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# $\mathscr{M}$-Theory compactification on $G_2$ manifolds

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**Abstract**

This is a short review article which was written for the ICTP diploma thesis.

We present in this report an introduction to $\mathscr{M}$-theory compactifications. In order to obtain a physical theory with $\mathscr{N} = 1$ supersymmetry on the resulting 4-dimensional spacetime, this compactification could be realized on a special kind of manifolds with exceptional holonomy group, $G_2$. Nonetheless, this compactification does not give rise to interesting physical phenomena, namely, non-Abelian gauge theories or chiral fermions, when the $G_2$ holonomy manifold is smooth. We present some ways which allow us obtaining these interesting physical aspects from the singularities.

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# Contents

# Introduction

During the XX century, the physical science was developed mainly due to two new branches of knowledge, General Relativity and Quantum Mechanics. The first is the best theory of gravitation we have by now and, of course, it is the theory of the macro physics (cosmological dimensions). The second is the theory of the elementary elements of the matter, and so the theory of the micro (atomic sides).

The major problem in modern physics arrived when physicists tried to unify both theories in an only "unified theory". The best approximation known, by now, is the Quantum Field Theory. QFT is no more than the union of the quantum mechanic with the Special Relativity.

Quantum field theory have succeed in the following aspects:

- Quantum Electro-Dynamic: The quantum theory of electromagnetism, is the best physical theory until now. It can be studied in perturbative approach.

- Electro-weak: It is the unified theory of electromagnetism and weak interaction.

- Standard Model: It is the theory which "unify" the electroweak interaction with the strong interaction (Quantum Chromo-Dynamic).

But quantum field theory fails in unifying gravity with the standard model.

At this point, (the supersymmetric version of) string theory, which was born as a theory of strong interaction, begins to play an important rôle in unification of interactions so that the quantum string theory is a quantum field theory which include a spin 2 particle which might be associated with the graviton in a quantum theory of gravity.

Other strong point is that as this theory includes supersymmetry, which solves the hierarchy[1]. SUSY can explain the problem because at some scale all the coupling constants (excluding gravitational one) coincide, this is called grand unified scale.

After the born of the superstring theory as a grand unified theory, it has found that:

1. Superstring theory lives in a 10-dimensional target spacetime.
2. There exist five different kinds of superstrings theories, as we know,
   - Type $I$
   - Type $IIA$ and $IIB$
   - Heterotic $SO(32)$ and $E_8 \times E_8$.

Referring to the first point, Candelas, Horowitz, Strominger and Witten [9] have shown that in order to keep one supersymmetry after the compactification of the 10-dimensional manifold as $M^{1,9} = M^{3,1} \times K$, with $M^{3,1}$ a maximally symmetric space, then $K$ should be a 3-fold Calabi-Yau manifold.

Respect to the second, after discovering duality relations among the different string theories, the existence of a "mother" theory was proposed, in a higher dimensional spacetime (a 11-dimensional target space), such that under certain limits, its effective theory in 10 dimensions give us each one of the five string theories. This theory is called $\mathscr{M}$-theory [23, 22].

After all, now we have a new theory in a higher dimensional spacetime, so in order to satisfy the condition of a remaining supersymmetry after the compactification, $M^{10,1} = M^{3,1} \times K$, $K$ should be a $G_2$ manifold. A $G_2$ manifold is a 7-dimensional, Ricci-flat manifold with holonomy group $G_2 \subset SO(7)$.

An important remark about $G_2$ manifolds is that, as in Calabi-Yau manifolds case, we do not know how many of those manifolds can exist. Despite this fact, many compact manifolds with $G_2$ holonomy have been constructed [24, 25].

Nevertheless, compactifying $\mathscr{M}$-theory on smooth $G_2$ manifolds gives us an effective 4-dimensional theory whose gauge group is an Abelian group, so does not describe realistic physics.

[1] That is explain the difference between electroweak mass and grand unified mass scale.

Now, it is understood that certain kinds of singularities give rise to interesting physics, such as chiral fermions [6, 34, 2] and non-Abelian gauge groups[4, 3].

The singularities which give rise to non-Abelian gauge groups are known as $ADE$ singularities, called like that because they are associated with the classification of finite subgroups of the Lie groups, which is described in terms of simple laced semi-simple Lie algebras, namely

$$\begin{array}{ccc} A_n & \rightsquigarrow & SU(n+1), \\ D_n & \rightsquigarrow & SO(2n), \\ E_6, & E_7, & E_8. \end{array}$$

Chiral fermions appear due to the existence of conical singularities in the manifold on which we are compactifying the theory.

The aim of this article is to review $\mathscr{M}$-theory compactification on $G_2$-holonomy manifolds. The report is organized as follows, the first chapter will introduces the relation between the holonomy groups and the supersymmetry. In the second, we shall present an introduction to $D = 11$ supergravity and finally an example of Kaluza-Klein compactification of $\mathscr{M}$-theory on smooth $G_2$-holonomy manifold is presented. As we have said, this method gives us non-realistic physics, for that reason in chapter 3 a set of ideas of how non-Abelian gauge groups for a realistic physics could be obtained from singularities are shown.

# Chapter 1

# Holonomy Groups and Supersymmetry

$\mathscr{M}$-theory lives in an 11-dimensional Riemannian manifold, $(M^{10,1}, g)$, and its low energy behavior is given by $D = 11$ supergravity. We want to compactify down to 4 dimensions, but we would like to preserve $\mathscr{N} = 1$ supersymmetry, because for extended supersymmetry, $\mathscr{N} \geq 2$, the 4-dimensional massless fermions transform in the real representation of the gauge group.

This condition is related to the existence of a covariantly constant spinor in the 7-dimensional manifold on which we are compactified. Now the problem turns into finding manifolds which have a covariant spinor. Our problem was solved by mathematicians by making use of the holonomy groups.

Then, a perfect start point of our subject is to define what these groups are and how they are related with physics.

## 1.1 Holonomy Groups and Supersymmetry

Let us consider a $n$-dimensional Riemannian manifold, $M$, with metric $g(M)$. Take the Levi-Civita connection, $\nabla$, and define the parallel transport of a vector, $v \in TM$, respect to the Levi-Civita connection along a curve $\gamma$

as

$$\nabla v|_{\dot{\gamma}} = 0.^1 \tag{1.1}$$

Let $m$ be a point in $M$, $v$ a vector in $TM$ and $\gamma$ a path all in $M$. When we parallel transport $v$ along $\gamma$ in general we obtain a different vector. Taking all possible closed path based on $m$, the holonomy group is said to be the smallest group of transformations which permits us to rotate the final vector into the initial $v$. It follows that $Hol(M) \subset O(n)$, and for oriented manifolds, $Hol(M) \subset SO(n)$.

In 1955, Berger [7] classified the holonomy groups of non-symmetric (*e.g.* non-homogeneous), Riemannian manifolds as follows

| Metric | Holonomy | Dimension |
|---|---|---|
| Generic | $SO(n)$ | $n$ |
| Kähler | $U(n/2)$ | $n = 2k$ |
| Calabi-Yau | $SU(n/2)$ | $n = 2k$ |
| Hyper-Kähler | $Sp(n/4)$ | $n = 4k$ |
| Quaternionic | $Sp(n/4) \cdot Sp(1)$ | $n = 4k$ |
| Exceptional | $G_2$ | $n = 7$ |
| Exceptional | $Spin(7)$ | $n = 8$ |

Being $k \in \mathbb{N}^*$.

Obviously, as the holonomy group is bigger, the geometry of the manifolds is less restricted. One can related this fact with the number of supersymmetries preserved after the compactification, as is shown in the next table

| Manifold | $T^n$ | | $CY_3$ | | $G_2$ | | $Spin(7)$ |
|---|---|---|---|---|---|---|---|
| $dim(M)$ | $n$ | | 6 | | 7 | | 8 |
| $Hol(M)$ | 1 | $\subset$ | $SU(3)$ | $\subset$ | $G_2$ | $\subset$ | $Spin(7)$ |
| SUSY | 1 | $>$ | 1/4 | $>$ | 1/8 | $>$ | 1/16 |

In fact these manifolds preserve supersymmetry because there exists a covariantle constant spinors, $\nabla\xi = 0$. This can be obtained from the supergravity equations of motion if we take of fields zero except the metric (see section 2.1). The existence of a covariant spinor implies a reduction in the holonomy groups, so $Hol(M) \subset SO(n)$. Let us explain in more detail this fact.

[1]This condition in term of components is written as $\frac{dx^a}{dt}\nabla_a v^b(t) = 0$,where $t$ is the parameter of the curve $\gamma$.

First of all note that as we are considering spinors. In general the holonomy group acting on spinors is $Hol(g) \subset Spin(n)$ in $n$-dimensions. Since $\nabla\xi = 0$, it follows that

$$[\nabla_m, \nabla_n]\,\xi = \frac{1}{4}\mathcal{R}_{mnpq}\Gamma^{pq}\xi = 0, \tag{1.2}$$

where we have used the definition of the curvature tensor as the Lie bracket in term of spinors.

By using the symmetries of $\mathcal{R}_{mnpq}$, we can interpret it as a matrix $\mathcal{R}_{mn}$ which lies in $\Lambda^2\mathfrak{so}(n)$,[2] and

$$\frac{1}{4}\mathcal{R}_{mnpq}\Gamma^{pq},$$

is an element of $\mathfrak{hol}(g)$.

Let $\mathfrak{g} \in \mathfrak{so}(n)$, our condition of invariant spinors becomes

$$\mathfrak{g}\xi = 0, \tag{1.3}$$

and by taking the exponential map, it can be written as,

$$e^{\mathfrak{g}}\xi = \xi \longrightarrow G\xi = \xi, \tag{1.4}$$

where $G \in Spin(n)$. (1.4) tells us that the holonomy group $Hol(g)$, is the subgroup of $Spin(n)$ formed by all the elements which act trivially on the spinor $\xi$. Obviously, for a generic enough element of $SO(n)$ the condition is not satisfied, then $Hol(g) \subset SO(n)$.

If $Hol(M) = G_2 \subset SO(7)$, it implies[3]

$$8 \to 7 \oplus 1.$$

All those manifold listed above are Ricci-flat, i.e., $R_{ij} = 0$, it guarantees that all backgrounds of the form

$$\mathbb{R}^{10-n,1} \times K$$

solve the 11-dimensional Einstein equations with vanishing sources.

[2]We have written $\mathfrak{so}(n)$ because $SO(n)$ has the same Lie algebra than $Spin(n)$.

[3]In fact, the spin irreducible representation of $SO(7)$ is 8-dimensional, if one of this is covariantly invariant, this 8 -dimensional representation splits into a 1-dimensional irrep. and an other 7-dimensional irrep.

Making use of the covariantly constant spinor, we can construct a set of invariant forms by

$$\omega_{(p)} = \xi^\dagger \Gamma_{i_1 \ldots i_p} \xi, \tag{1.5}$$

these forms are invariant under $Hol(K)$, because

$$\begin{aligned}\nabla \omega_{(p)} &= \nabla(\xi^\dagger \Gamma_{i_1 \ldots i_p} \xi) \\ &= (\nabla \xi^\dagger) \Gamma_{i_1 \ldots i_p} \xi + \xi^\dagger (\nabla \Gamma_{i_1 \ldots i_p}) \xi + \xi^\dagger \Gamma_{i_1 \ldots i_p} (\nabla \xi) \\ &= \xi^\dagger (\nabla \Gamma_{i_1 \ldots i_p}) \xi \\ &= 0. \end{aligned} \tag{1.6}$$

The invariant 3-form can be written locally as

$$\begin{aligned}\omega_{(3)} &= \mathbf{d}x^1 \wedge \mathbf{d}x^2 \wedge \mathbf{d}x^3 + \mathbf{d}x^1 \wedge \mathbf{d}x^4 \wedge \mathbf{d}x^5 + \mathbf{d}x^1 \wedge \mathbf{d}x^6 \wedge \mathbf{d}x^7 + \mathbf{d}x^2 \wedge \mathbf{d}x^4 \wedge \mathbf{d}x^6 \\ &\quad - \mathbf{d}x^2 \wedge \mathbf{d}x^5 \wedge \mathbf{d}x^7 - \mathbf{d}x^3 \wedge \mathbf{d}x^4 \wedge \mathbf{d}x^7 - \mathbf{d}x^3 \wedge \mathbf{d}x^5 \wedge \mathbf{d}x^6. \end{aligned} \tag{1.7}$$

In order to find all possible invariant forms, we need to decompose the space of differential forms on $K$ into irreducible representations of $Hol(K)$ and identify the singlets.

Since the Laplacian of $g(K)$ preserves this decomposition [24], the harmonic forms are also decomposed in this way. For $G_2$ this decomposition is given by

$$\begin{aligned} H^0(K,\mathbb{R}) &= \mathbb{R}, \\ H^1(K,\mathbb{R}) &= H^1_7(K,\mathbb{R}), \\ H^2(K,\mathbb{R}) &= H^2_7(K,\mathbb{R}) \oplus H^2_{14}(K,\mathbb{R}), \\ H^3(K,\mathbb{R}) &= H^3_1(K,\mathbb{R}) \oplus H^3_7(K,\mathbb{R}) \oplus H^3_{27}(K,\mathbb{R}), \end{aligned}$$

and $H^{7-n}(K,\mathbb{R}) = H^n(K,\mathbb{R})$ by Poincarè duality. $H^k_n(K,\mathbb{R})$ is the subspace of $H^k(K,\mathbb{R})$ with elements a $n$-dimensional irreducible representation of $G_2$.

The fact that $Hol(g(K))$ has irreducible representations in $G_2$ implies constrains on the subspaces, for example[4]

$$\boxed{\begin{array}{ccc} R_{ij} = 0 & \Rightarrow & \pi_1(K) = finite \\ & & \Downarrow \\ H^k_7(K,\mathbb{R}) = 0 \;\; \forall\, k = 1 \ldots 6 & \Leftarrow & H^1(K,\mathbb{R}) = 1 \end{array}} \tag{1.8}$$

[4]The proof of most of these statement are difficult, so cannot be presented here. For a proof of $\mathcal{R}_{ij} = 0$ see section 2.1.

So, the invariant forms can appear have only degree 3 or 4. These are called *associative* and *coassociative* forms respectively and are denoted by $\Phi$ and $^*\Phi$.

Since $\nabla\Phi = \nabla^*\Phi = 0$, then $\triangle\Phi = \triangle^*\Phi = 0$.

It is possible to reconstruct a $G_2$ metric from the associative 3-form [20]:

$$g_{ij} = det(B)^{-\frac{1}{9}} B_{ij} \tag{1.9}$$

$$B_{ij} = -\frac{1}{144}\Phi_{imn}\Phi_{jpq}\Phi_{rst}\epsilon^{mnpqrst}. \tag{1.10}$$

The invariant forms represent volume forms of minimal submanifolds in $K$, they are called *calibrations* and their corresponding submanifolds are called calibrated submanifolds.

## About Calibrations

A closed $p$-form, $\Psi$, is a calibration if it is less than or equal to the volume on each oriented $p$-dimensional submanifold $S \subset K$.

Using orientation of $S$ and the restriction of $g$ on $S$, $g|_S$, we can define the volume form, $vol(T_xS)$, on the tangent space of $S$ at $x$. Then $\Psi|_{T_xS} = \alpha \cdot vol(T_xS)$ for $\alpha \in \mathbb{R}$. If $\alpha \leq 1$,

$$\Psi|_{T_xS} \leq vol(T_xS). \tag{1.11}$$

If (1.11) is saturated for all $x \in S$, then $S$ is defined to be a calibrated submanifold with respect to the calibration $\Psi$.

Since

$$vol(S) = \int_{x\in S} \Psi|_{T_xS} = \int_S \Psi,$$

and $\mathbf{d}\Psi = 0$, so the right hand side depends only on the homology class of $S$. $S$ and $S'$ are said to be in the same homology class if

$$\int_S \Psi - \int_{S'} \Psi = \int_M \mathbf{d}\Psi, \tag{1.12}$$

where the boundary of $M$ is just $\partial M = S \amalg S'$.

So

$$\begin{aligned} vol(S) &= \int_S \Psi = \int_{S'} \Psi = \int_{x\in S'} \Psi|_{T_xS'} \\ &\leq \int_{x\in S'} vol(T_xS') = vol(S'), \end{aligned}$$

for all submanifold $S'$ in the same cohomology class.

This property of calibrated manifolds allows us to identify them with supersymmetric cycles, where the bound in volume becomes equivalent to the BPS bound. If a brane (which has tension) wraps a submanifold $S'$, the tension of the brane will deform the submanifold until arrive to its minimal volume, this process gives us dynamic of the submanifold $S'$ and finally it is deformed to $S$, which is the calibrated submanifold in the same homology class of $S'$. In particular, branes in string theory and $M$-theory wrapped over calibrated submanifolds can give rise to BPS states in the effective theory.

**Examples of Calibrated Geometries**

- Special Lagrangian cycles in a CY: It is a 3-dimensional submanifold in the CY calibrated with respect to $Re(\Omega)$ with $\Omega$ the holomorphic 3-form of CY.

- Holomorphic Subvarieties: such as holomorphic curves, surfaces, etc.

- On $G_2$: Submanifolds corresponding to the associative and coassociative forms, $\Phi$ and $^*\Phi$, are calibrated submanifolds.

## BPS States

Any system with supersymmetry has a special subspace of the full Hilbert space $\mathcal{H}$, the so called $BPS$ space

$$\mathcal{H}_{BPS} \subset \mathcal{H},$$

that consist of small supermultiplets.

More precisely, suppose we are dealing with some supersymmetry algebra with a set of $n$ supercharges $Q^\alpha$, the Hilbert space will decompose in irreducible representations of this algebra. Since the supersymmetry algebra will be of the general form

$$\left\{Q^\alpha, Q^\beta\right\} = \omega_i^{\alpha\beta} K^i, \tag{1.13}$$

with

$$\left[K^i, Q^\alpha\right] = 0, \quad \left[K^i, K^j\right] = 0, \tag{1.14}$$

where the $K^i$ are some set of bosonic charges, consisting of the translation operator $P_\mu$ and some extra set of central charges, usually denoted as $Z$.

Therefore, when we consider a representation where the operators $K^i$ have fixed generic eigenvalues $k^i$, so the total bilinear form $\omega = \omega_i k^i$ is non-degenerate, we are essentially dealing with a representation of a $n$-dimensional Clifford algebra. The dimension of the representation will therefore be $2^{n/2}$. However, for special values of the charges $k^i$, it might be the case that the bilinear form $\omega$ becomes accidentally degenerate. In that case there are certain linear combinations of the supercharges that annihilate the representation. If it satisfies the conditions

$$\epsilon_\alpha Q^\alpha \left|BPS\right\rangle = 0, \tag{1.15}$$

for $m$ independent spinors $\epsilon$, the rank of the Clifford algebra will be $n - m$ and therefore the dimension of the representation will be $2^{(n-m)/2}$.

For a general state with eigenvalues $P_M = M\delta_{M,0}$ and $Z$, one can derive a lower bound for the mass [35]

$$M^2 \geq |Z|^2. \tag{1.16}$$

In fact, states with $M^2 = |Z|^2$ are precisely the small $BPS$ representations.[12]

Let us calculate this bound explicitly in the 11-dimensional case[5]. In general we can write the supersymmetric algebra as [31]

$$\{Q_\alpha, Q_\beta\} = \left(C\Gamma^M\right)_{\alpha\beta} P_M + \frac{1}{2}\left(C\Gamma_{MN}\right)_{\alpha\beta} Z^{MN}, \tag{1.17}$$

where

$$Z^{MN} = Q \int \mathbf{d}X^M \wedge \mathbf{d}X^N, \tag{1.18}$$

and the integral is taken over the 2-cycle occupied by the membrane in spacetime, and $Q$ is the charge of the membrane. In this case, the $Z$ does not represent the central charges because they do not commute with the generetors of the Poincarè symmetry.

By supersymmetry algebra, we know that

$$\{Q_i, Q_j\} \geq 0, \tag{1.19}$$

in terms of its eigenvalues.

[5]For a proof of this statement in diverse dimensions see [32] and reference therein.

Taking now a membrane in the plane 12, and a representation in which $C = \Gamma^0$, the equation (1.17) becomes

$$P^0 + \Gamma^{012} Z_{12} \geq 0, \tag{1.20}$$

where $\Gamma_{012} = \Gamma_{[0}\Gamma_1\Gamma_{2]}$. It is, once more, talking in term of its eigenvalues. Since the eigenvalues of $\Gamma$ matrices are just $\pm 1$, then

$$P^0 \geq |Z_{12}|. \tag{1.21}$$

From (1.17) and (1.19), it follows that the saturation limit implies

$$\Gamma_{012}\varepsilon = \pm\varepsilon, \tag{1.22}$$

this $\varepsilon$ is a Killing spinor, since ${\Gamma_{012}}^2 = \mathbf{1}$ and $tr(\Gamma_{012}) = 0$, it follows that the total number of supersymmetries are halves.

By using the Killing spinor, we can construct an invariant 3-form, as in (1.5)

$$\varepsilon^\dagger \Gamma_{012} \varepsilon = \varepsilon^\dagger \varepsilon = \mathbf{1}, \tag{1.23}$$

so this 3-form can be associated to the volume form $vol(M^3)$ of the 3-dimensional submanifold, called $\mathscr{M}$2-brane. Equation (1.23) says that the 3-form restricts to the $\mathscr{M}$2-brane world volume is equal to its volume form. Hence the $BPS$ condition implies the world volume is calibrated and minimal.

### 1.1.1 Problems of Exceptional Holonomy

There are two problems which made difficult the study of exceptional holonomy manifolds.

The first is referent to the existence of an exceptional metric on a given manifold $K$, because unfortunately we have not an equivalent to the Yau's theorem [36, 24].

The second is related to the singularities in the manifold, so that the interesting physics is associated with types of singularities of maximal codimension, which exploit the geometry of the special holomony manifold to the fullest.

# Chapter 2

# Kaluza–Klein analysis of $\mathscr{M}$-theory compactification

In this chapter we shall introduce the compactification of $\mathscr{M}$-theory on smooth $G_2$ holonomy manifolds and its Kaluza-Klein analysis. As we shall see, compactifying $\mathscr{M}$-theory on these kind of manifolds gives rise to a theory which is not phenomenologically interesting because it has neither non-Abelian gauge symmetries nor chiral fermions, which are fundamental ingredients in the Standard Model.

## 2.1 Supergravity in $D = 11$

The low energy limit of $\mathscr{M}$-theory is $D = 11$ supergravity. In this section, we shall review a few facts on the subject.

Supergravity $D = 11$ contains the metric $g_{MN}$, a Majorana fermion of spin $3/2$ $\psi_M$ (vector-spinor), and a 3-form $C$, where $M = 0...10$.

The $\mathscr{N}=1$ supergravity action in $D=11$ is given by [10]

$$
\begin{aligned}
S =& \frac{1}{2\kappa^2}\int \left[\mathcal{R}^*1 - \frac{1}{2}G\wedge {}^*G - \frac{1}{6}C\wedge G\wedge G\right] \\
&+ \frac{1}{2\kappa^2}\int d^{11}z\sqrt{g}\bar{\psi}_M\Gamma^{MNP}\nabla_N\left(\frac{\omega+\hat{\omega}}{2}\right)\psi_P \\
&- \frac{1}{2\kappa^2}\int d^{11}z\sqrt{g}\left(\bar{\psi}_M\Gamma^{MNPQRS}\psi_N + 12\bar{\psi}^P\Gamma^{RS}\psi^Q\right)\left(G_{PQRS}+\hat{G}_{PQRS}\right),
\end{aligned}
\tag{2.1}
$$

where $G=\mathbf{d}C$, $\omega$ is the spin connection and $\hat{\omega}$ and $\hat{G}$ are the supercovariant connection and field strength respectively.

This theory has the following symmetries:

- *General covariance* respect to a parameter $\xi_M$,

$$
\begin{aligned}
\delta e^A{}_M &= e^A{}_N\partial_M\xi^N + \xi^N\partial_N e^A{}_M \\
\delta C_{MNP} &= 3C_{Q[MN}\partial_{P]}\xi^Q + \xi^Q\partial_Q C_{MNP} \\
\delta\psi_M &= \psi_N\partial_M\xi^N + \xi^N\partial_N\psi_M.
\end{aligned}
\tag{2.2}
$$

- Local $SO(1,10)$ respect to parameters $\alpha_{AB}=-\alpha_{BA}$

$$
\begin{aligned}
\delta e^A{}_M &= -\alpha^A{}_B e^B{}_M \\
\delta C_{MNP} &= 0 \\
\delta\psi_M &= -\frac{1}{4}\alpha_{AB}\Gamma^{AB}\psi_M.
\end{aligned}
\tag{2.3}
$$

- $\mathscr{N}=1$ supersymmetry respect to a Grassmann variable $\eta$

$$
\begin{aligned}
\delta e^A{}_M &= -\frac{1}{2}\bar{\eta}\Gamma^A\psi_M \\
\delta C_{MNP} &= -\frac{3}{2}\bar{\eta}\Gamma_{[MN}\psi_{P]} \\
\delta\psi_M &= \widetilde{\nabla}_M\eta
\end{aligned}
\tag{2.4}
$$

- Abelian gauge transformations respect to the 2-form $\Lambda$

$$
\begin{aligned}
\delta e^A{}_M &= 0 \\
\delta C &= \mathbf{d}\Lambda \\
\delta\psi_M &= 0
\end{aligned}
\tag{2.5}
$$

• Odd number of space time reflections together with $C_{MNP} \mapsto -C_{MNP}$,

in these equations, $e_M^A$ are the vielbeins with $A$ an index of the tangent space and $M$ an index of the target space, and

$$\widetilde{\nabla}_M \psi_N = \nabla_M \psi_N - \frac{1}{288} \left( \Gamma_M{}^{PQRS} - 8\delta_M^P \Gamma^{QRS} \right) \hat{G}_{PQRS} \psi_N.$$

The field equations are

$$\mathcal{R}_{MN} - \frac{1}{2} g_{MN} \mathcal{R} = \frac{1}{12} \left( G_{MPQR} G_N{}^{PQR} - \frac{1}{8} g_{MN} G_{PQRS} G^{PQRS} \right) \tag{2.6}$$

and

$$\mathbf{d}^* G + \frac{1}{2} G \wedge G = 0. \tag{2.7}$$

Making use of the equations of motion, we can define two conserved charges

$$Q_e = \int_{\partial M_8} \left( {}^*G + \frac{1}{2} C \wedge G \right), \tag{2.8}$$

$$Q_m = \int_{\partial M_5} G. \tag{2.9}$$

Let us study the equations (2.4) considering all fields zero except the metric. So, the condition of $\mathscr{N} = 1$ supersymmetry is just

$$\delta \psi = \nabla \eta = 0. \tag{2.10}$$

In compactification, this conditions is equivalent to saying that $G_2$ manifold is Ricci-flat. It can be seen as follows,

$$\nabla_m \eta = 0 \quad \Longrightarrow \quad [\nabla_m, \nabla_n] \eta = \frac{1}{4} \mathcal{R}_{mnpq} \Gamma^{pq} \eta = 0,$$

then

$$\begin{aligned} \mathcal{R}_{mnpq} \Gamma^n \Gamma^{pq} \eta &= 0, \\ \Rightarrow \mathcal{R}_{m(npq)} \Gamma^n \Gamma^{pq} \eta &= 0, \\ \Rightarrow \mathcal{R}_{mn} \Gamma^n \eta &= 0, \\ \Rightarrow \mathcal{R}_{mn} &= 0. \end{aligned}$$

It is clear that this solve the equations of motion of $D = 11$ supergravity. Also the equations of motion imply that $\mathcal{R}_{\mu\nu} = 0$, it means that the 4-dimensional manifold in $M^{3,1} \times K$ is just a Minkowski spacetime.

This is our main motivation for compactifying on manifolds with special holonomy.

## 2.2 Kaluza-Klein spectrum

As we mention before, the low energy limit of $\mathscr{M}$-theory is $D = 11$ supergravity, when the spacetime is smooth and large compared with the 11-dimensional Planck length. So, we can obtain the effective low energy description by consider Kaluza-Klein analysis [29, 1].

Basically, we shall consider the 11-dimensional background $M^{10,1}$ as a reducible manifolds,

$$M^{10,1} = M^{3,1} \times K,$$

where $M^{3,1}$ is a maximally symmetric spacetime and $K$ is a compact 7-dimensional manifold with holonomy group $G_2$.

The only bosonic fields in $D = 11$ supergravity are the metric $g$ and the 3-form $C$.

### 2.2.1 Expansion of the $C$-form

As we comment above, we shall consider vanishing vacuum expectation value for all the fields, except the metric. Then, considering fluctuations around the vacuum solution, it follows that

$$C(x, y) = \delta C(x, y). \tag{2.11}$$

Therefore, (2.7) can be written to first order in $\delta C$ as

$$\mathbf{d}^* G = 0. \tag{2.12}$$

As in Maxwell theory, we should impose the gauge condition $\mathbf{d}^* C = 0$. Since $\triangle = \{\mathbf{d}, \boldsymbol{\delta}\}$, finally we get the equations of motion

$$\triangle_{11} C = 0. \tag{2.13}$$

Now, splitting $\triangle_{11} = \triangle_4 + \triangle_7$, we note that for the effective 4-dimensional theory, $\triangle_7$ is like a mass term. So, let expand $C$ in term of the eigen-forms of $K$.

One of the assumptions of Kaluza-Klein approach is that as we want to consider extra dimensions with small radii, the only interesting modes in the low energy 4-dimensional theory are massless states. This is because after compactifying the 4-dimensional mass depends on radii as $m \sim \frac{1}{R}$, so if $R$ is really small, roughly speaking if it is of Planck length scale, which is $10^{-33}cm$, the massive states are too heavy and they will not be considered in the low energy limit.

This allows us to consider the relevant expansion of $C$ just in terms of the harmonic forms on $K$, as

$$C = \phi_I(x)\omega^I(y) + A_\alpha(x) \wedge \beta^\alpha(y) + (massive\ terms), \tag{2.14}$$

where $\omega^I$ are basis of the harmonic 3-forms on $K$, and $\beta^\alpha$ are basis of the harmonic 2-forms on $K$, so $I = 1...b_3(K)$ and $\alpha = 1...b_2(K)$, $x$ are coordinates along $M^4$ and $y$ are coordinates along $K$. Since $C$ is odd under parity, $\phi_I$ describe $b_3(K)$ pseudo-scalars. On the other hand, $A_\alpha$ are 1-forms in Minkowski space, so, they are Abelian gauge fields in 4 dimensions.

### 2.2.2 Expansion of the metric $g$

As before, let us consider small fluctuations around the exact solution. If we impose the 4-dimensional vacuum to be $SO(3,1)$ invariant and that $M^{10,1}$ has spin structure[1], so the equation (2.7) is satisfied trivially, and (2.6) is just

$$\mathcal{R}_{MN} = 0. \tag{2.15}$$

It implies that $M^{3,1}$ is nothing but 4-dimensional Minkowski spacetime, $\mathcal{M}^{3,1}$.

Let us consider the fluctuations around the exact solution as

$$g_{MN} = \langle g_{MN} \rangle + \delta g_{MN}. \tag{2.16}$$

So (2.15) can be written as

$$\begin{aligned}
\mathcal{R}_{MN}(g_{ST}) &= \mathcal{R}_{MN}\left(\langle g_{ST} \rangle\right) + \triangle_L \delta g_{MN} + \mathscr{O}(\delta {g_{ST}}^2) \\
&= -\nabla_{11}^2 \delta g_{MN} - 2\mathcal{R}_{MPNQ}(\langle g_{ST} \rangle)\delta g^{PQ} + 2\mathcal{R}_{(M}{}^{P}(\langle g_{ST} \rangle)\delta g_{N)P} \\
&= -\nabla_{11}^2 \delta g_{MN} - 2\mathcal{R}_{MPNQ}(\langle g_{ST} \rangle)\delta g^{PQ} \\
&= 0,
\end{aligned} \tag{2.17}$$

[1]Since $M^{10,1} = M^{3,1} \times K$ and $M^{3,1}$ is maximally symmetric, it implies that both, $M^{3,1}$ and $K$ have spin structure.

where $\triangle_L$ is the Lichnerowicz operator, and it is defined by the second line of the equation, we have dropped $\mathcal{R}_{MN}(\langle g_{ST}\rangle)$ because it is the exact solution so this vanishes.

Since the metric can be $g(M^{10,1}) = \eta \times g(K)$, where $\eta$ is the 4-dimensional Minkowski metric and $g(K)$ is the $G_2$-holonomy metric on $K$, we have to consider two possibilities, when $M, N = \mu, \nu = 0...3$ and $M, N = m, n = 4...10$.

Then, considering

$$g_{mn}(x, y) = \langle g_{mn}(x, y)\rangle + \delta g_{mn}(x, y),$$

the linearized Einstein equations are written as

$$\triangle_L \delta g_{mn} = -\nabla_{11}^2 \delta g_{mn} - 2\mathcal{R}_{mpnq}\delta g^{pq} = 0, \tag{2.18}$$

Now, splitting the 11-dimensional covariant derivative as $\nabla_{11}^2 = \nabla_4^2 + \nabla_7^2$, we get

$$-\nabla_4^2 \delta g_{mn} - \nabla_7^2 \delta g_{mn} - 2\mathcal{R}_{mpnq}\delta g^{pq} = \left(-\nabla_4^2 + \triangle_L^{(7)}\right)\delta g_{mn} = 0, \tag{2.19}$$

where we have defined $\triangle_L^{(7)}$ as the 7-dimensional Lichnerowicz operator. Then, making the Kaluza-Klein ansatz

$$\delta g_{mn}(x, y) = s_i(x) h^i_{mn}(y), \tag{2.20}$$

where $h^i_{mn}$ are eigenfunctions of the 7-dimensional Lichnerowicz operator with eigenvalue $\lambda^i$. Then,

$$\left(-\nabla_4^2 + \triangle_L^{(7)}\right) s_i(x) h^i_{mn}(y) = \left(-\nabla_4^2 + \lambda^i\right) s_i(x) h^i_{mn}(y), \tag{2.21}$$

the eigenvalues of the 7-dimensional Lichnerowicz operator are seen as mass term in the effective 4-dimensional physics.

Using the associative form of $G_2$ manifold, $\Phi$, we can construct a 3-form as follows

$$\omega_{mnp} = 3\Phi_{q[mn} h_{p]}{}^q, \tag{2.22}$$

such that

$$\triangle_L h = 0 \quad \Leftrightarrow \quad \triangle\omega = 0. \tag{2.23}$$

It, in fact, can be seen by computing explicitly as follows.

First of all we know that

$$\nabla_m \Phi_{abc} = 0,$$

is a property of the asoociative 3-form of a $G_2$ manifold. It follows that

$$[\nabla_m, \nabla_n]\,\Phi_{abc} = -3\mathcal{R}_{mnd[a}\,\Phi^d{}_{bc]} = 0. \tag{2.24}$$

It is well known that

$$\triangle\omega_{abc} = -\nabla^2\omega_{abc} + 6\mathcal{R}_{mn[ab}\,\omega_{c]}{}^{mn},$$

we have dropped the contribution given by the Ricci tensor because in our case $\mathcal{R}_{\mu\nu} = 0$. Next, we shall write down the last term of this equation.

$$\begin{aligned} 3\mathcal{R}_{mn[ab}\,\omega_{c]}{}^{mn} &= \mathcal{R}_{mnab}\omega_c{}^{mn} + \mathcal{R}_{mnbc}\omega_a{}^{mn} + \mathcal{R}_{mnca}\omega_b{}^{mn} \\ &= \mathcal{R}_{mnab}\left(h_{ec}\Phi^{mne} + \Phi^n{}_c{}^e h_e{}^m + \Phi_c{}^{me}h_e{}^n\right) \\ &\quad + \mathcal{R}_{mnbc}\left(h_{ea}\Phi^{mne} + \Phi^n{}_a{}^e h_e{}^m + \Phi_a{}^{me}h_e{}^n\right) \\ &\quad + \mathcal{R}_{mnca}\left(h_{eb}\Phi^{mne} + \Phi^n{}_b{}^e h_e{}^m + \Phi_b{}^{me}h_e{}^n\right). \end{aligned} \tag{2.25}$$

From the identity of the associative form of the $G_2$ structure (2.24)

$$\mathcal{R}_{mnk[a}\,\Phi_{bc]}{}^k = 0, \tag{2.26}$$

it follows that

$$\mathcal{R}_{mnkl}\Phi_b{}^{kl} = 0,$$

so, the equation (2.25) is reduced to

$$\begin{aligned} 3\mathcal{R}_{mn[ab}\,\omega_{c]}{}^{mn} &= \mathcal{R}_{mnab}\left(\Phi^n{}_c{}^e h_e{}^m + \Phi_c{}^{me}h_e{}^n\right) \\ &\quad + \mathcal{R}_{mnbc}\left(\Phi^n{}_a{}^e h_e{}^m + \Phi_a{}^{me}h_e{}^n\right) \\ &\quad + \mathcal{R}_{mnca}\left(\Phi^n{}_b{}^e h_e{}^m + \Phi_b{}^{me}h_e{}^n\right). \end{aligned} \tag{2.27}$$

By using the identity (2.26), we get

$$\begin{aligned} 3\mathcal{R}_{mn[ab}\,\omega_{c]}{}^{mn} &= -\mathcal{R}_{abcn}\Phi^{ne}{}_m h_e{}^m - \mathcal{R}_{ab}{}^e{}_n\Phi^n{}_{mc}h_e{}^m \\ &\quad -\mathcal{R}_{abcm}\Phi^{me}{}_n h_e{}^n - \mathcal{R}_{ab}{}^e{}_m\Phi^m{}_{nc}h_e{}^n \\ &\quad -\mathcal{R}_{bcan}\Phi^{ne}{}_m h_e{}^m - \mathcal{R}_{bc}{}^e{}_n\Phi^n{}_{ma}h_e{}^m \\ &\quad -\mathcal{R}_{bcam}\Phi^{me}{}_n h_e{}^n - \mathcal{R}_{bc}{}^e{}_m\Phi^m{}_{na}h_e{}^n \\ &\quad -\mathcal{R}_{cabn}\Phi^{ne}{}_m h_e{}^m - \mathcal{R}_{ca}{}^e{}_n\Phi^n{}_{mb}h_e{}^m \\ &\quad -\mathcal{R}_{cabm}\Phi^{me}{}_n h_e{}^n - \mathcal{R}_{ca}{}^e{}_m\Phi^m{}_{nb}h_e{}^n, \end{aligned}$$

as $h_{ab}$ is symmetric and $\Phi$ is a 3-form, i.e., it is totally antisymmetric, it follows that the first term in each line is zero, then

$$\begin{aligned} 3\mathcal{R}_{mn[ab}\omega_{c]}{}^{mn} &= -\mathcal{R}_{ab}{}^{e}{}_{n}\Phi^{n}{}_{mc}h_{e}{}^{m} - \mathcal{R}_{ab}{}^{e}{}_{m}\Phi^{m}{}_{nc}h_{e}{}^{n} \\ &\quad -\mathcal{R}_{bc}{}^{e}{}_{n}\Phi^{n}{}_{ma}h_{e}{}^{m} - \mathcal{R}_{bc}{}^{e}{}_{m}\Phi^{m}{}_{na}h_{e}{}^{n} \\ &\quad -\mathcal{R}_{ca}{}^{e}{}_{n}\Phi^{n}{}_{mb}h_{e}{}^{m} - \mathcal{R}_{ca}{}^{e}{}_{m}\Phi^{m}{}_{nb}h_{e}{}^{n} \\ &= 6\mathcal{R}^{e}{}_{n[ab}T_{c]e}{}^{n}, \end{aligned} \tag{2.28}$$

where we have defined

$$T^{mn}{}_{c} = \Phi^{mne}h_{ce}.$$

Now,

$$\begin{aligned} 6\mathcal{R}^{e}{}_{n[ab}T_{c]e}{}^{n} &= 2\mathcal{R}^{e}{}_{nab}T_{ce}{}^{n} + 2\mathcal{R}^{e}{}_{nbc}T_{ae}{}^{n} + 2\mathcal{R}^{e}{}_{nca}T_{be}{}^{n} \\ &= -2\mathcal{R}_{a}{}^{e}{}_{nb}T^{n}{}_{ce} - 2\mathcal{R}_{anb}{}^{e}T^{n}{}_{ce} \\ &\quad -2\mathcal{R}_{b}{}^{e}{}_{nc}T^{n}{}_{ae} - 2\mathcal{R}_{bnc}{}^{e}T^{n}{}_{ae} \\ &\quad -2\mathcal{R}_{c}{}^{e}{}_{na}T^{n}{}_{be} - 2\mathcal{R}_{cna}{}^{e}T^{n}{}_{be}, \end{aligned}$$

where we have used the first Bianchi identity. Then

$$\begin{aligned} 6\mathcal{R}^{e}{}_{n[ab}T_{c]e}{}^{n} &= -2\mathcal{R}_{a}{}^{e}{}_{nb}T^{n}{}_{ce} + 2\mathcal{R}_{an}{}^{e}{}_{b}T^{n}{}_{ce} \\ &\quad -2\mathcal{R}_{b}{}^{e}{}_{nc}T^{n}{}_{ae} + 2\mathcal{R}_{bn}{}^{e}{}_{c}T^{n}{}_{ae} \\ &\quad -2\mathcal{R}_{c}{}^{e}{}_{na}T^{n}{}_{be} + 2\mathcal{R}_{cn}{}^{e}{}_{a}T^{n}{}_{be}, \\ &= -4\mathcal{R}_{[a}{}^{e}{}_{|n|b]}T^{n}{}_{ce} - 4\mathcal{R}_{[b}{}^{e}{}_{|n|c]}T^{n}{}_{ae} - 4\mathcal{R}_{[c}{}^{e}{}_{|n|a]}T^{n}{}_{be} \\ &= -6\mathcal{R}_{[a|en|b}T_{c]}{}^{en}. \end{aligned}$$

on the other hand, let us analyze the term $\mathcal{R}_{cmjn}h^{mn}$, so

$$\begin{aligned} \mathcal{R}_{cmjn}h^{mn} &\Longrightarrow 3\Phi^{j}{}_{[ab}\mathcal{R}_{c]mjn}h^{mn} \\ &= -3\Phi^{j}{}_{[ab}\mathcal{R}_{c]jnm}h^{mn} - 3\Phi^{j}{}_{[ab}\mathcal{R}_{c]nmj}h^{mn}, \end{aligned}$$

by using the first Bianchi identity, additionally, the first term vanishes because the Riemann tensor is antisymmetric in the last two indeces but the $h$ tensor is symmetric. Therefore,

$$\begin{aligned} 3\Phi^{j}{}_{[ab}\mathcal{R}_{c]mjn}h^{mn} &= -\Phi^{j}{}_{ab}\mathcal{R}_{cnmj}h^{mn} - \Phi^{j}{}_{bc}\mathcal{R}_{anmj}h^{mn} - \Phi^{j}{}_{ca}\mathcal{R}_{bnmj}h^{mn} \\ &= \Phi^{j}{}_{bm}\mathcal{R}_{cnaj}h^{mn} + \Phi^{j}{}_{ma}\mathcal{R}_{cnbj}h^{mn} + \Phi^{j}{}_{cm}\mathcal{R}_{anbj}h^{mn} \\ &\quad +\Phi^{j}{}_{mb}\mathcal{R}_{ancj}h^{mn} + \Phi^{j}{}_{am}\mathcal{R}_{bncj}h^{mn} + \Phi^{j}{}_{mc}\mathcal{R}_{bnaj}h^{mn}, \end{aligned}$$

here we have just expanded the antisymmetrization and after that used the identity

$$\mathcal{R}_{mnk[a}\Phi_{bc]}{}^{k} = 0.$$

By using our definition of $T$ tensor, the last equation can be written as

$$\begin{aligned}
3\Phi^{j}{}_{[ab}\mathcal{R}_{c]mjn}h^{mn} &= T^{j}{}_{b}{}^{n}\mathcal{R}_{cnaj} + T^{j}{}_{c}{}^{n}\mathcal{R}_{anbj} + T^{j}{}_{a}{}^{n}\mathcal{R}_{bncj} \\
&\quad - T^{j}{}_{a}{}^{n}\mathcal{R}_{cnbj} - T^{j}{}_{b}{}^{n}\mathcal{R}_{ancj} - T^{j}{}_{c}{}^{n}\mathcal{R}_{bnaj} \\
&= 2T^{j}{}_{[b}{}^{n}\mathcal{R}_{|cn|a]j} + 2T^{j}{}_{c}{}^{n}\mathcal{R}_{[a|n|b]j} + T^{j}{}_{[a}{}^{n}\mathcal{R}_{b]ncj} \\
&= 6T^{j}{}_{[a}{}^{n}\mathcal{R}_{b|n|c]j}. \qquad (2.29)
\end{aligned}$$

Finally, the Lichnerovicz equation can be rewritten as,

$$\begin{aligned}
-\nabla^2 h_{cj} - 2\mathcal{R}_{cmjn}h^{mn} &\mapsto -3\nabla^2\Phi^{j}{}_{[ab}h_{c]j} - 6\Phi^{j}{}_{[ab}\mathcal{R}_{c]mjn}h^{mn} \\
&= -3\nabla^2\Phi^{j}{}_{[ab}h_{c]j} - 12\mathcal{R}_{[a|en|b}T^{n}{}_{c]}{}^{e} \\
&= -\nabla^2\omega_{abc} - 12\mathcal{R}_{[a|en|b}T^{n}{}_{c]}{}^{e} \\
&= -\nabla^2\omega_{abc} + 6\mathcal{R}_{mn[ab}\omega^{mn}{}_{c]} \\
&= \triangle\omega_{abc} \qquad (2.30)
\end{aligned}$$

Therefore, from (2.23) we have that $b_3(K)$ scalars in the 4-dimensional theory came from the metric. In general, the number of massless scalars coming from the $G_2$ metric is given by the dimension of the moduli space of $G_2$ metrics on $K$, which coincides with $b_3(K)$.

On the other hand, when we take into account the Kaluza-Klein expansion of $g$ on term as

$$\delta g_{\mu\nu}(x,y) = h^{i}_{\mu\nu}(x)t_i(y),$$

can appear, where $t_i(y)$ are eigenfunctions of $\triangle_7$, then as the 4-dimensional background is Minkowski, $\mathcal{R}_{\mu\nu\lambda\rho} = \mathcal{R}_{\mu\nu} = 0$, the fluctuation equations become

$$\sum_i \left(\triangle_4 + \lambda_i\right) h^{i}_{\mu\nu}(x)t_i(y) = 0. \qquad (2.31)$$

And the 4-dimensional massless particle is the graviton. There is only one because $b_0(K) = 1$.

The scalars and pseudo-scalars combine to give rise to $b_3(K)$ complex scalars, which are the lowest components of massless chiral supermultiplet in 4 dimensions. The fluctuations around 4-dimensional Minkowski give us 4-dimensional gravity, which due to supersymmetry implies that the effective

4-dimensional theory is locally supersymmetric. Since $A_\alpha$'s are 1-forms in Minkowski space, from them arise just Abelian gauge fields.

Finally, we can say that the low-energy effective 4-dimensional theory is an $\mathscr{N} = 1$ supergravity coupled to $b_2(K)$ Abelian vector multiplets and $b_3(K)$ massless, neutral chiral multiplets.

## 2.3 What shall we do now?

As we saw in the last section, if we compactify $\mathscr{M}$-theory on smooth $G_2$ manifolds we do not obtain a physically interesting theory, since in our 4-dimensional world (excluding gravity) interactions are explained by the Standard Model which contains light charged particles, chiral fermions, and non-Abelian gauge group.

# Chapter 3

# Physics from singularities

We have seen that compactifying $\mathscr{M}$-theory on smooth $G_2$ manifold does not give us realistic 4-dimensional physics, in this chapter we shall obtain some relevant physics by considering compactification on singular $G_2$ manifolds.

In recent years it has been studied, and it has been understood how to obtain non-Abelian gauge groups [3, 4] and chiral fermions [6, 34, 2] from the singularites. It is due to the fact that $\mathscr{M}$-branes become light at the singularity, allowing with this the enhancement of the gauge group [3, 4].

We know that such manifolds exist because Joyce has found a method to construct them [24], and the singularities are (in this case) orbifold singularities. Orbifold singularities can be represented locally by a quotient of $\mathbb{R}^n$ by a discrete group $\Gamma$.

One of the methods to obtain chiral fermion demands that the $G_2$ manifold must have additionally conical singularities [6].

The physics related to orbifold singularities is well understood in perturbative string theory, and it is extracted from the orbifold conformal field theory. Unfortunately that technique cannot be applied in studying $\mathscr{M}$-theory on singular $G_2$ manifolds. But, since we know some dualities between $\mathscr{M}$-theory and string theories, we shall use those dualities in order to extract the physical information.

## 3.1 Duality with heterotic string

The origin of the non-Abelian gauge groups in $\mathscr{M}$-theory is understood thanks to the duality between $\mathscr{M}$-theory on $K3$ and heterotic strings on $T^3$ [33]. This teaches us that the singularities should be the called $ADE$-singularities.

A $K3$ surface is defined as a compact complex Kähler manifold of complex dimension two, such that

$$h^{1,0}(K3) = 0, \quad \text{and} \quad c_1(TK3) = 0, \tag{3.1}$$

where $c_1(TK3)$ is the first Chern class of the holomorphic tangent bundle of $K3$. $K3$ is, up to diffeomorphisms, the only simply connected compact 4-dimensional manifold admitting metrics of $SU(2)$-holonomy.

The moduli space of $SU(2)$-holonomy metrics on $K3$ is locally a coset space

$$\mathfrak{M}(K3) = \frac{SO(3,19)}{SO(3) \times SO(19)} \times \mathbb{R}^+, \tag{3.2}$$

which has dimension 58.

An $SU(2)$ holonomy metric admits 2 parallel spinors, which tensored with the **8** constant spinors of 7-dimensional Minkowski space[1] give us 16 global supercharges. This correspond to minimal supersymmetry in 7 dimensions.

For an arbitrary smooth point in $\mathfrak{M}(K3)$ we can apply Kaluza-Klein analysis and we get 58 massless scalar particles in the 7-dimensional theory. Since $H^2(K3,\mathbb{R}) \cong \mathbb{R}^{22}$, it gives rise to a $U(1)^{22}$ gauge group in 7 dimensions.

In the heterotic string side, we have a 10-dimensional theory whose low energy behavior is described by $D = 10$ supergravity, which contains 3 massless bosonic fields, namely metric $g$, a 2-form $B$ and a dilaton $\phi$, and non-Abelian gauge fields of structure group $SO(32)$ or $E_8 \times E_8$.

Narain [28] showed by direct computation that this moduli space is actually also locally the same as $\mathfrak{M}(K3)$.

If $\mathscr{M}$-theory on $K3$ is equivalent to the heterotic string on $T^3$, it should exhibit non-Abelian symmetry enhancement at special points in the moduli space. These points are precisely the ones in which $K3$ develops orbifold

[1] Remember that the number of spinors in $n$ dimensions is $2^{[\frac{n}{2}]}$, where $[\cdot]$ is the integer part function

singularities. So, we should look at the $K3$ moduli space in a neighbourhood of this singularity, where all the interesting behaviour of the theory is occurring.

### 3.1.1 In which kind of singularities are we interested?

One important kind of singularity, which arises in string theory are the orbifold singularities. Those singularities in a Riemannian $n$-dimensional manifold can be described locally as $\mathbb{R}^n/\Gamma$, where $\Gamma$ is a finete subgroup of $SO(n)$. In particular, as we are interested in singularities of $K3$, its singularities will be described by $\mathbb{R}^4/\Gamma$ where $\Gamma \subset SO(4)$. For generic enough subgroup $\Gamma$, the only singular point of the orbifold is the origin[2].

In the heterotic string theory side, the moduli space has a complete unbroken supersymmetry, so by using the duality, we shall demand that also the moduli space in the $\mathscr{M}$-theory side contains one unbroken supersymmetry. In order that supersymmetry is to be preserved, $\Gamma$ should be not just subgroup of $SO(4)$ but subgroup of $SU(2)$[3], where $SU(2)$ is nothing but the holonomy group of the $K3$ surface. It is due to the decomposition

$$\begin{array}{ccc} Spin(4) & \longrightarrow & SU(2)_L \\ (2,1)+(1,2) & \rightarrow & 2+1+1. \end{array} \tag{3.3}$$

We can choose a set of complex coordinates so that $\mathbb{C}^2 = \mathbb{R}^4$. Then the $SU(2)$ acts on $\mathbb{C}^2$ in the standard way,

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{3.4}$$

The finite subgroups of $SU(2)$ have a classification described in terms of the simply laced semi-simple Lie algebrae: $A_n$, $D_n$, $E_6$, $E_7$ and $E_8$. And the subgroups, which we shall denote by $\Gamma_{A_n}$, $\Gamma_{D_n}$ and $\Gamma_{E_i}$ can be described explicitly

$\Gamma_{A_{n-1}}$ is isomorphic to $Z_n$, the cyclic group of order $n$, and it is generated by

$$\begin{pmatrix} e^{\frac{2\pi\imath}{n}} & 0 \\ 0 & e^{-\frac{2\pi\imath}{n}} \end{pmatrix}. \tag{3.5}$$

---

[2] It can be shown that the only singularities of $K3$'s are orbifold singularities.

[3] Remember that $SO(4) \cong (SU(2) \times SU(2))/Z_2$. In particular, that $SU(2)$ is one of these in the product and not a combination of them.

$\Gamma_{D_n}$ is isomorphic to $D_{k-2}$, the binary dihedral of order $4k-8$, and it has two generators,

$$\begin{pmatrix} e^{\frac{\pi\imath}{k-2}} & 0 \\ 0 & e^{-\frac{\pi\imath}{k-2}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix}. \tag{3.6}$$

$\Gamma_{E_6}$ is isomorphic to $T$, the binary tetrahedral group of order 24, and it has two generators,

$$\begin{pmatrix} e^{\frac{\pi\imath}{2}} & 0 \\ 0 & e^{-\frac{\pi\imath}{2}} \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{7\pi\imath}{4}} & e^{\frac{7\pi\imath}{4}} \\ e^{\frac{5\pi\imath}{4}} & e^{\frac{\pi\imath}{4}} \end{pmatrix}. \tag{3.7}$$

$\Gamma_{E_7}$ is isomorphic to $O$, the binary octohedral group of order 48, and it has three generators,

$$\begin{pmatrix} e^{\frac{\pi\imath}{4}} & 0 \\ 0 & e^{\frac{7\pi\imath}{4}} \end{pmatrix}, \quad \begin{pmatrix} e^{\frac{\pi\imath}{2}} & 0 \\ 0 & e^{-\frac{\pi\imath}{2}} \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{7\pi\imath}{4}} & e^{\frac{7\pi\imath}{4}} \\ e^{\frac{5\pi\imath}{4}} & e^{\frac{\pi\imath}{4}} \end{pmatrix}. \tag{3.8}$$

Finally, $\Gamma_{E_8}$is isomorphic to $I$, the icosahedral group of order 120, and it has two generators,

$$-\begin{pmatrix} e^{\frac{6\pi\imath}{5}} & 0 \\ 0 & e^{\frac{4\pi\imath}{5}} \end{pmatrix} \quad \text{and} \quad \frac{1}{e^{\frac{4\pi\imath}{5}} - e^{\frac{6\pi\imath}{5}}} \begin{pmatrix} e^{\frac{2\pi\imath}{5}} + e^{-\frac{2\pi\imath}{5}} & 1 \\ 1 & e^{\frac{2\pi\imath}{5}} - e^{-\frac{2\pi\imath}{5}} \end{pmatrix}. \tag{3.9}$$

### 3.1.2 Approximation to singular $K3$ manifolds

Now, we know that the singular points on $K3$ can be locally described by an orbifold $\mathbb{C}^2/\Gamma_{ADE}$. In order to obtain interesting physics, we can replace $K3$ by $\mathbb{C}^2/\Gamma_{ADE}$ and study $\mathscr{M}$-theory on $\mathbb{C}^2/\Gamma_{ADE} \times \mathbb{R}^{6,1}$. Since $K3$ went from smooth to singular, as we varied its moduli space, we expect that the singular orbifolds $\mathbb{C}^2/\Gamma_{ADE}$ are singular limits of non-compact smooth 4-manifolds $K^{ADE}$. Because of supersymmetry, $K^{ADE}$ should have $SU(2)$ holonomy. The metric of $SU(2)$-holonomy on $K^{ADE}$ are known as $ALE$ metric, since they asymptote to the locally Euclidean metric on $\mathbb{C}^2/\Gamma_{ADE}$.

Let us see how these singularities appear in the moduli space. We are considering $\mathscr{M}$-theory compactification on a compact 4-dimensional manifold with $SU(2)$-holonomy,

$$M^{10,1} = M^{6,1} \times K.$$

In general, $K$ may be either $T^4$ or $K3$. We shall be interested in $K3$ because its holonomy group is $SU(2)$. This means that $K3$ is a Calabi-Yau manifold and since $SU(2) \cong Sp(1)$, we have that it is also a Hyper-Kähler manifold. So that $K3$ is unique up to diffeomorphisms, let us just study a simple example of a homogeneous polynomial of $n$-order, $S$, embedded on a $\mathbb{C}P^3$. Using that

$$T_{\mathbb{C}P^3|_S} = T_S \oplus N_S,$$

where $T_S$ denotes the tangent bundle to $S$, $N_S$ is the normal bundle to $S$ and $T_{\mathbb{C}P^3|_S}$ is the restriction of the tangent bundle of the embedding $\mathbb{C}P^3$ to the hypersurface $S$. From the sum of Chern classes of a bundle $E$,

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots ,$$

it is possible to show that

$$c(T_{\mathbb{C}P^3}) = c(T_S) \wedge c(N_S).$$

Now, it is known that

$$c(T_{\mathbb{C}P^k}) = (1 + x)^{k+1},$$

where $x$ is the fundamental generator of $H^2(\mathbb{C}P^3, \mathbb{Z})$, which satisfies

$$\int_{\mathbb{C}p^3} x^3 = 1.$$

Stated in the fact that[4] $c_1(N_S) = nx$, we have that

$$\begin{aligned} c(T_S) &= \frac{(1+x)^4}{1+nx} && (3.10) \\ &= 1 + (4-n)x + (6 - 4n + n^2)x^2 + \cdots . && (3.11) \end{aligned}$$

Now, by requiring that $c_1(T_S) = 0$, it follows that $n = 4$. Now, we can calculate the Euler characteristic of $S$,

$$\begin{aligned} \chi(S) &= \int_S c_2(T_S) \\ &= \int_{\mathbb{C}P^3} c_2(T_S) \wedge l_S, && (3.12) \end{aligned}$$

[4]It can be shown by using the concept of divisors (in algebraic geometry) or by using the adjunction formula.

where $l_S$ is the 2-form which is the dual of the hypersurface $S$, then

$$\chi(S) = \int_{\mathbb{C}P^3} 6x^2 \cdot 4x = 24. \tag{3.13}$$

Since $b_0(S) = b_4(S) = 1$ and

$$\chi(S) = \sum_{n=0}^{4} (-1)^n b_n(S),$$

we get $b_2(S) = 22$.[5]

We can calculate the signature of $H^2(K3, \mathbb{R})$ by using the signature complex[6], which for Calabi-Yau manifolds is given by

$$\tau(M) = -\frac{2}{3}\chi(M) \ \Rightarrow \ \tau(S) = -16. \tag{3.14}$$

Thus, our 22-dimensional space has signature $(3, 19)$.

It allows us to split

$$H^2(K3, \mathbb{R}) = H^2_+(K3, \mathbb{R}) \oplus H^2_-(K3, \mathbb{R}), \tag{3.15}$$

where $H^2_+$ and $H^2_-$ are the subspace of the self-dual and anti-self-dual harmonic 2-forms on $K3$, and $b_2^+(K3) = 3$ and $b_2^-(K3) = 19$.

The first example of HyperKähler ALE space was written down explicitly by Eguchi and Hanson [15], and it describe a 4-dimensional manifold

$$M^4 = \mathbb{R}^2 \times S^2 = T^*S^2 \cong T^*\mathbb{C}P^1,$$

algebraically, it can be written as

$$M^4 = \left\{X, Y, Z \in \mathbb{C}^3 | X^2 + Y^2 + Z^2 = r^2\right\}.^7$$

[5]It is due to the fact that by definition, a $K3$ surface has $h^{1,0}(S) = 0$, which implies $b_1(S) = 0$.

[6]It is derived from the index theorem.

[7]As we have shown, whatever homogeneous function of order 4 embedded in a $\mathbb{C}P^3$ is algebraically a $K3$ surface. Then, in particular let us consider a $K3$ whose locus is given by

$$\left(X_1^2 + X_2^2 + X_3^2 + X_4^2\right) P_2(X_i) = 0,$$

where $P_2(X_i)$ represents an arbitrary polynomial of second order. So, calling $X_1 = X$, $X_2 = Y$, $X_3 = Z$ and $X_4^2 = -r^2$, and demanding that $P_2(X_i)|_{X_4^2=-r^2} \neq 0$, finally the locus equation is satisfies if $X^2 + Y^2 + Z^2 = r^2$.

Next, let us check that this metric describes the $\mathbb{R}^2 \times S^2$. First, let us take $Im(X, Y, Z) = 0$ and $r \in \mathbb{R}$, then

$$M^4 \to S^2 \quad \text{constrained to} \quad \sum_i Re(X^i) Im(X^i) = 0. \tag{3.16}$$

It is clear that the algebraic expression which describes the surface become a $S^2$ if we take $X$, $Y$, $Z$ and $r$ to be real variables. Note that this equation implies that $Im(X^i)$ are the tangent directions to the 2-sphere.

Taking the limit $r \to 0$, $S^2$ shrinks to a point, and the algebraic expression becomes

$$X^2 + Y^2 + Z^2 = 0. \tag{3.17}$$

Alternatively, we can change the coordinates as

$$X = u^2 - v^2, \tag{3.18}$$
$$Y = \imath(u^2 + v^2), \tag{3.19}$$
$$Z = 2uv, \tag{3.20}$$

such that

$$X^2 + Y^2 = -4u^2v^2 = -Z^2,$$

which locally maps

$$\mathbb{C}^3 \mapsto \mathbb{C}^2.$$

But by construction, when $u, v \to -u, -v$, the map is invariant, this transformation defines a $Z_2$ symmetry. Then, we get a map not just from $\mathbb{C}^3 \to \mathbb{C}^2$, but

$$\mathbb{C}^3 \mapsto \frac{\mathbb{C}^2}{Z_2} = \frac{\mathbb{C}^2}{\Gamma_{A_1}}. \tag{3.21}$$

This shows that $T^*S^2$ is mapped to $\mathbb{C}^2/\Gamma_{A_1}$ in the limit $r \to 0$.

### 3.1.3 $\mathscr{M}$-theory physics at the singularity

The Eguchi-Hanson metric has 3 parameters, which control the size and shape of the 2-sphere, that imply that considering $\mathscr{M}$-theory on the smooth non-compact space

$$\mathcal{M}^{3,1} \times \mathbb{R}^3 \times \mathbb{EH},$$

where $\mathbb{EH}$ is the Eguchi-Hanson space, which is the "blow-up" of

$$\mathcal{M}^{3,1} \times \mathbb{R}^3 \times \mathbb{C}^2/\Gamma_{A_1},$$

and tends to it when the radius of the 2-sphere goes to zero, we shall obtain three massless scalars in the 7-dimensional theory.

We can interpret the 2-sphere as a 2-cycle and contruct its dual harmonic 2-form. In fact, it is the only harmonic 2-form in $\mathbb{EH}$. In order for $\mathcal{M}^4 \times \mathbb{R}^3 \times \mathbb{EH}$ to be vacuum of $D = 11$ supergravity, we need to have $\langle G \rangle = \langle \psi \rangle = 0$, consistent with the requirements of compactifications on $G_2$ manifolds, it implies that from the Kaluza-Klein ansatz, we shall get a vector field in the 7-dimensional theory coming from the $C$-field.

But, from the $C$-fields, we get just a $U(1)$ gauge field theory in 7 dimensions. This gauge field combines with the scalars we have obtained from the 3 parameters of the metric, to form the bosonic part of a 7-dimensional Abelian vector multiplet.

Turning back into the singularities, we expect that when $r \to 0$, our target space looks locally as

$$\mathcal{M}^{3,1} \times \mathbb{R}^3 \times \frac{\mathbb{C}^2}{\Gamma_{A_1}}.$$

We expect that the theory contains some states which are massive as long as $r \neq 0$ and the 2-sphere is finite.

In order to enhance the symmetry, we must have that as well as the volume of the 2-sphere goes to zero, those massive states become massless. In $\mathscr{M}$-theory, it has a natural picture in term of the $\mathscr{M}2$-branes.

### 3.1.4 $\mathscr{M}$-branes

In the early 90's, two other solution of the equations of motion were found. They were called $\mathscr{M}2$- and $\mathscr{M}5$-branes [13, 19], these solutions appear when we consider (2.7) with an inhomogeneous term

$$\mathbf{d}^* G + \frac{1}{2} G \wedge G = \delta^8{}_{\mathscr{M}2}, \tag{3.22}$$

or the non-homogeneous gauge condition,

$$\mathbf{d} G = \delta^5{}_{\mathscr{M}5}, \tag{3.23}$$

respectively. In these equations, $\delta^n{}_{\mathscr{M}p}$ must be understood as an $n$-dimensional form functional with support on a $p$-dimensional submanifold.

The $\mathscr{M}2$-brane solution can be written as follows

$$\langle g\rangle_{MN} = H^{1/3}(r)\left[H^{-1}(r)\left(\eta^{(3)}_{ij}\delta^i_M\delta^j_N\right) + \mathbf{d}r_M\mathbf{d}r_N + r^2\mathbf{d}\Omega^{(7)}_M\mathbf{d}\Omega^{(7)}_N\right], \quad (3.24)$$

where $\eta$ is the 3-dimensional Minkowski metric, $i, j = 0, 1, 2$, $r = x^3$, and $\left(\mathbf{d}\Omega^{(7)}\right)^2$ is the 7-dimensional angular line element. Also we have

$$\langle C_{012}\rangle = H^{-1}(r), \quad (3.25)$$
$$\langle \psi_M\rangle = 0. \quad (3.26)$$

From (3.25) follows that

$$G = \frac{\partial_r H(r)}{H^2(r)}\mathbf{d}x^0\mathbf{d}x^1\mathbf{d}x^2\mathbf{d}x^3, \quad (3.27)$$

and of course

$$G \wedge G = 0.$$

Then, we get $G^{0123} = -H^{-1/3}(r)\partial_r H(r)$, so the field equations are

$$\begin{aligned} G^{0123} &\neq 0, \\ {}^*G_{456789\ell} &= \sqrt{g}\epsilon_{0123456789\ell}G^{0123}, \\ \mathbf{d}^*G_{456789\ell} &= \mathbf{d}\left(\sqrt{g}\epsilon_{0123456789\ell}G^{0123}\right) \end{aligned}$$

Since $\epsilon = \text{const}$ and $\sqrt{g} = \sqrt{g(r)}$, it follows that

$$\partial_r\left(\sqrt{g}G^{0123}\right) = 0, \quad (3.28)$$

this gives us, by using that $\sqrt{g} = H^{1/3}r^7$,

$$H''(r)r^7 + 7H'(r)r^6 = 0. \quad (3.29)$$

This equation can be solved, and by consistency with the Einstein equation of $D = 11$ supergravity (2.6), we obtain

$$H(r) = 1 + \frac{a}{r^6}. \quad (3.30)$$

The solution admits Killing spinors of the form

$$\varepsilon = H^{-1/6}\eta,$$

with $\eta$ a constant spinor satisfying

$$\Gamma_{012}\eta = \eta, \tag{3.31}$$

where $\Gamma_{012}$ represents the antisymmetric product of the three $\Gamma$'s 0, 1 and 2. This condition is nothing but the condition of $BPS$ states, as we refer in the section 1.1.

Similarly, we can construct the $\mathscr{M}5$-brane solution, by trying the ansatz

$$\langle g\rangle_{MN} = H^{2/3}(r)\left[H^{-1}(r)\left(\eta^{(6)}_{ij}\delta^i_M\delta^j_N\right) + \mathbf{d}r_M\mathbf{d}r_N + r^2\mathbf{d}\Omega^{(4)}_M\mathbf{d}\Omega^{(4)}_N\right], \tag{3.32}$$

where $\eta^{(6)}$ is the 6-dimensional Minkowski metric, $i, j = 0...5$, $r = x^6$ and $\left(\mathbf{d}\Omega^{(4)}\right)^2$ is the line element of the 4-dimensional angular metric. Together with

$$G_{x^7x^8x^9x^\ell} = \epsilon_{rx^7x^8x^9x^\ell}\partial_r H(r). \tag{3.33}$$

It also admits Killing spinors given by $\varepsilon = H^{-1/12}\eta$ where $\eta$ satisfies the projection

$$\Gamma_{01234}\eta = \eta, \tag{3.34}$$

where $\Gamma_{01234} = \Gamma_{[0}\Gamma_1\Gamma_2\Gamma_3\Gamma_{4]}$ and for that reason the $\mathscr{M}5$-brane solution is another $BPS$ state of the theory.

This solution is compatible with the Einstein equation of the $D = 11$ supergravity if

$$H(r) = 1 + \frac{a}{r^4}. \tag{3.35}$$

Now, we can think that if we wrap the 2-sphere with $\mathscr{M}2$-branes their will appear as particles from the 7-dimensional point of view.

Having the same intuition as in Maxwell's theory, in which the gauge field $A$ couples to the 1-dimensional world-line $\gamma$ that is swept out by an electron via

$$\int_\gamma A.$$

In 11-dimensional supergravity, the 3-form $C$ can couple to a (2+1)-object, the $\mathscr{M}2$-brane, via

$$\int_{\mathscr{M}2} C. \tag{3.36}$$

Let us assume

$$\mathscr{M}2 = S^2 \times \mathbb{R},$$

where $\mathbb{R}$ represents the temporal coordinate and $S^2$ is the one of the Eguchi-Hanson metric. Since

$$C = A \wedge \beta, \tag{3.37}$$

where $\beta$ is the harmonic 2-form on $S^2$, (3.36) can be written as

$$\int_{\mathscr{M}2} C = \int_{S^2\times\mathbb{R}} \beta \wedge A = \int_{S^2} \beta \int_{\mathbb{R}} A, \tag{3.38}$$

it follows by using Poincarè duality that[8]

$$\int_{S^2} \beta = 1. \tag{3.39}$$

Then,

$$\int_{\mathscr{M}2} C = \int_{\mathbb{R}} A, \tag{3.40}$$

which represent a charged particle under $U(1)$.

Similarly, for the anti-branes[9], we have

$$\int_{\overline{\mathscr{M}2}} C = -\int_{S^2\times\mathbb{R}} \beta \wedge A = -\int_{\mathbb{R}} A, \tag{3.41}$$

which represent particles with opposite charge under $U(1)$.

These $\mathscr{M}2$-brane solutions are associated to the $W^+$ gauge boson and $\overline{\mathscr{M}2}$-brane describes the $W^-$ gauge boson.[10]

It can be shown that $\mathscr{M}2$-branes are BPS states[11], for BPS states is known that its mass is proportional to its charge, in general the proportionality is given by a function depending of the parameters of the moduli space,

$$M_{BPS} = f(moduli)Q. \tag{3.42}$$

In the case of $\mathscr{M}2$-branes, that function is just the volume of the 2-sphere $vol(S^2)$.

[8]It is because $\beta$ is the harmonic 2-form associated to the 2-cycle $S^2$.

[9]Our meaning of anti-brane is just a brane with opposite orientation.

[10]Due to supersymmetry, they are not just the gauge bosons but gauge bosons multiplet.

[11]As we said before, a BPS is invariant under half of supersymmetries, and we know that the state associated with the least volume 2-sphere is BPS because is a calibrated cycle. This $S^2$ is the one given by $Im(X, Y, Z) = 0$ in the Eguchi-Hanson metric.

Finally, as we are reconsidering the discussion about the enhancement of symmetry, this picture shows us that when the radius of the 2-sphere goes to zero, the mass of the gauge bosons tend to zero. So, in this limit we obtain two additional massless states of opposite charge. These combine with the $U(1)$ gauge fields to enhance the symmetry group from $U(1)$ to $SU(2)$. Therefore, we expect that the effective theory in 7 dimensions is a super Yang-Mills theory with gauge group $SU(2)$.

More generally, we expect the effective 7-dimensional theory of $\mathscr{M}$-theory on $\mathcal{M}^{3,1} \times \mathbb{R}^3 \times \mathbb{C}^2/\Gamma_{ADE}$ to be super Yang-Mills theory with the corresponding $ADE$ gauge group.

### 3.1.5 $ADE$-singularities on $G_2$-manifolds

In the last section, we have studied the $ADE$-singularities on $K3 \times \mathbb{R}^{6,1}$ manifolds. In this section, we shall consider a more general case of $ADE$-singularities in an 11-dimensional manifold reducible as $\mathbb{C}^2/\Gamma \times Y^{6,1}$.

As before, the 7-dimensional super Yang-Mills theory on $Y^{6,1}$ has a gauge group determined by the $ADE$-singularity living along $Y^{6,1}$, and in general its group symmetry is given by $SO(3) \times SO(6,1)$, where the $SO(3)$ is the $R$-symmetry and the second factor is actually the Lorentz group in the 7-dimensional spacetime.

This theory contains scalars, fermions and gauge fields in the following representations:

| | Representation in $SO(3) \times SO(6,1)$ |
|---|---|
| Scalars | $(3,1)$ |
| Fermions | $(2,8)$ |
| Gauge fields | $(1,7)$ |

The supercharges are in fact fermionic fields, so they transform in the $(2,8)$ representation.

As a first glance, we shall just consider $Y^{6,1} = W \times \mathcal{M}^4$, in this case, the symmetry group is broken to $SO(3) \times SO(3') \times SO(3,1)$. Therefore, the fields content of the theory transform as

| | Representation in $SO(3) \times SO(3') \times SO(3,1)$ |
|---|---|
| Scalars | $(3,1,1)$ |
| Fermions | $(2,2,2) + (2,2,\bar{2})$ |
| Gauge fields | $(1,3,1) + (1,1,4)$ |

The $SO(3')$ represents the group structure of the tangent bundle on $W$. Similarly the $SO(3)$ symmetry can be seen as the group structure of the normal bundle on $W$. Then, the covariance of the theory requires that there must exist a gauge field transforming under $SO(3)$ and other transforming under $SO(3')$.

For generic $W$, the 4-dimensional theory is not supersymmetric, so we must demand that in the 7-dimensional manifold, which looks locally as $\mathbb{C}^2/\Gamma_{ADE} \times W$, we can define a $G_2$-structure. In a curved $W$ manifold, we cannot write the metric of the $G_2$-structure as a product of the metric on $\mathbb{C}^2/\Gamma_{ADE}$ times the metric on $W$, because in that case the metric is warped and it looks like a bundle metric in which the metric of $\mathbb{C}^2$ change while we are moving on $W$. For this we shall concentrate our attention in a locally flat frame on $W$.

In a locally flat frame, we can write down a formula for the $G_2$ structure on $\mathbb{C}^2/\Gamma_{ADE} \times W$,

$$\Phi = \omega_i \wedge e_j \delta^{ij} + e_1 \wedge e_2 \wedge e_3, \tag{3.43}$$

where $e_i$ are a flat frame on $W$ and the $\omega_i$ define the Hyper Kähler structure of the $\mathbb{C}^2/\Gamma_{ADE}$ and are defined in a locally coordinate system as

$$\omega_1 = \mathbf{d}x^0 \wedge \mathbf{d}x^1 + \mathbf{d}x^2 \wedge \mathbf{d}x^3 \tag{3.44}$$

$$\omega_2 = \mathbf{d}x^0 \wedge \mathbf{d}x^2 + \mathbf{d}x^3 \wedge \mathbf{d}x^1 \tag{3.45}$$

$$\omega_3 = \mathbf{d}x^0 \wedge \mathbf{d}x^3 + \mathbf{d}x^1 \wedge \mathbf{d}x^2. \tag{3.46}$$

We can show that (3.43) is an invariant 3-form as (1.7) by making the explicit calculation and rotate our coordinate system.

The $SO(3)$ symmetry rotates the complex structures of the Hyper-Kähler manifold, i.e., acts on the $\omega_i$. In order to have a well defined $G_2$-structure the $SO(3)$ must act in the same way on the $e_i$. Nonetheless, the natural action on $e_i$ is the action of $SO(3')$. Thus, if $\mathbb{C}^2/\Gamma_{ADE} \times W$ admits a $G_2$-holonomy metric, we must identify $SO(3)$ with $SO(3')$.

Identifying $SO(3)$ with $SO(3')$ breaks down the symmetry group to $SO(3'')\times SO(3,1)$. Now the effective 4-dimensional theory is supersymmetric and this is because there exists a covariantly contant spinor, thus, the field content of the 4-dimensional theory is given by

| | Representation in $SO(3'') \times SO(3,1)$ |
|---|---|
| Scalars | $(3,1)$ |
| Fermions | $(1,2) + (3,2) + cc$ |
| Gauge fields | $(3,1) + (1,4)$ |

Obviously, the $(1,2)$ and its complex conjugate give us the constant spinors on $W$ and by this we are dealing with a supersymmetric 4-dimensional theory.

Thus the fields which are scalars under the 4-dimensional Lorentz group are two copies of the 3 of $SO(3'')$. These may be interpreted as two 1-forms on $W$. These will be massless if they are zero modes of the Laplacian on $W$ (with respect to its induced metric from the $G_2$ manifold). There will be precisely $b_1(W)$ of them. Their supersymmetric partners are the $(3,2)+cc$ fermions, which will be massless by supersymmetry. This is the field content of $b_1(W)$ chiral supermultiplets of the supersymmetry algebra in 4 dimensions.

All these fields transform in the adjoint representation of the 7-dimensional gauge group. Thus the final result for massless fields is that they are described by the $\mathscr{N}=1$ super Yang-Mills theory with $b_1(W)$ massless adjoint chiral multiplets.

## 3.2 Duality with Type $IIA$ strings

There is another method to study the physics coming from singularities of $\mathscr{M}$-theory. It comes from the duality between type $IIA$ string theory and $\mathscr{M}$-theory compactified on a circle $S^1$ [30, 33].

From this relation, we should expect that some states have their geometrical origin in the lifted of the 11 dimension. It can be seen by writing the 11-dimensional metric as

$$g_{MN} = e^{-\frac{2}{3}\phi} g_{\hat{\mu}\hat{\nu}} \delta^{\hat{\mu}}_{M} \delta^{\hat{\nu}}_{N} + e^{\frac{4}{3}\phi} \left(\mathbf{d}x_{11} + A_{\hat{\mu}} \mathbf{d}x^{\hat{\mu}}\right)_M \left(\mathbf{d}x_{11} + A_{\hat{\mu}} \mathbf{d}x^{\hat{\mu}}\right)_N , \quad (3.47)$$

where $\hat{\mu}, \hat{\nu} = 0...9$, $\phi$ is the dilaton, $g_{\hat{\mu}\hat{\nu}}$ is the 10-dimensional metric, and $A_{\hat{\mu}}$ is the Ramond-Ramond 1-form.

The explicit form for the metric (3.47) can be interpreted as a $S^1$ fibration over the 10-dimensional spacetime, such as the topology of the fibration is determined by the Ramond-Ramond 1-form.

As a first glance, let us consider $\mathscr{M}$-theory on Taub-NUT. The Taub-NUT space is a non-compact 4-dimensional manifold with $SU(2)$ holonomy and metric [15, 14]

$$g^{TN}{}_{\mu\nu} = H \mathbf{d}\vec{x}_\mu \mathbf{d}\vec{x}_\nu + H^{-1} \left(\mathbf{d}x_{11} + A_i \mathbf{d}x^i\right)_\mu \left(\mathbf{d}x_{11} + A_i \mathbf{d}x^i\right)_\nu , \quad (3.48)$$

with

$$\vec{\nabla} \times \vec{A} = -\vec{\nabla} H \quad \text{and} \quad H = 1 + \frac{1}{2|\vec{x}|}. \tag{3.49}$$

This metric describes obviously the $S^1$ fibration, which for a sphere of constant radius inside the $\mathbb{R}^3$, the winding number of the fibration is one over the sphere. This indicates that there exist a topological defect localized at $\vec{x} = 0$, where the $S^1$ fibre degenerates. This topological defect is a $D6$-brane.

This drives us to think that there is a duality between $\mathscr{M}$-theory on Taub-NUT space and type $IIA$ string theory in flat spacetime with a $D6$-brane.

This duality can be extendend to more general manifolds that admit a smooth $U(1)$ action, as follows, if $X$ is a space with $U(1)$ isometry, such that $X/U(1)$ is smooth, then the fixed point set, $L$, of the action $U(1)$ must be a codimension 4 inside $X$ [27, 26]. Then, it gives us a duality between $\mathscr{M}$-theory on $X$ and type $IIA$ string theory on $X/U(1)$ with $D6$-branes on $L$. Some constructions based on these ideas can be found in [5, 6].

In these dualities, physics of the singularities on $\mathscr{M}$-theory side is obtained from the $D6$-branes on type $IIA$ strings side.

## 3.3 Chiral Fermions from $G_2$-manifolds

There are three different ways to show that chiral fermions could be obtained from singularities of conical type on $G_2$ holonomy manifolds. A $n$-dimensional manifold $(M, g)$ is said to have a conical singularity if its line element can be written asymptotically as

$$ds^2 \sim dr^2 + r^2 \widetilde{g}_{ij} dx^i dx^j, \tag{3.50}$$

where $r$ is the radial coordinate and $\widetilde{g}_{ij}$ is the metric of the $n-1$-dimensional manifold. The conical singularity is localized at $r = 0$.

The ways for obtaining chiral fermions are

- Considering duality with heterotic string theory [2],
- Considering duality with type $IIA$ string theory [6] and
- By doing anomalies analysis [34].

We shall no present details of neither of these mechanism, because the necessary tools for our complete understanding of the subject are not still developed.

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