imath}{2}\int\mathbf{d}^3x\left(\cancel{\partial_i\phi_1\partial^i\phi_2}-\cancel{\partial^i\phi_1\partial_i\phi_2}+\partial_i(\phi_1\partial^i\phi_2-\partial^i\phi_1\phi_2)\right)\\
&= 0.
\end{aligned} \tag{13.31}$$

Here, we have applied the definition of the right-left derivative and then act with ∂_0. After that, using the KG eqs. of motion (remember that both scalars have the same mass) and integrating by parts, we get down to the last line, where we argue that $\phi \to 0$, if $|x| \to \infty$.

We conclude that the Klein-Gordon inner product is constant in time.

13.7 Nöther charge for translation

We already know that the Nöther current for tranlation is the energy momentum tensor, so in order to compute its associate charge, we fix the first index to 0, i.e.,

$$P^\mu = \int \mathbf{d}^3x\, \Theta^{0\mu} = \int \mathbf{d}^3x\, (\partial^0\phi\partial^\mu - \eta^{0\mu}\mathscr{L}. \tag{13.32}$$

If $\mu = i$, we have

$$\begin{aligned} P^i &= -\frac{\imath}{2}\int \mathbf{d}^3x\, \left(\partial_0\phi\imath\partial^i\phi + \partial_0\phi\imath\partial^i\phi\right) \\ &= -\frac{\imath}{2}\int \mathbf{d}^3x\, \left(\partial_0\phi\imath\partial^i\phi - \partial_0\phi\imath\partial^i\phi\right) \\ &= \left\langle\phi|\imath\partial^i|\phi\right\rangle, \end{aligned} \tag{13.33}$$

in the intermiddle step, we've dropped the divergence, so that (as usual) we demand $phi \to 0, |x| \to \infty$.

For $\mu = 0$,

$$\begin{aligned} P^0 &= \int \mathbf{d}^3x\left(\partial^0\phi\partial^0\phi - \frac{1}{2}((\partial^0\phi)^2 - (\partial^i\phi)^2 - m^2\phi^2)\right) \\ &= \frac{1}{2}\int \mathbf{d}^3x\, \left((\partial_0\phi)^2 + \phi(\partial_i\partial^i + m^2)\phi\right) \\ &= \frac{\imath}{2}\int \mathbf{d}^3x\, \left(\phi\imath\partial^0\partial_0\phi - \partial_0\phi\imath\partial^0\phi\right) \\ &= \left\langle\phi|\imath\partial^0|\phi\right\rangle. \end{aligned} \tag{13.34}$$

Thus,

$$P^\mu = \langle\phi|\imath\partial^\mu|\phi\rangle\,. \tag{13.35}$$

13.8 Spinorial Lagrangian

Consider the lagrangian for left-handed Weyl spinors,

$$\mathscr{L} = (cte)\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L. \tag{13.36}$$

One would like the lagrangian to be real, so

$$
\begin{aligned}
(cte)\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L &= (cte)^* \psi_L^t \bar{\sigma}^{*\mu} \partial_\mu \psi_L^* \\
&= -(cte)^* \psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + \cancel{\text{Boundary term}}\,,
\end{aligned}
\tag{13.37}
$$

Therefore,

$$
(cte)^* = -(cte). \tag{13.38}
$$

Chapter 14

Does it need a name?

14.1 Wick's Theorem

By using the Wick's theorem, compute:

1. $\left\langle 0|T\{\phi^4(x)\phi^4(y)\}|0\right\rangle$
2. $T\{:\phi^4(x)::\phi^4(y):\}$
3. $\langle 0|T\{\bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y)\}|0\rangle$.

Answer

a Wick's theorem tells that one should consider all possible contractions of the fields, for this case, there are three possibilities, all x and y contracted with themselves, or with each other and finally a couple of crossed contractions and a pair of self-contractions.

$$\begin{aligned}\left\langle 0|T\{\phi^4(x)\phi^4(y)\}|0\right\rangle &= 9(D(x-x))^2(D(y-y))^2 \\ &+72(D(x-y))^2D(x-x)D(y-y) \\ &+24(D(x-y))^4. \end{aligned} \tag{14.1}$$

The factors came from

- 3 different ways of contracting four $\phi(x)$ or $\phi(y)$ among themselves.
- 6 different ways of contracting a pair of $\phi(x)$ or $\phi(y)$ among themselves, times 2 ways of contract the remaining fields but crossed.
- 4! (no question)

b Since inside the time-ordering operator there are normal products, one must apply the non-equal-time-contraction Wick's theorem. So,

$$\begin{aligned} T\{:\phi^4(x)::\phi^4(y):\} &= :\phi^4(x)\phi^4(y): +16 :\phi^3(x)\phi^3(y): D(x-y) \\ &+72 :\phi^2(x)\phi^2(y): (D(x-y))^2 \\ &+96 :\phi(x)\phi(y): (D(x-y))^3 \\ &+4!(D(x-y))^4. \end{aligned} \tag{14.2}$$

The symmetry factors are

- 16 ways of taking a pair from two set of four.
- 72... just like before, or $72 = \frac{4^2 3^2}{2!}$.
- $96 = \frac{4^2 3^2 2^2}{3!}$.
- 4!.

c Finally

$$\begin{aligned} \langle 0|T\{\bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y)\}|0\rangle &= \left\langle 0|T\{\bar{\psi}_\alpha(x)\psi_\alpha(x)\bar{\psi}_\beta(y)\psi_\beta(y)\}|0\right\rangle \\ &= S_{\alpha\alpha}(x-x)S_{\beta\beta}(y-y) \\ &\quad -S_{\beta\alpha}(y-x)S_{\alpha\beta}(x-y). \end{aligned} \tag{14.3}$$

14.2 2 → 2 Scattering in $g\phi^3$

In the theory $g\psi^3$ calculate, directly from the Dyson expansion and by using the Wick's theorem, the first non-trivial contribution to the matrix element of S for a $2 \to 2$ scattering.

Answer

Let ϕ be a real scalar field with lagrangian

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m}{2}\phi^2 - \frac{g}{3!}\phi^3, \tag{14.4}$$

then,

$$H_{int} = \frac{g}{3!} :\phi^3: . \tag{14.5}$$

The Dyson expansion is

$$S = \mathbb{1} + (-\imath)\int d^4x\mathscr{H}_I(x) + \frac{(-\imath)^2}{2!}\int d^x d^4y T\{\mathscr{H}_I(x)\mathscr{H}_I(y)\} + \cdots . \tag{14.6}$$

The 2 → 2 scattering is given by

$$\left\langle \vec{p}_1\vec{p}_2|S|\vec{k}_1\vec{k}_2\right\rangle, \tag{14.7}$$

with $\{\vec{p}\} \neq \{\vec{k}\}$. Thus,

$$\left\langle \vec{p}_1\vec{p}_2|S^{(0)}|\vec{k}_1\vec{k}_2\right\rangle = \left\langle \vec{p}_1\vec{p}_2|\mathbb{1}|\vec{k}_1\vec{k}_2\right\rangle = 0. \tag{14.8}$$

$$\begin{aligned}\left\langle \vec{p}_1\vec{p}_2|S^{(1)}|\vec{k}_1\vec{k}_2\right\rangle &= (-\imath)^2(\imath)^2\left(-\imath\frac{g}{3!}\right)\int d^3z_1d^3z_2d^3y_1d^3y_2e^{\imath p_1\cdot z_1}e^{\imath p_2\cdot z_2}e^{-\imath k_1\cdot y_1}e^{-\imath k_2\cdot y_2}\\ &\overleftrightarrow{\partial}_{z_1^0}\overleftrightarrow{\partial}_{z_2^0}\overleftrightarrow{\partial}_{y_1^0}\overleftrightarrow{\partial}_{y_2^0}\int d^4x\left\langle 0|\phi(z_1)\phi(z_2):\phi^3(x):\phi(y_1)\phi(y_2)|0\right\rangle\\ &= 0,\end{aligned} \tag{14.9}$$

by the Wick's theorem, because there is a non-paired field.

$$\begin{aligned}\left\langle \vec{p}_1\vec{p}_2|S^{(2)}|\vec{k}_1\vec{k}_2\right\rangle &= (-\imath)^2(\imath)^2\left(-\imath\frac{g}{3!}\right)^2\int d^3z_1d^3z_2d^3y_1d^3y_2e^{\imath p_1\cdot z_1}e^{\imath p_2\cdot z_2}e^{-\imath k_1\cdot y_1}e^{-\imath k_2\cdot y_2}\\ &\overleftrightarrow{\partial}_{z_1^0}\overleftrightarrow{\partial}_{z_2^0}\overleftrightarrow{\partial}_{y_1^0}\overleftrightarrow{\partial}_{y_2^0}\\ &\int d^4x_1d^4x_2\left\langle 0|\phi(z_1)\phi(z_2)T\{:\phi^3(x_1)::\phi^3(x_2):\}\phi(y_1)\phi(y_2)|0\right\rangle\\ &= (-\imath)^2(\imath)^2\left(-\imath\frac{g}{3!}\right)^2\lim_{z_i\to+\infty}\lim_{y\to-\infty}\int d^3z_1d^3z_2d^3y_1d^3y_2\\ &e^{\imath p_1\cdot z_1}e^{\imath p_2\cdot z_2}e^{-\imath k_1\cdot y_1}e^{-\imath k_2\cdot y_2}\overleftrightarrow{\partial}_{z_1^0}\overleftrightarrow{\partial}_{z_2^0}\overleftrightarrow{\partial}_{y_1^0}\overleftrightarrow{\partial}_{y_2^0}\\ &\int d^4x_1d^4x_2\left\langle 0|T\{\phi(z_1)\phi(z_2):\phi^3(x_1)::\phi^3(x_2):\phi(y_1)\phi(y_2)\}|0\right\rangle\\ &= (-\imath)^2(\imath)^2\left(-\imath\frac{g}{3!}\right)^2\lim_{z_i\to+\infty}\lim_{y\to-\infty}\int d^3z_1d^3z_2d^3y_1d^3y_2\\ &e^{\imath p_1\cdot z_1}e^{\imath p_2\cdot z_2}e^{-\imath k_1\cdot y_1}e^{-\imath k_2\cdot y_2}\overleftrightarrow{\partial}_{z_1^0}\overleftrightarrow{\partial}_{z_2^0}\overleftrightarrow{\partial}_{y_1^0}\overleftrightarrow{\partial}_{y_2^0}(3!)^2\int d^4x_1d^4x_2\\ &\{D(z_1-x_1)d(z_2-x_1)D(x_1-x_2)D(x_2-y_1)D(x_2-y_2)\\ &D(z_1-x_1)D(z_2-x_2)D(x_1-x_2)D(x_1-y_1)D(x_2-y_2)\\ &+D(z_1-x_1)D(z_2-x_2)D(x_1-x_2)D(x_2-y_1)D(x_1-y_2)\}.\end{aligned} \tag{14.10}$$

Since

$$\lim_{z_i^0\to\infty}\int d^3z_i e^{\imath p_i\cdot z_i}\overleftrightarrow{\partial}_{z_i^0}D(z_i-x) = (-\imath)e^{\imath p_i\cdot x} \tag{14.11}$$

$$\lim_{y_i^0\to-\infty}\int d^3y_i e^{-\imath k_i\cdot y_i}\overleftrightarrow{\partial}_{y_i^0}D(x-y_i) = (\imath)e^{-\imath k_i\cdot x}, \tag{14.12}$$

then,

$$\begin{aligned}\left\langle \vec{p}_1\vec{p}_2|S^{(2)}|\vec{k}_1\vec{k}_2\right\rangle &= -g^2\int d^4x_1d^4x_2\frac{d^4q}{(2\pi)^4}\left\{e^{\imath(p_1+p_2)\cdot x_1}e^{-\imath(k_1+k_2)\cdot x_2}\right.\\ &\quad\left.+e^{\imath(p_1-k_1)\cdot x_1}e^{\imath(p_2-k_2)\cdot x_2}+e^{\imath(p_1-k_2)\cdot x_1}e^{\imath(p_2-k_1)\cdot x_2}\right\}\\ &\quad\imath\frac{e^{\imath q\cdot(x_1-x_2)}}{q^2-m^2+\imath\epsilon}\\ &= -g^2\delta^{(4)}(p_1+p_2-k_1-k_2)\left\{\frac{\imath}{(k_1+k_2)^2-m^2+\imath\epsilon}\right.\\ &\quad\left.+\frac{\imath}{(k_1-p_1)^2-m^2+\imath\epsilon}+\frac{\imath}{(k_1-p_2)^2-m^2+\imath\epsilon}\right\}\end{aligned} \tag{14.13}$$

14.3 $2\to 2$ Scattering in $\lambda\phi^4$

Rederive the result obtained in class of the first order contribution, in λ, to the S matrix in the $2\to 2$ scattering, without using the Wick's theorem.

Answer

Since

$$\phi(x)=\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\left\{a_qe^{-\imath q\cdot x}+a_q^\dagger e^{\imath q\cdot x}\right\}, \tag{14.14}$$

it follows that,

$$\begin{aligned}:\phi^4(x): &= \int\frac{d^3q_1}{(2\pi)^3}\frac{1}{\sqrt{2E_1}}\frac{d^3q_2}{(2\pi)^3}\frac{1}{\sqrt{2E_2}}\frac{d^3q_3}{(2\pi)^3}\frac{1}{\sqrt{2E_3}}\frac{d^3q_4}{(2\pi)^3}\frac{1}{\sqrt{2E_4}}\\ &\quad\left\{a_1a_2a_3a_4e^{-\imath(q_1+q_2+q_3+q_4)\cdot x}+4a_1^\dagger a_2a_3a_4e^{-\imath(-q_1+q_2+q_3+q_4)\cdot x}\right.\\ &\quad+6a_1^\dagger a_2^\dagger a_3a_4e^{-\imath(-q_1-q_2+q_3+q_4)\cdot x}+4a_1^\dagger a_2^\dagger a_3^\dagger a_4e^{\imath(q_1+q_2+q_3-q_4)\cdot x}\\ &\quad\left.+a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger e^{\imath(q_1+q_2+q_3+q_4)\cdot x}\right\}.\end{aligned} \tag{14.15}$$

The only one term that contribute is the one with equal number of a's and $a^\dagger$'s, i.e., $6a_1^\dagger a_2^\dagger a_3a_4e^{-\imath(-q_1-q_2+q_3+q_4)\cdot x}$.

Therfore,

$$\left\langle \vec{p}_1\vec{p}_2|S^{(1)}|\vec{k}_1\vec{k}_2\right\rangle = -\frac{\imath\lambda}{4!}\int d^4x \frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}\frac{d^3q_3}{(2\pi)^3}\frac{d^3q_4}{(2\pi)^3}\left(\frac{E_{p_1}E_{p_2}E_{k_1}E_{k_2}}{E_1E_2E_3E_4}\right)^{\frac{1}{2}}$$
$$3!e^{\imath(q_1+q_2-q_3-q_4)\cdot x}\left\langle 0|a_{p_1}a_{p_2}a_1^\dagger a_2^\dagger a_3a_4a_{k_1}^\dagger a_{k_2}^\dagger|0\right\rangle. \quad (14.16)$$

Now,

$$\begin{aligned}\left\langle 0|a_{p_1}a_{p_2}a_1^\dagger a_2^\dagger a_3a_4a_{k_1}^\dagger a_{k_2}^\dagger|0\right\rangle &= \left\langle 0|a_{p_1}a_{p_2}a_1^\dagger a_2^\dagger a_3\left((2\pi)^3\delta^{(3)}(q_4-k_1)+a_{k_1}^\dagger a_4\right)a_{k_2}^\dagger|0\right\rangle \\ &= \left\langle 0|a_{p_1}a_{p_2}a_1^\dagger a_2^\dagger a\left((2\pi)^3(2\pi)^3\delta^{(3)}(q_4-k_1)\delta^{(3)}(q_3-k_2)\right.\right. \\ &\quad \left.\left. +a_{k_2}^\dagger a_3(2\pi)^3\delta^{(3)}(q_4-k_1)+a_3a_{k_1}^\dagger a_{k_2}^\dagger a_4\right)|0\right\rangle \\ &= \left\langle 0|a_{p_1}a_{p_2}a_1^\dagger a_2^\dagger|0\right\rangle(2\pi)^3(2\pi)^3 \\ &\quad \left\{\delta^{(3)}(q_4-k_2)\delta^{(3)}(q_3-k_1)\right. \\ &\quad \left. +\delta^{(3)}(q_4-k_1)\delta^{(3)}(q_3-k_2)\right\}. \end{aligned} \quad (14.17)$$

Thus,

$$\begin{aligned}\left\langle 0|a_{p_1}a_{p_2}a_1^\dagger a_2^\dagger a_3a_4a_{k_1}^\dagger a_{k_2}^\dagger|0\right\rangle &= (2\pi)^3(2\pi)^3(2\pi)^3(2\pi)^3 \\ &\quad \left\{\delta^{(3)}(p_1-q_1)\delta^{(3)}(p_2-q_2)\delta^{(3)}(q_3-k_1)\delta^{(3)}(q_4-k_2)\right. \\ &\quad \left. +\delta^{(3)}(p_1-q_2)\delta^{(3)}(p_2-q_1)\delta^{(3)}(q_3-k_1)\delta^{(3)}(q_4-k_2)+k_1\leftrightarrow k_2\right\}. \end{aligned} \quad (14.18)$$

And so,

$$\begin{aligned}\left\langle \vec{p}_1\vec{p}_2|S^{(1)}|\vec{k}_1\vec{k}_2\right\rangle &= -\frac{\imath\lambda}{4!}\int d^4x \frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}\frac{d^3q_3}{(2\pi)^3}\frac{d^3q_4}{(2\pi)^3}\left(\frac{E_{p_1}E_{p_2}E_{k_1}E_{k_2}}{E_1E_2E_3E_4}\right)^{\frac{1}{2}} \\ &\quad 4!e^{\imath(q_1+q_2-q_3-q_4)\cdot x}\delta^{(3)}(p_1-q_1)\delta^{(3)}(p_2-q_2)\delta^{(3)}(q_3-k_1)\delta^{(3)}(q_4-k_2) \\ &= -\imath\lambda\int d^4x e^{\imath(p_1+p_2-k_1-k_2)\cdot x} \\ &= -\imath\lambda(2\pi)^4\delta^{(4)}(p_1+p_2-k_1-k_2) \end{aligned} \quad (14.19)$$

14.4 Identities for $u_s(p)$ and $v_s(p)$ Spinors

Use th explicit form of the spinors $u_s(p)$ and $v_s(p)$ for showing

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p}+m \quad (14.20)$$
$$\sum_s v_s(p)\bar{v}_s(p) = \not{p}-m. \quad (14.21)$$

What normalisation should the spinors have? Show that these identities are consistent with the equations which define the spinors $u_s(p)$ and $v_s(p)$. Are these identities invariant under a change on the representation of the γ-matrices?

Answer

Remind that

$$u_s(p) = \sqrt{E+m}\begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s \end{pmatrix} \quad (14.22)$$

$$v_s(p) = \sqrt{E+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s \\ \chi_s \end{pmatrix}, \quad (14.23)$$

where

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (14.24)$$

Since

$$\frac{\vec{\sigma}\cdot\vec{p}}{E+m} = \frac{1}{E+m}\begin{pmatrix} p_z & p_x - \imath p_y \\ p_x + \imath p_y & -p_z \end{pmatrix}, \quad (14.25)$$

then,

$$u_1(p) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+\imath p_y}{E+m} \end{pmatrix}, \qquad u_2(p) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-\imath p_y}{E+m} \\ -\frac{p_z}{E+m} \end{pmatrix}, \quad (14.26)$$

and

$$\bar{u}_1(p) = \sqrt{E+m}\begin{pmatrix} 1 & 0 & -\frac{p_z}{E+m} & -\frac{p_x+\imath p_y}{E+m} \end{pmatrix}, \quad (14.27)$$

$$\bar{u}_2(p) = \sqrt{E+m}\begin{pmatrix} 0 & 1 & -\frac{p_x-\imath p_y}{E+m} & \frac{p_z}{E+m} \end{pmatrix}, \quad (14.28)$$

where the standard representation of Dirac matrices have been used.

Similarly,

$$v_1(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+\imath p_y}{E+m} \\ 1 \\ 0 \end{pmatrix}, \qquad v_2(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_x-\imath p_y}{E+m} \\ -\frac{p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}, \quad (14.29)$$

and

$$\bar{v}_1(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} & \frac{p_x+\imath p_y}{E+m} & -1 & 0 \end{pmatrix}, \quad (14.30)$$

$$\bar{v}_2(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_x-\imath p_y}{E+m} & -\frac{p_z}{E+m} & 0 & -1 \end{pmatrix}. \quad (14.31)$$

Now, by simple matrix multiplication, we get,

$$u_1(p)\bar{u}_1(p) = \begin{pmatrix} E+m & 0 & -p_z & -p_x+\imath p_y \\ 0 & 0 & 0 & 0 \\ p_z & 0 & -\frac{p_z^2}{E+m} & \frac{-p_z p_x + \imath p_z p_y}{E+m} \\ p_x+\imath p_y & 0 & \frac{-p_x p_z + \imath p_y p_z}{E+m} & \end{pmatrix}, \quad (14.32)$$

$$u_2(p)\bar{u}_2(p) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & E+m & -p_x+\imath p_y & p_z \\ 0 & p_x+\imath p_y & -\frac{p_x^2+p_y^2}{E+m} & \frac{p_z p_x - \imath p_z p_y}{E+m} \\ 0 & -p_z & \frac{p_x p_z - \imath p_y p_z}{E+m} & -\frac{p_z^2}{E+m} \end{pmatrix}, \tag{14.33}$$

so that

$$\begin{aligned} \sum_s u_s(p)\bar{u}_s(p) &= \begin{pmatrix} E+m & 0 & -p_z & -p_x+\imath p_y \\ 0 & E+m & -p_x-\imath p_y & p_z \\ p_z & p_x-\imath p_y & -E+m & 0 \\ p_x+\imath p_y & -p_z & 0 & -E+m \end{pmatrix} \\ &= \not{p}+m. \end{aligned} \tag{14.34}$$

Repeating the above process, we get,

$$\begin{aligned} \sum_s v_s(p)\bar{v}_s(p) &= \begin{pmatrix} E-m & 0 & -p_z & -p_x+\imath p_y \\ 0 & E-m & -p_x-\imath p_y & p_z \\ p_z & p_x-\imath p_y & -E-m & 0 \\ p_x+\imath p_y & -p_z & 0 & -E-m \end{pmatrix} \\ &= \not{p}-m. \end{aligned} \tag{14.35}$$

From (14.26) and (14.29), we obtain

$$u_i^\dagger(p)u_i(p) = 2E, \qquad v_i^\dagger(p)v_i(p) = 2E, \tag{14.36}$$

for $i = 1, 2$.

Of course, properties (14.34) and (14.35) are consistent with the equation of motion for spinors $u_s(p)$ and $v_s(p)$, so that,

$$\begin{aligned} (\not{p}-m)\sum_s u_s(p)\bar{u}_s(p) &= (\not{p}-m)(\not{p}+m) \\ &= p^2-m^2 \\ &= 0 \end{aligned} \tag{14.37}$$

$$\begin{aligned} (\not{p}+m)\sum_s v_s(p)\bar{v}_s(p) &= (\not{p}+m)(\not{p}-m) \\ &= p^2-m^2 \\ &= 0, \end{aligned} \tag{14.38}$$

as it was expected from the equation of motion,

$$(\not{p}-m)u_s(p) = 0, \qquad (\not{p}+m)v_s(p). \tag{14.39}$$

Additionally, (14.34) and (14.35) are independents on the representation of the gamma matrices used, as will see next,

$$
\begin{aligned}
\sum_s u_s(p)\bar{u}_s(p) &= \not{p} + m \\
\sum_s u_s(p)u_s^\dagger(p)\gamma^0 &= p_\mu \gamma^m u + m \\
\sum_s U u_s(p)u_s^\dagger(p)\gamma^0 U^\dagger &= p_\mu \gamma'^m u + m \\
\sum_s U u_s(p)u_s^\dagger(p)U^\dagger U\gamma^0 U^\dagger &= p_\mu \gamma'^m u + m \\
\sum_s u'_s(p)u'^\dagger_s(p)\gamma'^0 &= p_\mu \gamma'^m u + m \\
\sum_s u'_s(p)\bar{u}'_s(p) &= \not{p}' + m
\end{aligned} \tag{14.40}
$$

with $u'_s(p) = U u_s(p)$, $u'_s(p) = U u_s(p)$ and $\not{p}' = p_\mu \gamma'^m u$.

14.5 Dirac Propagator

Show that the Feynman propagator for the Dirac field,

$$
S(x-y)_{\alpha\beta} = \left\langle 0|T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\}|0\right\rangle \tag{14.41}
$$

is given by

$$
S(x-y) = \int \frac{d^3p}{(2\pi)^3 2E_p}\left\{\vartheta(x^0-y^0)(\not{p}+m)e^{-\imath p\cdot(x-y)} - \vartheta(y^0-x^0)(\not{p}-m)e^{\imath p\cdot(x-y)}\right\}. \tag{14.42}
$$

Show that this expression is equivalent to

$$
S(x-y) = \int \frac{d^4p}{(2\pi)^4}\tilde{S}(p)e^{-\imath p\cdot(x-y)}, \tag{14.43}
$$

where

$$
\tilde{S}(p) = \frac{\imath(\not{p}+m)}{p^2-m^2+\imath\epsilon} \tag{14.44}
$$

is the propagator in the momentum space. Show that $S(x-y)$ is a Green's function of the free Dirac operator, $\imath\,\not{\partial}_x - m$.

Answer

In class was shown eq. (14.42), so let us show the equivalence between (14.42) and (14.43).

$$S(x-y) = \int \frac{d^4p}{(2\pi^4)} \frac{\imath(\not{p}+m)}{(p^0)^2 - E_p^2 + \imath\epsilon} e^{-\imath p\cdot(x-y)}. \tag{14.45}$$

Consider

$$\begin{aligned} I &= \int_{-\infty}^{+\infty} \frac{dp^0}{2\pi} \frac{\imath(\not{p}+m)}{(p^0)^2 - E_p^2 + \imath\epsilon} e^{-\imath p^0\cdot(x^0-y^0)} \\ &= \int_{-\infty}^{+\infty} \frac{dp^0}{2\pi} \frac{\imath(p^0\gamma^0 - \vec{p}\cdot\vec{\gamma} + m)}{(p^0)^2 - E_p^2 + \imath\epsilon} e^{-\imath p^0\cdot(x^0-y^0)}, \end{aligned} \tag{14.46}$$

changing to complex variable,

$$p^0 \mapsto z = z_r + \imath z_i, \tag{14.47}$$

therefore, if $x^0 - y^0 < 0$ one must close the integral by below, and if $x^0 - y^0 > 0$ one must close the path by above.

Also, the poles are

$$z_\pm = \pm\sqrt{E_p^2 - \imath\epsilon} \cong \pm E_p \mp \imath\epsilon. \tag{14.48}$$

Then,

$$\begin{aligned} I &= \vartheta(x^0-y^0) \oint \frac{dz}{2\pi} \frac{\imath(z\gamma^0 - \vec{p}\cdot\vec{\gamma} + m)}{(z-z_+)(z-z_-)} e^{-\imath z(x^0-y^0)} \\ &\quad +\vartheta(y^0-x^0) \oint \frac{dz}{2\pi} \frac{\imath(z\gamma^0 - \vec{p}\cdot\vec{\gamma} + m)}{(z-z_+)(z-z_-)} e^{-\imath z(x^0-y^0)} \\ &= -\vartheta(x^0-y^0) 2\pi\imath \frac{1}{2\pi} \frac{\imath(z_+\gamma^0 - \vec{p}\cdot\vec{\gamma} + m)}{(z_+ - z_+)} e^{-\imath z_+(x^0-y^0)} \\ &\quad +\vartheta(y^0-x^0) 2\pi\imath \frac{1}{2\pi} \frac{\imath(z_-\gamma^0 - \vec{p}\cdot\vec{\gamma} + m)}{(z_- - z_+)} e^{-\imath z_-(x^0-y^0)} \\ &= \vartheta(x^0-y^0) \frac{(E_p - \imath\epsilon)\gamma^0 - \vec{p}\cdot\vec{\gamma} + m}{2(E_p - \imath\epsilon)} e^{-\imath(E_p-\imath\epsilon)(x^0-y^0)} \\ &\quad +\vartheta(y^0-x^0) \frac{(-E_p + \imath\epsilon)\gamma^0 - \vec{p}\cdot\vec{\gamma} + m}{2(E_p - \imath\epsilon)} e^{\imath(E_p-\imath\epsilon)(x^0-y^0)} \\ &= \vartheta(x^0-y^0) \frac{\not{p}+m}{2E_p} e^{-\imath p_0(x^0-y^0)} \qquad (14.49) \\ &\quad +\vartheta(y^0-x^0) \frac{-E_p\gamma^0 - \vec{p}\cdot\vec{\gamma} + m}{2E_p} e^{\imath E_p(x^0-y^0)}. \qquad (14.50) \end{aligned}$$

Thus,

$$S(x-y) = \int \frac{d^3p}{(2\pi)^3 2E_p} \left\{ \vartheta(x^0 - y^0)(\not{p} + m)e^{-\imath p\cdot(x-y)} - \vartheta(y^0 - x^0)(\not{p} - m)e^{\imath p\cdot(x-y)} \right\}, \qquad (14.51)$$

were in the second term the variable have been change $\vec{p} \mapsto -\vec{p}$.

Additionally,

$$\begin{aligned}
(\imath \not{\partial}_x - m)S(x-y) &= \int \frac{d^4p}{(2\pi)^4} \imath \frac{\not{p} - m)(\not{p} + m}{p^2 - m^2 + \imath\epsilon} e^{\imath p\cdot x} \\
&= \int \frac{d^4p}{(2\pi)^4} \imath \frac{p^2 - m^2 + \imath\epsilon}{p^2 - m^2 + \imath\epsilon} e^{\imath p\cdot x} \\
&\quad - \int \frac{d^4p}{(2\pi)^4} \imath \frac{\imath\epsilon}{p^2 - m^2 + \imath\epsilon} e^{\imath p\cdot x} \\
&= \imath\delta^{(4)}(x-y) + \epsilon(\cdots) \\
&= \imath\delta^{(4)}(x-y). \qquad (14.52)
\end{aligned}$$

Chapter 15

... 2nd part

15.1 Kinematic restrictions for scattering and decay processes

a. Show that the following processes are not kinematically possible:

- Spontaneous emission of a photon by a free electron.
- A photon absorption by a free electron.
- A pair electron-positron from a single photon.
- A split of a photon into two.

b. Give a general argument that include all the above cases.

c. Obtain the condition that permit the decay of a particle of mass M decay into a triplet of particles of masses m_1, m_2 and m_3.

d. Is there any restriction on the masses m_1, m_2, m_3 and m_4 in order for scattering $1 + 2 \to 3 + 4$ to occur?

e. Apply the above result to enumerate all permit processes whose initial and final states include, a pair of photons and a couple of electron and/or positron.

Answer

a.

- Consider a free electron, and choose the CM frame. The energy in this frame is just $E_i = m$, but after the emission, $E_f = m + K_e + E_\gamma > E_i$, i.e., this process doesn't conserve energy and so it's not allowed.

- Once more choosing the CM. frame, $E_i = m + K_e + E_\gamma > E_f = m$, therefore it's not allowed.

- In the CM. frame, $P_f = P_{e^-} + P_{e^+} = 0$, but $P_i \neq 0$ because $m_\gamma = 0$.
- This last process can (eventually) conserve both, energy and momentum, but doesn't conserve helicity, so it's neither allow.

b. As it was seen in the above examples, processes of three particles which involve at least a massless one, are not kinematically allowed.

c. In order for a particle of mass M to decay into a triplet of particles with masses m_1, m_2 and m_3, energy and momentum must be conserved. It can be seen from the CM frame as

$$m_1 + T_1 + m_2 + T_2 + m_3 + T_3 = M, \quad (15.1)$$
$$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0, \quad (15.2)$$

where $\vec{p}_i$ are the momenta of the particles after the decay.

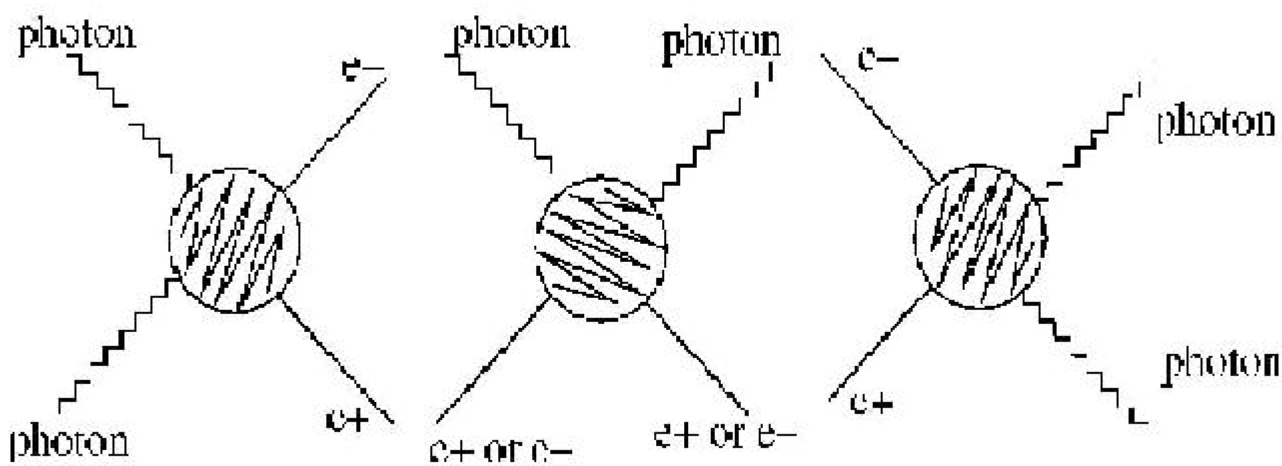
d. From the CM frame, the restriction to a $2 \to 2$ particles process is given by

$$m_1 + T + m_2 + T_2 = m_3 + T_3 + m_4 + T_4 \quad (15.3)$$

if the particles are massive.

If one of they is massless, the process is not allowed because of momentum (and helicity) conservation. Thus, a pair number of photons must be involved.

e.



15.2 Bhabha scattering

For the Bhabha scattering, $e^- + e^+ \to e^- + e^+$,

a. Show that the lower order contribution to the S matrix corresponding to this process is

$$S^{(2)}(e^- + e^+ \to e^- + e^+) = S_a + S_b, \tag{15.4}$$

with

$$S_a = -e^2 \int d^4x_1 d^4x_2 : (\bar{\psi}^- \gamma^\mu \psi^+)_{x_1} (\bar{\psi}^+ \gamma^\nu \psi^-)_{x_2} : D_{\mu\nu}(x_1 - x_2) \tag{15.5}$$

$$S_b = -e^2 \int d^4x_1 d^4x_2 : (\bar{\psi}^- \gamma^\mu \psi^-)_{x_1} (\bar{\psi}^+ \gamma^\nu \psi^+)_{x_2} : D_{\mu\nu}(x_1 - x_2). \tag{15.6}$$

b. From (15.5) and (15.6) get the Feynman amplitude for the process

$$e^+(p_1, s_1) + e^-(p_2, s_2) \to e^+(p_1', s_1') + e^-(p_2', s_2'). \tag{15.7}$$

Answer

a. For QED, the interaction Hamiltonian is

$$\mathscr{H}_I = -e : \bar{\psi} \not{A} \psi : . \tag{15.8}$$

Obviously, for just fermionic initial and final states

$$\left\langle \vec{p}_1 \vec{p}_2 \left| S^{(1)} \right| \vec{k}_1 \vec{k}_2 \right\rangle = 0. \tag{15.9}$$

Then, the first non-trivial contribution to the S matrix is second order on e, $\left\langle \vec{p}_1 \vec{p}_2 \left| S^{(2)} \right| \vec{k}_1 \vec{k}_2 \right\rangle$, with

$$S^{(2)} = -e^2 \int d^4x_1 d^4x_2 T \{: (\bar{\psi} \not{A} \psi)_{x_1} :: (\bar{\psi} \not{A} \psi)_{x_2} :\} \tag{15.10}$$

For the Bhabha scattering, no photons are involved at initial or final state, so the only possibility is that they'd be contracted, i.e.,

$$S^{(2)} = -e^2 \int d^4x_1 d^4x_2 : (\bar{\psi} \gamma^\mu \psi)_{x_1} (\bar{\psi} \gamma^\nu \psi)_{x_2} : D_{\mu\nu}(x_1 - x_2), \tag{15.11}$$

but

$$\bar{\psi} = \bar{\psi}^+ + \bar{\psi}^- \quad \text{and} \quad \psi = \psi^+ + \psi^-. \tag{15.12}$$

Since, ψ^+ annihilates an electron, ψ^- creates a positron, $\bar{\psi}^-$ annihilates an electron and $\bar{\psi}^+$ creates a positron, the set up should have all them once. It gives 4 different possibilities,

1. $(\bar{\psi}^- \gamma^\mu \psi^-)_{x_1} (\bar{\psi}^+ \gamma^\nu \psi^+)_{x_2}$

2. $(\bar{\psi}^{-}\gamma^{\mu}\psi^{+})_{x_1}(\bar{\psi}^{+}\gamma^{\nu}\psi^{-})_{x_2}$

3. $(\bar{\psi}^{+}\gamma^{\mu}\psi^{-})_{x_1}(\bar{\psi}^{-}\gamma^{\nu}\psi^{+})_{x_2}$

4. $(\bar{\psi}^{+}\gamma^{\mu}\psi^{+})_{x_1}(\bar{\psi}^{-}\gamma^{\nu}\psi^{-})_{x_2}$.

Obviously, 1 and 4 (2 and 3) are equals under interchange $x_1 \leftrightarrow x_2$. This gives a factor of 2 that cancels the $1/2$ coming from the Dyson's expansion.

Therefore, the two different contributions for Bhabha Scattering are,

$$\left\langle \vec{p}_1\vec{p}_2 \,|(S_a + S_b)|\, \vec{k}_1\vec{k}_2 \right\rangle, \tag{15.13}$$

with

$$S_a = -e^2 \int d^4x_1 d^4x_2 : (\bar{\psi}^{-}\gamma^{\mu}\psi^{-})_{x_1}(\bar{\psi}^{+}\gamma^{\nu}\psi^{+})_{x_2} : D_{\mu\nu}(x_1 - x_2) \tag{15.14}$$

$$S_b = -e^2 \int d^4x_1 d^4x_2 : (\bar{\psi}^{-}\gamma^{\mu}\psi^{+})_{x_1}(\bar{\psi}^{+}\gamma^{\nu}\psi^{-})_{x_2} : D_{\mu\nu}(x_1 - x_2). \tag{15.15}$$

b. By using the Feynman rules, the Feynman amplitude to the Bhabha scattering up to second order is

$$\begin{aligned} \mathscr{M}^{(2)} &= -\imath\frac{\eta_{\mu\nu}}{(p_1+p_2)^2 + \imath\epsilon}\bar{v}_{s_1}(p_1)\gamma^{\mu}u_{s_2}(p_2)\bar{u}_{s_2'}(p_2')\gamma^{\nu}v_{s_1'}(p_1') \\ &\quad -\imath\frac{\eta_{\mu\nu}}{(p_1+p_2)^2 + \imath\epsilon}\bar{u}_{s_2'}(p_2')\gamma^{\mu}u_{s_2}(p_2)\bar{v}_{s_1}(p_1)\gamma^{\nu}v_{s_1'}(p_1') \end{aligned} \tag{15.16}$$

15.3 Scattering by an external potential

Consider a real scalar field whose dynamic is described by the lagrangian

$$\mathscr{L}(x) = \mathscr{L}_0(x) + \mu U(\vec{x})\phi^2(x), \tag{15.17}$$

where $\mathscr{L}_0$ is the lagrangian of a free scalar field with mass μ and $U(\vec{x})$ is an static external potential.

a. Derive the equation of motion,

$$(\Box + \mu^2)\phi(x) = 2\mu U(\vec{x})\phi(x). \tag{15.18}$$

b. Show that to coupling this external potential gives rise to processes of scattering from a particle with initial momentum $k = (E, \vec{k})$ to final momentum $p = (E', \vec{p})$. Show that, the lower order contribution to the S matrix for this transition is given by

$$\left\langle \vec{p} \,|S(1)|\, \vec{k} \right\rangle = 2\imath\mu\frac{2\pi\delta(E - E')}{\sqrt{2VE}\sqrt{2VE'}}\widetilde{U}(\vec{p} - \vec{k}), \tag{15.19}$$

where

$$\widetilde{U}(\vec{q}) = \int d^3x U(\vec{x})e^{-\imath\vec{q}\cdot\vec{x}}. \tag{15.20}$$

Answer

In order to get the eqs. of motion, one apply the Euler-Lagrange eqs.,

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) = \partial_\mu \partial^\mu \phi, \tag{15.21}$$

and

$$\frac{\partial \mathscr{L}}{\partial \phi} = -\mu^2 \phi + 2\mu U(\vec{x})\phi, \tag{15.22}$$

so,

$$(\Box + \mu^2)\phi(x) = 2\mu U(\vec{x})\phi(x). \tag{15.23}$$

Since $\mathscr{L}_I$ is time-independent, then

$$H_I(x) = -\mu U(\vec{x}) : \phi^2(x) : . \tag{15.24}$$

The initial and final states are obtained from the vacuum via

$$|\vec{k} > \;=\; \sqrt{2E} a_k^\dagger |0 > \tag{15.25}$$

$$|\vec{p} > \;=\; \sqrt{2E'} a_p^\dagger |0 > . \tag{15.26}$$

Also, the Dyson expansion for the S matrix is,

$$S = \mathbb{1} + (-\imath) \int d^4 x \mathscr{H}_I(x) + \frac{(-\imath)^2}{2!} \int d^x d^4 y T\{\mathscr{H}_I(x)\mathscr{H}_I(y)\} + \cdots , \tag{15.27}$$

therefore, for (15.24) and $\vec{p} \neq \vec{k}$, one get,

$$\begin{aligned}
\left\langle \vec{p} \,|S\,(1)|\, \vec{k} \right\rangle &= \imath\mu \sqrt{2E} \sqrt{2E'} \int d^4 x U(\vec{x}) \left\langle 0 \left| a_p : \phi^2(x) : a_k^\dagger \right| 0 \right\rangle \\
&= \imath\mu \lim_{z^0 \to \infty} \lim_{y^0 \to \infty} \int d^4 x d^3 z d^3 y e^{\imath p \cdot z} e^{-\imath k \cdot y} \overleftrightarrow{\partial}_{z^0} \overleftrightarrow{\partial}_{y^0} U(\vec{x}) \left\langle 0 \left| \phi(z) : \phi^2(x) : \phi(y) \right| 0 \right\rangle \\
&= \imath\mu \lim_{z^0 \to \infty} \lim_{y^0 \to \infty} \int d^4 x d^3 z d^3 y e^{\imath p \cdot z} e^{-\imath k \cdot y} \overleftrightarrow{\partial}_{z^0} \overleftrightarrow{\partial}_{y^0} U(\vec{x}) 2D(z - x) D(x - y) \\
&= 2\imath\mu \int d^4 x e^{\imath p \cdot x} e^{-\imath k \cdot x} U(\vec{x}) \\
&= 2\imath\mu \int d^3 x e^{-\imath(\vec{p} - \vec{k}) \cdot \vec{x}} U(\vec{x}) \int dx^0 e^{\imath(E - E')x^0} \\
&= 2\imath\mu (2\pi) \delta(E - E') \widetilde{U}(\vec{p} - \vec{k}).
\end{aligned} \tag{15.28}$$

Note that if the particle state is defined by

$$|\vec{p} >= a_p^\dagger |0 >, \tag{15.29}$$

the result would be

$$\left\langle \vec{p} \,|S\,(1)|\, \vec{k} \right\rangle = 2\imath\mu \frac{(2\pi)\delta(E - E')}{\sqrt{2E} \sqrt{2E'}} \widetilde{U}(\vec{p} - \vec{k}). \tag{15.30}$$

15.4 Feynman rules for a pseudo-scalar mesons theory

Consider a theory for pseudo-scalar mesons, defined by the lagrangian

$$\mathscr{L}(x) = \mathscr{L}_0(x) + \mathscr{L}_{int}(x), \tag{15.31}$$

where

$$\mathscr{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\mu^2}{2}\phi^2 + \bar{\psi}(\imath\, \not{\partial} - m)\psi, \tag{15.32}$$

and

$$\mathscr{L}_{int} = -\imath g\bar{\psi}\gamma^5\psi\phi. \tag{15.33}$$

Use the similarity between this theory and QED for writing its Feynman rules.

Answer

Remind that in order to construct the Feynman diagram, one assign different lines to different sort of particles, i.e., two kind of propagators in the described set up, say, straight lines for fermions and dashed ones for scalar field. Also, the vertex is a convergent point of as many propagators as the order of the interaction.

Next are listed the Feynman rules for the set up theory.

1. Draw all graphs with a given number of vertices, n. Remember, fermion lines are oriented and are closed or infinite (it means begin and end at $\pm\infty$).

2. For each vertex, write a factor $g\gamma^5$.

3. For each internal bosonic line, labelled by the momentum k, write a factor

$$D(k) = \frac{i}{k^2 - \mu^2 + \imath\epsilon}. \tag{15.34}$$

4. For each internal fermion line, labelled by the momentum p, write a factor

$$S(p) = \imath\frac{\not{p} + m}{p^2 - m^2 + \imath\epsilon}. \tag{15.35}$$

5. Add one of the following factors for each external line:

    - initial electron: $u_s(p)$.
    - initial positron: $\bar{v}_s(p)$.
    - final electron: $\bar{u}_s(p)$.
    - final positron: $v_s(p)$.

Also each external boson has an associated momentum.

6. The spinor factors for each fermion line are ordered so that, reading from right to left, they occur in the same sequence as following the fermion line in the direction of its arrow.

7. For each fermion loop, take trace and multiply by a factor (-1).

8. For each momentum, q, which is not fixed by energy-momentum conservation, carry out the integration

$$\frac{1}{(2\pi)^4}\int d^4q.$$

9. Multiply the expression by a phase factor δ_P which is equal to +1 or -1 if an ever or odd number of interchanges of neighboring fermion operators is required to write the fermion operators in the correct normal order.

The above rules give the Feynman amplitude, $\mathscr{M}^{(n)}$, of order n (number of vertices). From here, the n-th order of the S matrix is given by

$$\left\langle f\left|S^{(n)}\right|i\right\rangle = (2\pi)^4\delta^{(4)}(P_f - P_i)\mathscr{M}^{(n)}. \tag{15.36}$$

Chapter 16

QFT II 3

16.1 $\gamma - \gamma$ Scattering

Unlike classical electrodynamics, QED predicts that a couple of light shafts that cross, suffer certain scattering,

a. Identify and write down the lower order, in the Dyson expansion for the S-matrix, which allows the process $2\gamma \to 2\gamma$. Write it as compact as possible, having special care with the final coefficients.

b. Draw the Feynman diagrams -in the momentum space- associated to the above process, and by using the Feynman rules write the expression of the Feynman amplitude.

Answer

Since,

$$H_I = -e \int d^3x : \bar{\psi} \not{A} \psi :, \tag{16.1}$$

and we have four photons in both, initial and final states, in order to get a non-trivial contribution we must consider the Dyson expansion to order fourth,

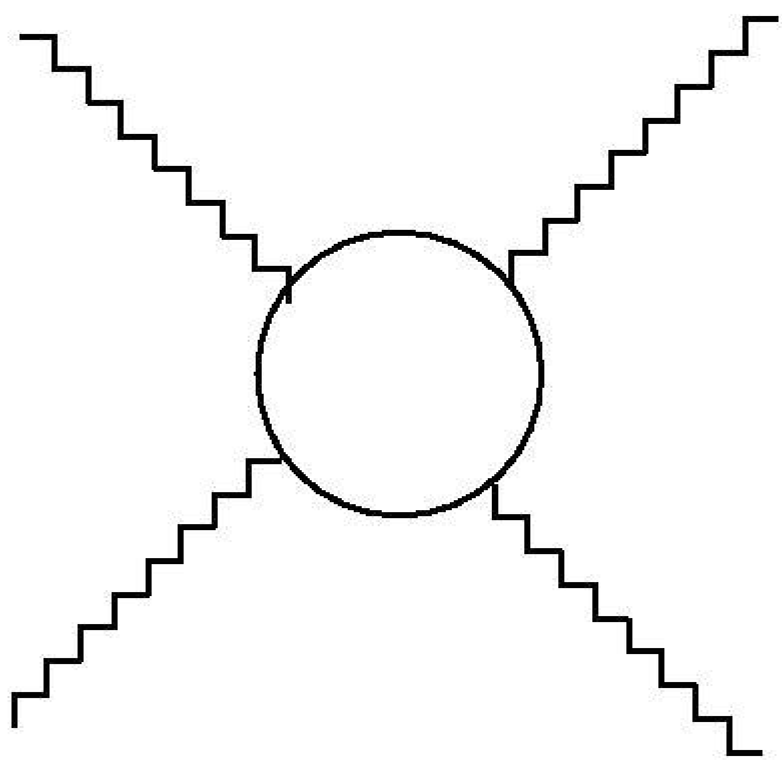
$$\begin{aligned}
\langle f|S|i\rangle &= \frac{(\imath e)^4}{4!} \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \\
&\quad \langle f|T\{: (\bar{\psi} \not{A}\psi)_{x_1} :: (\bar{\psi} \not{A}\psi)_{x_2} :: (\bar{\psi} \not{A}\psi)_{x_3} :: (\bar{\psi} \not{A}\psi)_{x_4} :\}|i\rangle \\
&= \frac{(\imath e)^4}{4!} \frac{1}{2VE_{p_1}} \frac{1}{2VE_{p_2}} \frac{1}{2VE_{p_3}} \frac{1}{2VE_{p_4}} \epsilon_\mu(p_1)\epsilon_\nu(p_2)\epsilon^*_\lambda(p_3)\epsilon^*_\rho(p_4) \\
&\quad \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{\imath p_1\cdot x_1} e^{\imath p_2\cdot x_2} e^{\imath p_3\cdot x_3} e^{\imath p_4\cdot x_4} \\
&\quad \left\langle f|T\left\{: (\bar{\psi}\gamma^\mu\psi)_{x_1} :: (\bar{\psi}\gamma^\nu\psi)_{x_2} :: (\bar{\psi}\gamma^\lambda\psi)_{x_3} :: (\bar{\psi}\gamma^\rho\psi)_{x_4} :\right\}|i\right\rangle .
\end{aligned} \tag{16.2}$$

Now, we must contract all the fermionic fields by using the Wicks thm. Of course, since all fermions are contracted, they'll form a loop, so we get a minus sign. Furthermore, they must not form disconnect groups, so that this will generate disconnected Feynman diagrams (it can be shown that disconnected diagrams can be neglected, by the cluster decomposition principle[1]). Additionally, the total number of permutations between the integration variables gives us a 4!.

Then,

$$\begin{aligned}\langle f|S|i\rangle &= -(\imath e)^4 \frac{1}{2VE_{p_1}}\frac{1}{2VE_{p_2}}\frac{1}{2VE_{p_3}}\frac{1}{2VE_{p_4}}\epsilon_\mu(p_1)\epsilon_\nu(p_2)\epsilon^*_\lambda(p_3)\epsilon^*_\rho(p_4)\\ &\quad (2\pi)^4\delta^{(4)}(p_i - p_f) \\ &\quad Tr\left\{\int \frac{d^4q}{(2\pi)^4} S(q)\gamma^\lambda S(q+p_2)\gamma^\rho S(q+p_1+p_2)\gamma^\mu S(q-p_4)\gamma^\nu\right\}.\end{aligned} \tag{16.3}$$

The Feynman graph for this process is



[1]See S. Weinberg, Quantum Theory of fields, Vol. I.

Clearly, from the diagram it follows that

$$\begin{aligned}\mathscr{M} &= -(\imath e)^4 \epsilon_\mu(p_1)\epsilon_\nu(p_2)\epsilon^*_\lambda(p_3)\epsilon^*_\rho(p_4) \\ & Tr\left\{\int \frac{d^4q}{(2\pi)^4} S(q)\gamma^\lambda S(q+p_2)\gamma^\rho S(q+p_1+p_2)\gamma^\mu S(q-p_4)\gamma^\nu\right\}\end{aligned} \tag{16.4}$$

16.2 Rutherford's scattering

Peskin-Shoeder problem 4.4

Answer

Since

$$\psi^+(x)|e^-(\vec{p},s)>=|0>\frac{1}{(2VE_p)^{1/2}}u_s(p)e^{-\imath p\cdot x}, \tag{16.5}$$

it follows that, at first order in perturbations,

$$\begin{aligned}\left\langle p'\left|\imath e\int d^4x : \bar{\psi}\gamma^\mu\psi : A_\mu\right| p\right\rangle &= \left\langle p'\left|\imath e\int d^4x : \bar{\psi}^-\gamma^\mu\psi^+ : A_\mu\right| p\right\rangle \\ &= \imath e\frac{1}{\sqrt{2VE_p}}\frac{1}{\sqrt{2VE_{p'}}}\bar{u}_{s'}(p')\gamma^m u u_s(p)\int d^4x e^{-\imath(p-p')\cdot x}A_\mu \\ &= \imath e\frac{1}{\sqrt{2VE_p}}\frac{1}{\sqrt{2VE_{p'}}}\bar{u}_{s'}(p')\gamma^m u u_s(p)\widetilde{A}_\mu(p-p').\end{aligned} \tag{16.6}$$

If A_μ does not depend on t,

$$\left\langle p'\left|\imath e\int d^4x : \bar{\psi}\gamma^\mu\psi : A_\mu\right| p\right\rangle = \imath e\widetilde{A}_\mu(\vec{p}-\vec{p}')2\pi\delta(E-E')\frac{1}{\sqrt{2VE_p}}\frac{1}{\sqrt{2VE_{p'}}}\bar{u}_{s'}(p')\gamma^\mu u_s(p). \tag{16.7}$$

Now, the cross-section for this process is given by

$$d\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{d^3p^f}{(2\pi)^3 2E_f}|\mathscr{M}(p_i\to p_f)|^2(2\pi)\delta(E_i-E_f), \tag{16.8}$$

it can be written as,

$$d\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{|p_f|^2 d|p^f|d\Omega}{(2\pi)^3 2E_f}|\mathscr{M}(p_i\to p_f)|^2(2\pi)\delta(E_i-E_f). \tag{16.9}$$

Since,

$$\delta(F(x)-F(x_0)) = \frac{1}{|F'(x_0)|}\delta(x-x_0), \tag{16.10}$$

and

$$\frac{dE_f}{d|p_f|} = \frac{p_f}{E_f}, \tag{16.11}$$

then, we can integrate over the modulus of the momentum,

$$\int \frac{|p_f|^2 d|p_f|}{(2\pi)^3 2E_f} 2\pi\delta(E_i - E_f) = \frac{p_i}{(2\pi)^2}\frac{1}{2}. \tag{16.12}$$

Thus,

$$d\sigma = \frac{1}{2(2\pi)^2}\frac{p_i}{v_i}\frac{1}{2E_i}|\mathscr{M}|^2 d\Omega, \tag{16.13}$$

next, we'll use the equation

$$E_i = \frac{p_i}{v_i}, \tag{16.14}$$

we get,

$$d\sigma = \frac{1}{(4\pi)^2}|\mathscr{M}|^2 d\Omega. \tag{16.15}$$

Let's now restrict ourselves to the Coulombian potential,

$$A_0 = \frac{Ze}{4\pi}\frac{1}{r}. \tag{16.16}$$

Therefore,

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{(4\pi)^2}|\mathscr{M}|^2 \\ &= \left(\frac{e^2}{4\pi}\right)^2 Z^2 \frac{1}{|\vec{p} - \vec{p}'|^4}\frac{1}{2}\sum_{r,s}|\bar{u}_r(p')\gamma^0 u_s(p)|^2 \\ &= \frac{1}{2}\frac{(\alpha Z)}{|\vec{q}|^4} tr\left\{(\not{p}'\gamma^0\,\not{p}\gamma^0) + m^2\mathbb{1}\right\} \\ &= \frac{1}{2}\frac{(\alpha Z)}{|\vec{q}|^4} tr\left\{p'_\mu p_\nu(\gamma^\mu\gamma^0\gamma^\nu\gamma^0) + m^2\mathbb{1}\right\} \\ &= \frac{1}{2}\frac{(\alpha Z)}{|\vec{q}|^4}\left\{4p'_\mu p_\nu(2\eta^{\mu 0}\eta^{\nu 0} - \eta^{\mu\nu}) + 4m^2\right\} \\ &= 2\frac{(\alpha Z)}{|\vec{q}|^4}\left(m^2 + E^2 + \vec{p}\cdot\vec{p}'\right). \end{aligned} \tag{16.17}$$

In the above calculations, we've used the relations

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m \tag{16.18}$$

$$tr(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho) = 4\left[\eta^{\mu\nu}\eta^{\lambda\rho} - \eta^{\mu\lambda}\eta^{\rho\nu} + \eta^{\mu\rho}\eta^{\nu\lambda}\right]. \tag{16.19}$$

Finally, by using

$$\vec{p} \cdot \vec{p'} = |p| \cos\theta, \tag{16.20}$$

and

$$|\vec{p} - \vec{p'}|^2 = 2|p|^2(1 - \cos\theta) = 4|p|^2 \sin^2(\theta/2), \tag{16.21}$$

we obtain,

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{8}\frac{(\alpha Z)^2}{|p|^4}\frac{(m^2 + E^2 + |p|^2\cos\theta)}{\sin^4(\theta/2)} \\ &= \frac{1}{4}\frac{(\alpha Z)^2}{E^2 v^4}\frac{(1 - v^2\sin^2(\theta/2))}{\sin^4(\theta/2)}. \end{aligned} \tag{16.22}$$

In the non-relativistic case, $v \ll 1$ and $E^2 \cong m^2$, then

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\frac{(\alpha Z)^2}{E^2 v^4}\frac{1}{\sin^4(\theta/2)}. \tag{16.23}$$

16.3 Crossing symmetry

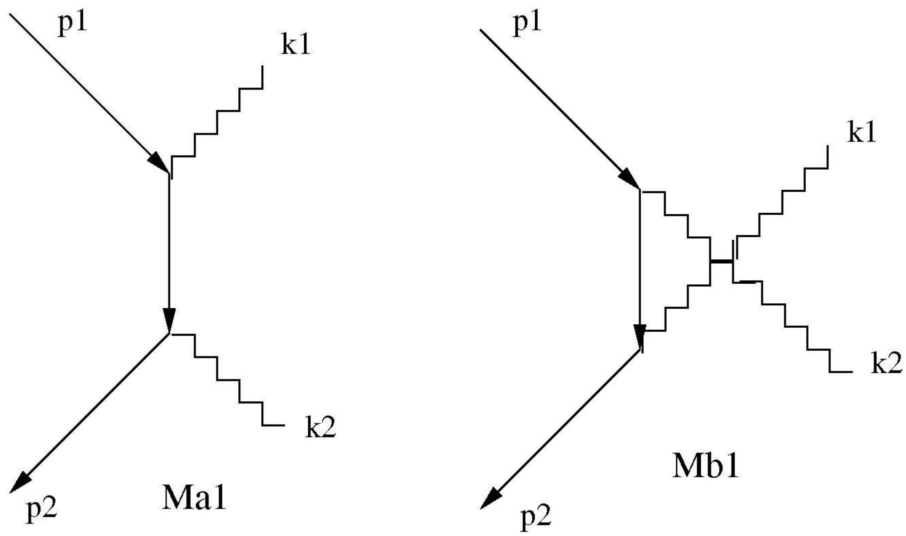
a. Draw the Feynman diagrams, at tree level, in the momentum space for $e^+(p_2) + e^-(p_1) \to (k_1) + \gamma(k_2)$. Use the Feynman rules in order to write the Feynman amplitudes associated to each diagram.

b. Draw the Feynman diagrams, at tree level, in the momentum space for $e^- + \gamma \to e^- + \gamma$. Use the Feynman rules in order to write the Feynman amplitudes associated to each diagram.

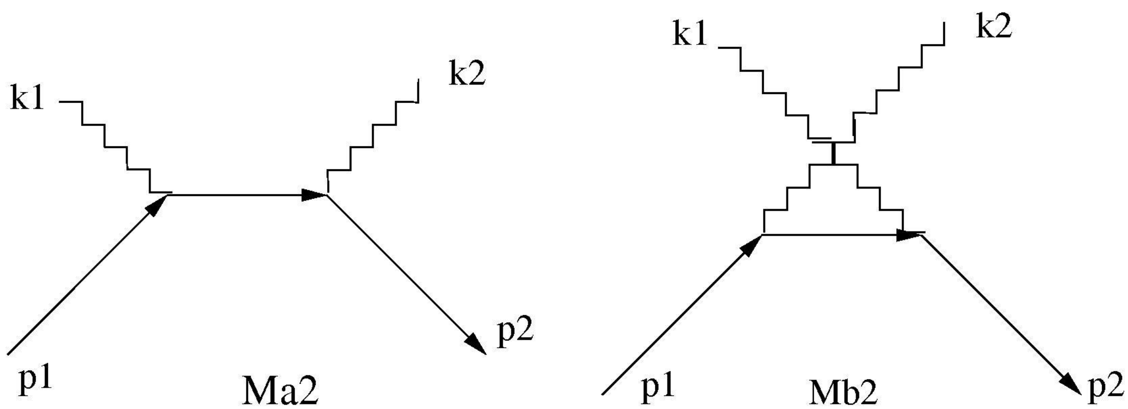
c. Show that the amplitudes of each one of the diagrams of the annihilation (a.) can be obtained from the ones of the Compton effect (b.) through certain identification of the momentum variables.

d. Explain what crossing symmetry consist on.

Answer



Annihilation process.



Compton Scattering.

And from the diagrams,

$$\mathscr{M}_{a1} = (\imath e)^2 \bar{v}_r(p_2)\gamma^\mu S(p_1 - k_1)\gamma^\nu u_s(p_1)\varepsilon^*_\mu(k_2)\varepsilon^*_\nu(k_1) \tag{16.24}$$

$$\mathscr{M}_{b1} = (\imath e)^2 \bar{v}_r(p_2)\gamma^\mu S(p_1 - k_2)\gamma^\nu u_s(p_1)\varepsilon^*_\mu(k_1)\varepsilon^*_\nu(k_2) \tag{16.25}$$

$$\mathscr{M}_{a2} = (\imath e)^2 \bar{u}_r(p_2)\gamma^\mu S(p_1 + k_1)\gamma^\nu u_s(p_1)\varepsilon^*_\mu(k_2)\varepsilon_\nu(k_1) \tag{16.26}$$

$$\mathscr{M}_{b2} = (\imath e)^2 \bar{u}_r(p_2)\gamma^\mu S(p_1 - k_2)\gamma^\nu u_s(p_1)\varepsilon_\mu(k_1)\varepsilon^*_\nu(k_2). \tag{16.27}$$

Let's change $p_2 \mapsto -p_2$ and $k_1 \mapsto -k_1$ in the amplitudes $\mathscr{M}_1$, then,

$$\mathscr{M}'_{a1} = (\imath e)^2 \bar{v}_r(-p_2)\gamma^\mu S(p_1 + k_1)\gamma^\nu u_s(p_1)\varepsilon^*_\mu(k_2)\varepsilon^*_\nu(-k_1) \tag{16.28}$$

$$\mathscr{M}'_{b1} = (\imath e)^2 \bar{v}_r(-p_2)\gamma^\mu S(p_1 - k_2)\gamma^\nu u_s(p_1)\varepsilon^*_\mu(-k_1)\varepsilon^*_\nu(k_2). \tag{16.29}$$

But the eq. of motion for spinos are,

$$(\not{p} + m)u_s(p) = 0 \tag{16.30}$$

$$(-\not{p} + m)v_s(p) = 0, \tag{16.31}$$

in the above equations, we can change $u_s(-p)$ by $v_s(p)$. Additionally, the polarization of a photon is complex just if it's elliptically polarised, therefore, under a change in the momentum $k \mapsto -k$, also the orientation of the polarization must change, i.e.,

$$\varepsilon_\mu(k) = \varepsilon^*_\mu(-k). \tag{16.32}$$

Finally, we get

$$\mathscr{M}'_{a1} = (\imath e)^2 \bar{u}_r(p_2)\gamma^\mu S(p_1 + k_1)\gamma^\nu u_s(p_1)\varepsilon^*_\mu(k_2)\varepsilon_\nu(k_1) \tag{16.33}$$

$$\mathscr{M}'_{b1} = (\imath e)^2 \bar{u}_r(p_2)\gamma^\mu S(p_1 - k_2)\gamma^\nu u_s(p_1)\varepsilon_\mu(k_1)\varepsilon^*_\nu(k_2), \tag{16.34}$$

that are nothing but the Feynman amplitudes for the Compton scattering.

Let's try now of explaining what this crossing symmetry is.

Both process, $e^+ + e^- \to 2\gamma$ and Compton scattering, (at tree level) are obtained from the second order expansion of the S-matrix for QED. Nonetheless, for each process we decide to take the positive or negative frecuency part of the fields, depending on the initial and final particles of the process. Since the total numbers of photons and leptons are the same, the Feynman graph for a process can be obtained from the topologically equivalents Feynman diagrams of the other, 'cause it's just a different choice of the positive and negative frecuency of the field. Furthermore, the combinatoric ceofficients are equal for both processes.

In the crossing symmetry, just as we've seen, if we take a lepton from initial to final (or viceversa) we should change the particle by it'a anti-particle. Similarly, if we change a photon, we should change it polarization.

Chapter 17

QGT II 4

17.1 Relativistic form for the flux factor

Answer

Consider

$$I = \sqrt{(\vec{p}_1 \cdot p_2)^2 - m_1^2 m_2^2}, \tag{17.1}$$

then, for $p_1 = (E_2, \vec{p}_1)$ and $p_2 = (E_2, \vec{p}_2)$, it follows that,

$$\begin{aligned} I &= \sqrt{(E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2)^2 - (E_1^2 - \vec{p}_1^2)(E_2^2 - \vec{p}_2^2)} \\ &= \sqrt{(\vec{p}_1 \cdot \vec{p}_2)^2 - 2E_1 E_2((\vec{p}_1 \cdot \vec{p}_2) - \vec{p}_1^2 \vec{p}_2^2 + E_1^2 \vec{p}_2 + E_2^2 \vec{p}_1} \\ &= E_1 E_2 \sqrt{(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \times \vec{v}_2)^2}. \end{aligned} \tag{17.2}$$

From (17.1), I is a Lorents invariant, therefore (17.2) is so. Furthermore, the density transform as a zeroth component of a 4-vector, as the energy, thus, if (17.2) is a lorentz invariant, $I' = n_1 n_2 \sqrt{(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \times \vec{v}_2)^2}$ is also a Lorentz invariant.

in the CM. frame, where $p_1 = (E, \vec{p})$ and $p_2 = (E', -\vec{p})$,

$$\begin{aligned} I &= \sqrt{(EE' + |\vec{p}|^2)^2 - ((E^2 - |\vec{p}|^2)(E'^2 - |\vec{p}|^2))} \\ &= \sqrt{E^2 E'^2 + |\vec{p}|^4 + 2|\vec{p}|^2 EE' - E^2 E'^2 - |\vec{p}|^4 + |\vec{p}|^2 (E^2 + E'^2)} \\ &= |\vec{p}| \sqrt{s}. \end{aligned} \tag{17.3}$$

17.2 Two bodies phase space in the lab frame

Maggiore, 6.2.

Answer

Since the velocity of the second particle in the initial frame is v_2, we should boost our set up by

$$gamma(v_2) = \frac{1}{\sqrt{1-v_2^2}}, \tag{17.4}$$

then,

$$V_{2,lab} = \frac{v_2 - v_2}{1 - v_2^2} = 0. \tag{17.5}$$

Also,

$$\begin{aligned} E_{lab} &= \gamma_2(E' + v_2 p'_x) \\ &= \gamma_2(E' + v_2|p'|\cos(\theta)). \end{aligned} \tag{17.6}$$

Thus,

$$dE_{lab} = \gamma_2 v_2 |p'| d(\cos(\theta)). \tag{17.7}$$

Moreover,

$$\begin{aligned} d\Phi^{(2)}_{CM} &= \frac{1}{16\pi^2}\frac{|\vec{p}'|}{\sqrt{s}} d\Omega \\ &= \frac{1}{16\pi^2}\frac{|\vec{p}'|}{\sqrt{s}} d\phi d(\cos(\theta)), \end{aligned} \tag{17.8}$$

next, by integrate on ϕ, we get,

$$d\Phi^{(2)}_{lab} = \frac{dE_{lab}}{8\pi\gamma_2 v_2 \sqrt{s}}. \tag{17.9}$$

17.3 Elastic $e^-\mu^-$ scattering

Mandl and Shaw, 8.2.

Answer

17.4 High energy elastic e^-e^- scattering

Mandl and Shaw, 8.6.

Answer

In here we'll work on the CM. frame, then

$$\begin{aligned} d\sigma &= \frac{1}{4I}|\mathscr{M}_a + \mathscr{M}_b|^2 d\Phi^{(2)} \\ &= \frac{1}{64\pi^2 s}\frac{1}{4}\overline{|\mathscr{M}_a + \mathscr{M}_b|^2} d\Omega, \end{aligned} \tag{17.10}$$

where,

$$\mathscr{M}_a = -\imath\frac{e^2}{(p_1 - p_1')^2}\bar{u}_{r_1}(p_1')\gamma^\mu u_{s1}(p_1)\bar{u}_{r_2}(p_2')\gamma^\mu u_{s2}(p_2) \tag{17.11}$$

$$\mathscr{M}_b = \imath\frac{e^2}{(p_1 - p_2')^2}\bar{u}_{r_2}(p_2')\gamma^\mu u_{s1}(p_1)\bar{u}_{r_1}(p_1')\gamma^\mu u_{s2}(p_2), \tag{17.12}$$

where the index 1 and 2 denote electrons and muons respectively, and also prime variables denote final momentum. Bellow, we'll drop all contribution coming from electron mass because $E \gg m_e$.

Note that,

$$|\mathscr{M}_a + \mathscr{M}_b|^2 = |\mathscr{M}_a|^2 + |\mathscr{M}_b|^2 + \mathscr{M}_a^*\mathscr{M}_b + \mathscr{M}_a\mathscr{M}_b^*, \tag{17.13}$$

then,

$$\begin{aligned} \frac{1}{4}\overline{|\mathscr{M}_a|^2} &= \frac{e^4}{4[(p_1 - p_1')^2]^2}\sum_{s_i,r_i}\bar{u}_{r_1}(p_1')\gamma^\mu u_{s1}(p_1)\bar{u}_{s_1}(p_1)\gamma^\nu u_{r1}(p_1') \\ &\quad \bar{u}_{r_2}(p_2')\gamma_\mu u_{s2}(p_2)\bar{u}_{s_2}(p_2)\gamma_\nu u_{r2}(p_2') \end{aligned} \tag{17.14}$$

$$= \frac{e^4}{4[(p_1 - p_1')^2]^2}Tr(\not{p}_1'\gamma^\mu \not{p}_1\gamma^\nu)Tr(\not{p}_2'\gamma^\mu \not{p}_2\gamma^\nu) \tag{17.15}$$

Part VI

Appendixes

Chapter 18

Geometry: Conics

This chapter gain importance when one is interested in Kepler's problem, i.e., a particle in a potential $U(\vec{r}) \sim |\vec{r}|^{-1}$. Here we give a brief review of conics, further treatment can be found in a book on analytical geometry.

18.1 Circumference

A circumference is described by the equation[1]

$$x^2 + y^2 = R^2, \tag{18.1}$$

where R is the radius.

18.2 Ellipse

An ellipse is described by the equation,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{18.2}$$

with a and b the major and minor radii, respectively[2], i.e., $a > b$. Foci lie at the major diameter at a distance c, such that, $c^2 = a^2 - b^2$. eccentricity, ϵ, is defined by

$$\epsilon = \frac{c}{a}. \tag{18.3}$$

Obviously, for a circumference $\epsilon = 0$.

We can also characterize a point on the ellipse by the radii from the foci,

$$r_1 = a + \epsilon x, \quad r_2 = a - \epsilon x. \tag{18.4}$$

[1]Next we consider just figures centered at the origin of coordinates, without lost of generality.
[2]We've chosen the large side of the ellipse oriented in x-axis.

18.3 Hyperbola

A hyperbola is described by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{18.5}$$

where a is the distance from the origin to the vertice and b such that the hyperbola goes asymptotically as

$$y = \pm\frac{b}{a}x. \tag{18.6}$$

The foci are located at $c = \sqrt{a^2 + b^2}$, again the eccentricity is $\epsilon = \frac{c}{a}$ but now since $c > a \Rightarrow \epsilon > 1$. In term of the focal radii,

$$r_1 = a + \epsilon x, \quad r_2 = a - \epsilon x. \tag{18.7}$$

18.4 Parabola

A parabola is described by the equation

$$y^2 = 2px, \tag{18.8}$$

where focus is located at $p/2$ from the vertice. The focal radius is $r = x + p/2$ and eccentricity is $\epsilon = 1$.

18.5 Conics in polar coordinates

All the conics named before can be written in polar coordinates by the equation

$$\rho = \frac{p}{1 - \epsilon\cos\theta}. \tag{18.9}$$

Chapter 19

Special Functions

Special functions play an important rôle when one wants to solve the equation of motion for certain systems, of course we don't pretend nor try to cover the subject, but showing a punctual useful formulae such as definitions, special values and so on.

19.1 Gamma Function

Integral forms for the gamma function are

$$\Gamma(z) = \int_0^\infty dt e^{-t} t^{z-1} \tag{19.1}$$

$$= x^z \int_0^\infty dt e^{-xt} t^{z-1} \tag{19.2}$$

$$= \int_0^\infty dt \ln(t) e^{-t} (t-z) e^{z-1} \tag{19.3}$$

$$= \int_{-\infty}^\infty dt \exp(zt - e^t). \tag{19.4}$$

Also,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}, \quad \Gamma(n+1) = n\Gamma(n). \tag{19.5}$$

19.2 Beta Function

Beta function can be defined in integral form by

$$\beta(x) = \int_0^1 dt \frac{t^{x-1}}{1+t} \tag{19.6}$$

$$= \int_0^\infty dt \frac{e^{-xt}}{1+e^{-t}}. \tag{19.7}$$

There exists the beta function for two parameters, defined by

$$B(x,y) = \int_0^1 dt t^{x-1}(1-t)^{y-1} \tag{19.8}$$

$$= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{19.9}$$

$$= B(y,x). \tag{19.10}$$

19.3 Bessel Functions

Bessel functions are solutions to the ordinary differential equation

$$\frac{d^2}{dx^2} Z_\nu(x) + \frac{1}{x}\frac{d}{dx} Z_\nu(x) + \left(1 - \frac{\nu^2}{x^2}\right) Z_\nu(x) = 0. \tag{19.11}$$

They have a lot of properties that we won't discuss here.

19.4 Associated Legendre Polynomials

Associated Legendre polynomials are solutions to the ordinary differential equation

$$(1-x^2)\frac{d}{dx} P_\nu^\mu(x) - 2x\frac{d}{dx} P_\nu^\mu(x) + \left(\nu(\nu+1) - \frac{\mu^2}{1-x^2}\right) P_\nu^\mu(x) = 0. \tag{19.12}$$

Legendre polynomials are the particular case $\mu = 0$, and they are defined by the relation

$$P_\nu(x) = \frac{1}{2^\nu \nu!}\frac{d^\nu}{dx^\nu}(x^2-1)^\nu. \tag{19.13}$$

Also,

$$P_\nu^\mu(x) = (-1)\mu(1-x^2)^{\mu/2}\frac{d^\mu}{dx^\mu} P_\nu(x). \tag{19.14}$$

19.5 Hermite Polynomials

Hermite polynomials are the solution to the ordinary differential equation

$$\frac{d^2}{dx^2}H_n(x) - 2x\frac{d}{dx}H_n(x) + 2nH_n(x) = 0. \tag{19.15}$$

Their Rodrigues' formula is

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}, \tag{19.16}$$

and they're a basis for the $L^2(x)$ satisfying

$$\int_{-\infty}^{\infty} dx e^{-x^2} H_n(x)H_m(x) = \begin{cases} 0; & m \neq n \\ \sqrt{\pi}2^n n!; & m = n \end{cases} \tag{19.17}$$

19.6 Laguerre polynomials

Laguerre polynomials are solution to the ordinary differential equation

$$x\frac{d^2}{dx^2}L_n^\alpha(x) + (\alpha + 1 - x)\frac{d}{dx}L_n^\alpha(x) + nL_n^\alpha(x) = 0. \tag{19.18}$$

Their Rodrigues' formula is

$$L_n^\alpha(x) = \frac{1}{n!}e^x x^{-\alpha}\frac{d^n}{dx^n}\left(e^{-x}x^\alpha\right) \tag{19.19}$$

$$= \sum_{m=0}^{n}(-1)^m \binom{n+\alpha}{n-\alpha}\frac{x^m}{m!} \tag{19.20}$$

They're also a basis for $L^2(x)$, and its orthogonality is given by

$$\int_0^{\infty} dx e^{-x} x^\alpha L_m^\alpha(x) L_n^\alpha(x) = \begin{cases} 0; & m \neq n \\ \binom{n+\alpha}{n}\Gamma(1+\alpha); & m = n \end{cases}. \tag{19.21}$$

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