An Integro-differential Equation for Pricing CDO

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Abstract: Expected value of CDO satisfies a simple integro-differential equation if cumulative default loss process is Markovian. We show how it is derived and is solved numerically.

1. Derivation

Copula models equate expected values of the protection leg (V) and the premium leg (W) to back out CDO tranche premium. Following standard treatment,

(1) $$V(l,t) = E\left(\int_t^T B(t,s)dM(s)\right) = B(t,T)E\big(M(T)\big) + \int_t^T E\big(M(s)\big)f(t,s)B(t,s)ds$$

(2) $$W(l,t) = \sum_{i=1}^{n} \omega \cdot p \cdot B(t,t_i) \cdot E\big(N(t_i)\big)$$

where $M(t) = \big(l(t)-\alpha\big)^+ - \big(l(t)-\beta\big)^+$, $N(t) = \big(\beta - l(t)\big)^+ - \big(\alpha - l(t)\big)^+$; $l(t)$ is the cumulative default loss at time t; α and β denote the tranche attachment and detachment points , respectively; $B(t,s)$ is the time-t price of a pure discount bond maturing at time-s, and $f(t,s)$ the instantaneous forward rate; ω is the payment interval for tranche premium p; $t_0 = t$, $t_n = T$; and wlog, all notionals are unitary.

Let $l(t)$ be Markovian, and consider replacing the copula in $E(\cdot)$ with transition probability $P(L,s \mid l,t)$. By the backward Chapman-Kolmogorov equation,

(3) $$P(L,s \mid l,t) = \int P(L,s \mid l+\xi, t+dt)P(l+\xi, t+dt \mid l,t)d\xi$$

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where the propagator, $P(l+\xi, t+dt \mid l, t)$, conditional on whether jump occurs, can be written as

$$P(l+\xi, t+dt \mid l, t) = \big(1-h(l,t)dt\big)\cdot\delta(l) + h(l,t)dt\cdot w(\xi \mid l,t)\,. \tag{4}$$

In (4), h is the jump intensity, δ is the Dirac delta function, and w is the conditional distribution density of jump size ξ. Applying (4) in (3) leads to the backward master equation [2]

$$-\frac{\partial}{\partial t}P(L,s \mid l,t) = h(l,t)\int P(L,s \mid l+\xi,t)w(\xi \mid l,t)d\xi - h(l,t)P(L,s \mid l,t)\,. \tag{5}$$

For simplicity, assume constant riskfree rate r. Differentiate both sides of (1) with respect to t and use (5); it is straightforward to derive

$$\frac{\partial}{\partial t}V(l,t) = (r+h)V(l,t) - h\int_0^{1-l}V(l+\xi,t)w(\xi \mid l,t)d\xi - r\big[(l-\alpha)^+ - (l-\beta)^+\big]. \tag{6}$$

Similarly,

$$\frac{\partial}{\partial t}W(l,t) = (r+h)W(l,t) - h\int_0^{1-l}W(l+\xi,t)w(\xi \mid l,t)d\xi\,. \tag{7}$$

Boundary and terminal (jump) conditions for Equations (6) and (7) are

$$\begin{aligned}
&V(\beta,s) = \beta-\alpha && (t \le s \le T)\\
&V(L,T) = (L-\alpha)^+ - (L-\beta)^+ && (l \le L \le 1)\\
&W(\beta,s) = 0 &&\\
&W(L,T) = \omega p\big[(\beta-L)^+ - (\alpha-L)^+\big] &&\\
&W(L,t_j-) = W(L,t_j+) + \omega p\big[(\beta-L)^+ - (\alpha-L)^+\big] && (j=1,2,\ldots,n-1)
\end{aligned} \tag{8}$$

By linear superposition, expected value of the tranche to a buyer, $U := V - W$, follows

$$\frac{\partial}{\partial t}U(l,t) = (r+h)U(l,t) - h\int_0^{1-l}U(l+\xi,t)w(\xi \mid l,t)d\xi - r\big[(l-\alpha)^+ - (l-\beta)^+\big] \tag{9}$$

subject to

$$U(\beta,s) = \beta-\alpha \quad \text{and} \tag{10}$$

$$U(L,T) = \big[(L-\alpha)^+ - (L-\beta)^+\big](1+\omega p) - (\beta-\alpha)\omega p\,. \tag{11}$$

$$U(L,t_j-) = U(L,t_j+) - \omega p\big[(\beta-L)^+ - (\alpha-L)^+\big] \qquad (j=1,2,\ldots,n-1) \tag{12}$$

[2] See Gillespie (1992).

Fair premium p can be backed out by adjusting the terminal condition (11) so that solution to $U(l,t)$ is zero.

2. Numerical Solution

In general, Equation (9) needs to be solved numerically. Discretize the spatial domain $[l,\beta)$ to $l+i\Delta$, $i=0,1,...,N$ with $\Delta=(\beta-l)/N$, and use simple trapezoidal rule to resolve the integral [3]; we have

$$\text{(13)}\quad \begin{aligned} \frac{\partial}{\partial t}U_i &= (r+h_i)U_i - h_i\Delta\left(\frac{1}{2}w_0^{(i)}U_i + \sum_{j=i+1}^{N-1} w_{j-i}^{(i)}U_j + \frac{1}{2}w_{N-i}^{(i)}U_N\right) \\ &\quad - h_i(\beta-\alpha)\int_{\beta-l-i\Delta}^{1-l-i\Delta} w(\xi \mid l+i\Delta,t)\,d\xi - r\left[(l+i\Delta-\alpha)^+ - (l+i\Delta-\beta)^+\right] \end{aligned}$$

where $U_i=U(l+i\Delta,t)$, $h_i=h(l+i\Delta)$ and $w_j^{(i)}=w(j\Delta \mid l+i\Delta,t)$ for $i=0,1,...,N-1$. This is a system of N first-order ODEs that can be easily solved once the functions h and $w^{(i)}$ ($i=0,1,...,N-1$) are specified. To back out p, we can use a root searching algorithm such as Brent's method.

For a toy example, consider $h(l,t)=h$ and $w(\xi \mid l,t)=(1-l)^{-1}$. Then

$$\text{(14)}\quad \frac{\partial}{\partial t}\tilde{U} = \left((r+h)\tilde{I} - h\Delta\tilde{A}\right)\cdot\tilde{U} - h(\beta-\alpha)\left(\Delta+2(1-\beta)\right)\tilde{B} - r\tilde{C},$$

where $\tilde{U}=\begin{pmatrix}U_{N-1} & U_{N-2} & \cdots & U_1 & U_0\end{pmatrix}^T$, $\tilde{I}$ the identity matrix, $\tilde{A}=[A_{ij}]_{N\times N}$ with $A_{ij}=\left(1-\left(l+(N-i)\Delta\right)\right)^{-1}$ if $i>j$, $A_{ii}=\frac{1}{2}\left(1-\left(l+(N-i)\Delta\right)\right)^{-1}$ and $A_{ij}=0$ if $i<j$, $\tilde{B}=diag(\tilde{A})$, $\tilde{C}=\left[C_i\right]_{N\times 1}$ with $C_i=\left(l+(N-i)\Delta-\alpha\right)^+ - \left(l+(N-i)\Delta-\beta\right)^+$, and $\tilde{U}(T)=(1+\omega p)\tilde{C}-(\beta-\alpha)\omega p$. Table 1 reports premium estimates to various tranches. Figure 1 gives the solution profile to the junior mezzanine tranche.

Table 1: Tranche premium

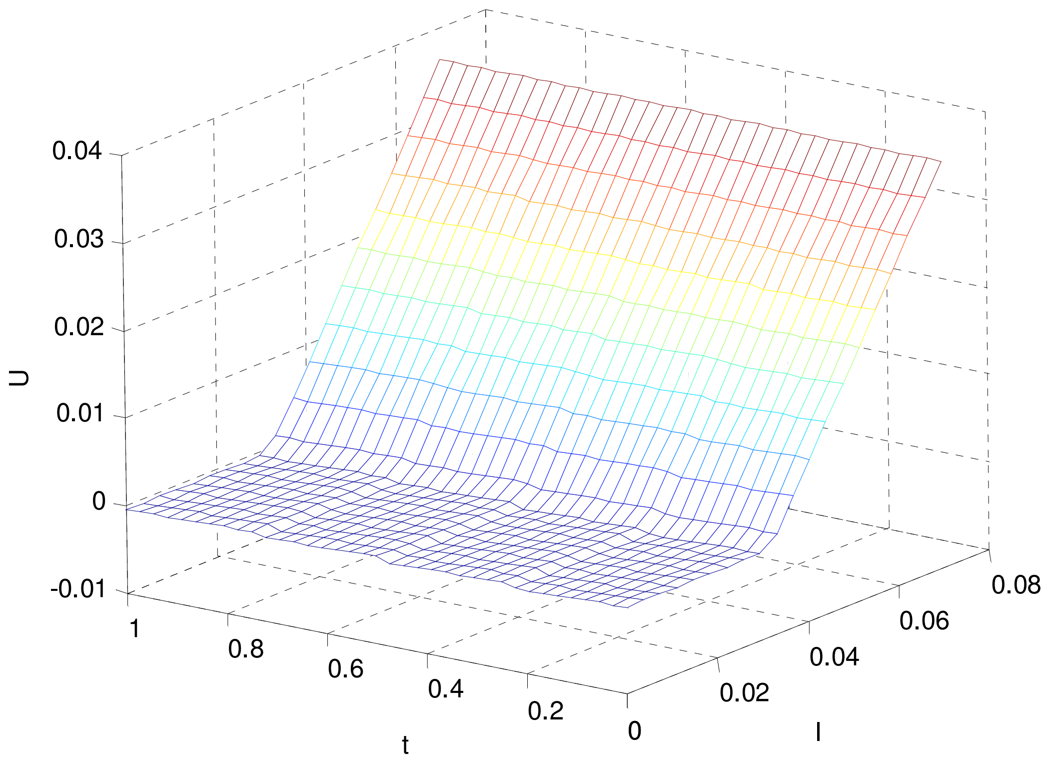
tranche	[0,0.03)	[0.03,0.07)	[0.07,0.1)	[0.1,0.15)	[0.15,0.3)	[0,1]
Premium (bps)	491	471	456	430	355	2

Note: $r=5\%$, $h=0.05$, $t=0$, $l(0)=0$, $T=1$ and $\omega=0.25$.

[3] For sophisticated higher order methods, see Linz (1985).

Realistic estimates might be found by choosing appropriate functional forms (or dynamics) for h and w. Potentially, CDO option could also be priced if we discretize the time dimension to allow for dynamic programming.

Figure 1: Solution profile to tranche $[0.03, 0.07)$



3. CDS

Similar integrodifferential equations exist for expected values of the protection leg (V) and the premium leg (W) of CDS. Using the following equalities,

(15.1) $DL = (1-R)B(t,\tau)$ (default/protection leg)

(15.2) $PL = \sum_{i=1}^{n} \omega \cdot p \cdot B(t,t_i) \cdot 1(\tau > t_i)$ (premium leg [4])

(15.3) $F(t) = E(1(\tau < t))$

(15.4) $l(t) = (1-R) \cdot 1(\tau < t)$

[4] Omitting the accrual.

where τ is default time, R the recovery [5] and $1(\cdot)$ the indicator function [6], we have

$$
\begin{aligned}
V &= E(DL) = \int_t^T (1-R)B(t,s)dF(s) \\
&= (1-R)B(t,T)F(T) + \int_t^T (1-R)f(t,s)B(t,s)F(s)ds \\
&= B(t,T)E(l(T)) + \int_t^T E(l(s))f(t,s)B(t,s)ds
\end{aligned} \tag{16}
$$

and

$$W = E(PL) = \sum_{i=1}^{n} \omega \cdot p \cdot B(t,t_i)\cdot(1-F(t_i)) = \sum_{i=1}^{n} \omega \cdot p \cdot B(t,t_i)\cdot\left(1-\frac{E(l(t_i))}{1-R}\right). \tag{17}$$

With the notion that $E(l(s)) = \int L \cdot P(L,s \mid l,t)dL$ [7], we can differentiate V (W) with respect to t and apply (5) [8], which gives

$$\frac{\partial}{\partial t}V(l,t) = (r+h)V(l,t) - h\int_0^{1-l} V(l+\xi,t)w(\xi \mid l,t)d\xi - rl \tag{18}$$

and

$$\frac{\partial}{\partial t}W(l,t) = (r+h)W(l,t) - h\int_0^{1-l} W(l+\xi,t)w(\xi \mid l,t)d\xi\,. \tag{19}$$

Boundary and terminal (jump) conditions are found as:

$$
\begin{aligned}
&V(L,T) = L && (l \le L \le 1) \\
&V(1-R,s) = 1-R && (t \le s \le T) \\
&W(L,T) = \omega p\left(1-\frac{L}{1-R}\right) && \\
&W(1-R,s) = 0\,. && \\
&W(L,t_j-) = W(L,t_j+) + \omega p\left(1-\frac{L}{1-R}\right) && (j = 1,2,\ldots,n-1)
\end{aligned} \tag{20}
$$

Naturally, expected value of CDS to a buyer, $U := V - W$, satisfies

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[5] Assume R is fixed for now.

[6] Other variables/functions are similarly defined as before.

[7] Abbreviate $l(t)$ to l.

[8] After replacing h and w in (5) with jump intensity and jump distribution for the individual obligor.

(21) $$\frac{\partial}{\partial t}U(l,t) = (r+h)U(l,t) - h\int_0^{1-l} U(l+\xi,t)w(\xi \mid l,t)d\xi - rl$$

subject to

(22) $$U(L,T) = L(1+\frac{\omega p}{1-R}) - \omega p \quad \text{and}$$

(23) $$U(1-R,s) = 1-R\ .$$

(24) $$U(L,t_j-) = U(L,t_j+) - \omega p(1-\frac{L}{1-R}) \qquad (j = 1,2,...,n-1)$$

Again, we back out CDS spread p by making $U(l,t)=0$. Jump distribution w in (18), (19) and (21) does not have to be Dirac delta, since recovery R can be stochastic. Therefore, boundary conditions could be slackened to $V(1,s)=1$, $W(1,s)=0$ and $U(1,s)=1$.

Denote h and w in (9) as h_Ω and w_Ω, and those in (21) as $h_{(i)}$ and $w_{(i)}$ for obligor i. Clearly there should be some constraint(s) between the aggregate functions h_Ω and w_Ω and the individual $h_{(i)}$ and $w_{(i)}$ ($i \in \Omega$).[9] These relations may help identifying their appropriate functional forms (or dynamics) to fit tranche prices.

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[9] For instance, $h_\Omega = \sum_i h_{(i)}$ since $1-h_\Omega dt = \prod_i (1-h_{(i)}dt)$.