# An Integro-differential Equation for Pricing CDO

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**Abstract**: Expected value of CDO satisfies a simple integro-differential equation if cumulative default loss process is Markovian. We show how it is derived and is solved numerically.

## 1. Derivation

Copula models equate expected values of the protection leg ($V$) and the premium leg ($W$) to back out CDO tranche premium. Following standard treatment,

$$(1) \qquad V(l,t) = E\left(\int_t^T B(t,s)dM(s)\right) = B(t,T)E\big(M(T)\big) + \int_t^T E\big(M(s)\big)f(t,s)B(t,s)ds$$

$$(2) \qquad W(l,t) = \sum_{i=1}^{n} \omega \cdot p \cdot B(t,t_i) \cdot E\big(N(t_i)\big)$$

where $M(t) = \big(l(t)-\alpha\big)^+ - \big(l(t)-\beta\big)^+$, $N(t) = \big(\beta - l(t)\big)^+ - \big(\alpha - l(t)\big)^+$; $l(t)$ is the cumulative default loss at time $t$; $\alpha$ and $\beta$ denote the tranche attachment and detachment points , respectively; $B(t,s)$ is the time-$t$ price of a pure discount bond maturing at time-$s$, and $f(t,s)$ the instantaneous forward rate; $\omega$ is the payment interval for tranche premium $p$; $t_0 = t$, $t_n = T$; and wlog, all notionals are unitary.

Let $l(t)$ be Markovian, and consider replacing the copula in $E(\cdot)$ with transition probability $P(L,s \mid l,t)$. By the backward Chapman-Kolmogorov equation,

$$(3) \qquad P(L,s \mid l,t) = \int P(L,s \mid l+\xi, t+dt) P(l+\xi, t+dt \mid l,t) d\xi$$

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where the propagator, $P(l+\xi, t+dt \mid l, t)$, conditional on whether jump occurs, can be written as

$$P(l+\xi, t+dt \mid l, t) = \big(1-h(l,t)dt\big)\cdot \delta(\xi) + h(l,t)dt \cdot w(\xi \mid l, t)\,. \tag{4}$$

In (4), $h$ is the jump intensity, $\delta$ is the Dirac delta function, and $w$ is the conditional distribution density of jump size $\xi$. Applying (4) in (3) leads to the backward master equation [3]

$$-\frac{\partial}{\partial t}P(L,s \mid l,t) = h(l,t)\int P(L,s \mid l+\xi,t)w(\xi \mid l,t)d\xi - h(l,t)P(L,s \mid l,t)\,. \tag{5}$$

For simplicity, assume constant riskfree rate $r$. Differentiate both sides of (1) with respect to $t$ and use (5); it is straightforward to derive

$$\frac{\partial}{\partial t}V(l,t) = (r+h)V(l,t) - h\int_0^{1-l} V(l+\xi,t)w(\xi \mid l,t)d\xi - r\big[(l-\alpha)^+ - (l-\beta)^+\big]. \tag{6}$$

Similarly,

$$\frac{\partial}{\partial t}W(l,t) = (r+h)W(l,t) - h\int_0^{1-l} W(l+\xi,t)w(\xi \mid l,t)d\xi\,. \tag{7}$$

Boundary and terminal (jump) conditions for Equations (6) and (7) are

$$\begin{aligned} &V(\beta,s) = \beta - \alpha && (t \le s \le T) \\ &V(L,T) = (L-\alpha)^+ - (L-\beta)^+ && (l \le L \le 1) \\ &W(\beta,s) = 0 && \\ &W(L,T) = \omega p\big[(\beta - L)^+ - (\alpha - L)^+\big] && \\ &W(L,t_j -) = W(L,t_j +) + \omega p\big[(\beta - L)^+ - (\alpha - L)^+\big] && (j = 1,2,\dots,n-1). \end{aligned} \tag{8}$$

By linear superposition, expected value of the tranche to a buyer, $U := V - W$, follows

$$\frac{\partial}{\partial t}U(l,t) = (r+h)U(l,t) - h\int_0^{1-l} U(l+\xi,t)w(\xi \mid l,t)d\xi - r\big[(l-\alpha)^+ - (l-\beta)^+\big] \tag{9}$$

subject to

$$U(\beta,s) = \beta - \alpha \tag{10}$$

$$U(L,T) = \big[(L-\alpha)^+ - (L-\beta)^+\big](1+\omega p) - (\beta-\alpha)\omega p \tag{11}$$

$$U(L,t_j -) = U(L,t_j +) - \omega p\big[(\beta - L)^+ - (\alpha - L)^+\big] \qquad (j = 1,2,\dots,n-1). \tag{12}$$

[3] See Gillespie (1992).

Fair premium $p$ can be backed out by adjusting the terminal condition (11) so that solution to $U(l,t)$ is zero.

## 2. Numerical Solution

In general, Equation (9) needs to be solved numerically. Discretize the spatial domain $[l,\beta)$ to $l+i\Delta$, $i=0,1,...,N$ with $\Delta=(\beta-l)/N$, and use simple trapezoidal rule to resolve the integral [4]; we have

$$\text{(13)}\quad \begin{aligned}\frac{\partial}{\partial t}U_i &= (r+h_i)U_i - h_i\Delta\left(\frac{1}{2}w_0^{(i)}U_i + \sum_{j=i+1}^{N-1} w_{j-i}^{(i)}U_j + \frac{1}{2}w_{N-i}^{(i)}U_N\right)\\ &\quad - h_i(\beta-\alpha)\int_{\beta-l-i\Delta}^{1-l-i\Delta} w(\xi \mid l+i\Delta,t)\,d\xi - r\left[(l+i\Delta-\alpha)^+ - (l+i\Delta-\beta)^+\right]\end{aligned}$$

where $U_i = U(l+i\Delta,t)$, $h_i = h(l+i\Delta,t)$ and $w_j^{(i)} = w(j\Delta \mid l+i\Delta,t)$ for $i=0,1,...,N-1$. This is a system of $N$ first-order ODEs that can be easily solved once the functions $h$ and $w^{(i)}$ ($i=0,1,...,N-1$) are specified. To back out $p$, we can use a root searching algorithm such as Brent's method.

For a toy example, consider $h(l,t)=h$ and $w(\xi \mid l,t)=(1-l)^{-1}$. Then

$$\text{(14)}\quad \frac{\partial}{\partial t}\tilde{U} = \left((r+h)\tilde{I} - h\Delta\tilde{A}\right)\cdot\tilde{U} - h(\beta-\alpha)\left(\Delta+2(1-\beta)\right)\tilde{B} - r\tilde{C},$$

where $\tilde{U} = \begin{pmatrix}U_{N-1} & U_{N-2} & \ldots & U_1 & U_0\end{pmatrix}^T$, $\tilde{I}$ the identity matrix, $\tilde{A}=[A_{ij}]_{N\times N}$ with $A_{ij} = \left(1-\left(l+(N-i)\Delta\right)\right)^{-1}$ if $i>j$, $A_{ii} = \frac{1}{2}\left(1-\left(l+(N-i)\Delta\right)\right)^{-1}$ and $A_{ij}=0$ if $i<j$, $\tilde{B} = diag(\tilde{A})$, $\tilde{C} = \left[C_i\right]_{N\times 1}$ with $C_i = \left(l+(N-i)\Delta-\alpha\right)^+ - \left(l+(N-i)\Delta-\beta\right)^+$, and $\tilde{U}(T) = (1+\omega p)\tilde{C} - (\beta-\alpha)\omega p$. Table 1 reports premium estimates to various tranches. Figure 1 gives the solution profile to the junior mezzanine tranche.

Table 1: Tranche premium

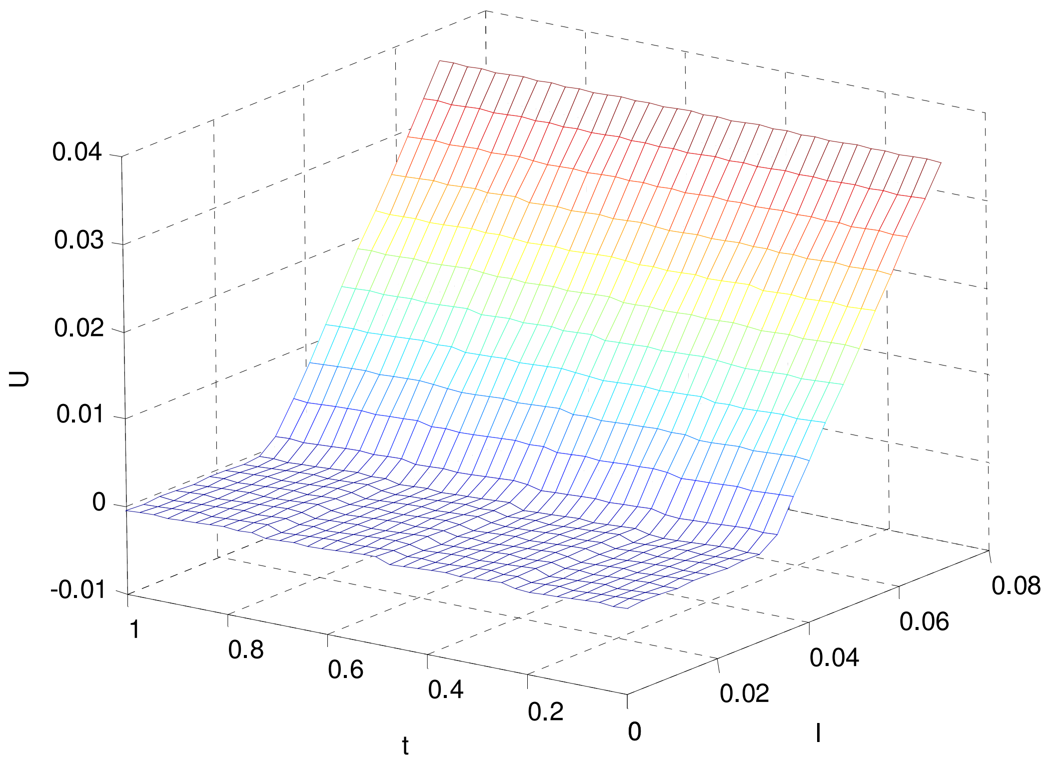
| tranche | [0,0.03) | [0.03,0.07) | [0.07,0.1) | [0.1,0.15) | [0.15,0.3) | [0,1] |
|---|---|---|---|---|---|---|
| Premium (bps) | 491 | 471 | 456 | 430 | 355 | 2 |

Note: $r=5\%$, $h=0.05$, $t=0$, $l(0)=0$, $T=1$ and $\omega=0.25$.

[4] For sophisticated higher order methods, see Linz (1985).

Realistic estimates might be found by choosing appropriate functional forms (or dynamics) for $h$ and $w$.

Figure 1: Solution profile to tranche [0.03,0.07)




## 3. CDS

Similar integrodifferential equations exist for expected values of the protection leg ($V$) and the premium leg ($W$) of CDS. Using the following equalities,

(15.1) $DL = (1-R)B(t,\tau)$ (default/protection leg)

(15.2) $PL = \sum_{i=1}^{n} \omega \cdot p \cdot B(t,t_i) \cdot 1(\tau > t_i)$ (premium leg [5])

(15.3) $F(t) = E\big(1(\tau < t)\big)$

(15.4) $l(t) = (1-R) \cdot 1(\tau < t)$

[5] Omitting the accrual.

where $\tau$ is default time, $R$ the recovery [6] and $1(\cdot)$ the indicator function [7], we have

$$\begin{aligned} V = E(DL) &= \int_t^T (1-R)B(t,s)dF(s) \\ &= (1-R)B(t,T)F(T) + \int_t^T (1-R)f(t,s)B(t,s)F(s)ds \\ &= B(t,T)E\big(l(T)\big) + \int_t^T E\big(l(s)\big)f(t,s)B(t,s)ds \end{aligned} \tag{16}$$

and

$$W = E(PL) = \sum_{i=1}^{n} \omega \cdot p \cdot B(t,t_i)\cdot\big(1-F(t_i)\big) = \sum_{i=1}^{n} \omega \cdot p \cdot B(t,t_i)\cdot\left(1-\frac{E\big(l(t_i)\big)}{1-R}\right). \tag{17}$$

With the notion that $E\big(l(s)\big) = \int L \cdot P(L,s \mid l,t)dL$, we can differentiate $V$ $(W)$ with respect to $t$ and apply (5) [8], which gives

$$\frac{\partial}{\partial t}V(l,t) = (r+h)V(l,t) - h\int_0^{1-l} V(l+\xi,t)w(\xi \mid l,t)d\xi - rl \tag{18}$$

and

$$\frac{\partial}{\partial t}W(l,t) = (r+h)W(l,t) - h\int_0^{1-l} W(l+\xi,t)w(\xi \mid l,t)d\xi\,. \tag{19}$$

Boundary and terminal (jump) conditions are:

$$\begin{aligned} &V(L,T) = L && (l \le L \le 1) \\ &V(1-R,s) = 1-R && (t \le s \le T) \\ &W(L,T) = \omega p(1-\frac{L}{1-R}) && \\ &W(1-R,s) = 0 && \\ &W(L,t_j-) = W(L,t_j+) + \omega p(1-\frac{L}{1-R}) && (j = 1,2,...,n-1). \end{aligned} \tag{20}$$

Naturally, expected value of CDS to a buyer, $U := V - W$, satisfies

$$\frac{\partial}{\partial t}U(l,t) = (r+h)U(l,t) - h\int_0^{1-l} U(l+\xi,t)w(\xi \mid l,t)d\xi - rl \tag{21}$$

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[6] Assume $R$ is fixed for now.
[7] Other variables/functions are similarly defined as before.
[8] After replacing $h$ and $w$ in (5) with jump intensity and jump distribution for the individual obligor.

subject to

$$U(L,T) = L(1+\frac{\omega p}{1-R}) - \omega p \tag{22}$$

$$U(1-R,s) = 1-R \tag{23}$$

$$U(L,t_j-) = U(L,t_j+) - \omega p(1-\frac{L}{1-R}) \qquad (j=1,2,...,n-1). \tag{24}$$

Jump distribution $w$ in (18), (19) and (21) does not have to be Dirac delta, as recovery $R$ can be stochastic. Given a distribution $\psi(R)$ for $R$ independent of $\tau$, (16) and (17) become

$$\begin{aligned} V = E(DL) &= \int_0^1 \int_t^T (1-R)B(t,s)dF(s)\cdot\psi(R)dR \\ &= B(t,T)\int_0^1 (1-R)F(T)\psi(R)dR + \int_t^T \int_0^1 (1-R)F(s)\psi(R)dR\cdot f(t,s)B(t,s)ds \\ &= B(t,T)E\big(l(T)\big) + \int_t^T E\big(l(s)\big)f(t,s)B(t,s)ds \end{aligned} \tag{25}$$

and

$$W = E(PL) = \sum_{i=1}^{n} \omega\cdot p\cdot B(t,t_i)\cdot\left(1-\frac{E\big(l(t_i)\big)}{\int_0^1 (1-R)\psi(R)dR}\right), \tag{26}$$

which still solve (18) and (19), respectively. Note that upon default, $U(l,t)=l \ \ \forall l \in (0,1]$ and $\forall t$ [9]; (21) is hence essentially the ODE

$$\frac{\partial}{\partial t}U(0,t) = (r+h)U(0,t) - h\int_0^{1-l} \xi\cdot w(\xi\,|\,0,t)d\xi \tag{27}$$

with terminal restriction $U(0,T) = -\omega p$ and jump requirement $U(0,t_j-) = U(0,t_j+) - \omega p$ ($j=1,2,...,n-1$). [10] This shows $w$ has the form

$$w(\xi\,|\,l,t) = \big(1-H(l)\big)w(\xi\,|\,0,t) + H(l)\delta(\xi) \tag{28}$$

where $H(\cdot)$ is the Heaviside function.

## 4. Risk-neutral Pricing and Conservation of Probability Mass

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[9] i.e., protection leg receives default loss and premium leg vanishes.
[10] By (27), CDS spread is related to expected recovery only, not its higher moments.

Rewriting (9) to

$$(29)\qquad \underbrace{\frac{\partial}{\partial t}U(l,t)}_{1}+\underbrace{h\int_0^{1-l}\left(U(l+\xi,t)-U(l,t)\right)w(\xi\mid l,t)d\xi}_{2}+\underbrace{r\left[(l-\alpha)^{+}-(l-\beta)^{+}\right]}_{3}=rU(l,t)$$

reveals the familiar risk-neutral pricing principle. Term 1 is time value holding everything else the same, term 2 represents expected p&l due to jump and term 3 is interest earned from money-in-hand (which was paid out instantaneously when jump occurs); the sum of these increments grows at riskfree rate to exclude arbitrage. Equation (21) can be interpreted in the same way.

Expectedly, a credit derivative which generates no instantaneous cash flow contingent on default(s) should be governed by equation

$$(30)\qquad \frac{\partial}{\partial t}U(l,t)+h\int_0^{1-l}\left(U(l+\xi,t)-U(l,t)\right)w(\xi\mid l,t)d\xi=rU(l,t)\,.$$

This may apply to a CDO option which gives the holder the right to enter a CDO (swap) at the cost of $K$ bps for example. A typical option could be priced if we discretize the time dimension in the numerical analysis to allow for dynamic programming.

Denote the $h$ and $w$ in (9) as $h_\Omega$ and $w_\Omega$, and those in (21) as $h_i$ and $w_i$ for obligor $i$. Clearly we expect certain constraint(s) between the aggregate functions $h_\Omega$ and $w_\Omega$ and the individual $h_i$ and $w_i$ ($i\in\Omega$). To conserve probability mass, it should be true that

$$(31)\qquad h_\Omega w_\Omega(\xi\mid l=\sum_i l_i,t)=\sum_i h_i w_i(\xi\mid l_i,t)\,.$$

This is because the probability mass at point $(t,\xi)$, $(h_\Omega dt)(w_\Omega d\xi)$, i.e., the probability that the underlying process has jumped at time $t$ *and* jumped by a size of $\xi$, has to be contributed by the individual obligors, each giving $(h_i dt)(w_i d\xi)$ $\forall i$.[11] (31) implies

$$(32)\qquad \begin{aligned} h_\Omega &= \sum_i h_i\int_0^{1-\sum_j l_j} w_i(\xi\mid l_i,t)d\xi=\sum_i h_i\int_0^{\sum_j \kappa_j-l_j} w_i(\xi\mid l_i,t)d\xi \\ &= \sum_i h_i\int_0^{\kappa_i-l_i} w_i(\xi\mid l_i,t)d\xi=\sum_i h_i \end{aligned}$$

where $\kappa_i$ $\forall i$ is weight of the $i^{th}$ obligor in total notional.

[11] If obligor $i$ has defaulted, it contributes probability mass only to the 'origin' $l$, which won't affect term 2 of equation (28) or therefore the solution. (31) implicitly excludes simultaneous defaults.

Since $h$ and $w$ appear in the joint form of a product in pricing equations, the identification (or specification) of both functions for each CDS should uniquely determine the price of CDO.

## 5. NtD BDS

Building upon the above results, we propose the following set of ODEs for valuation of a $n^{th}$-to-default basket default swap, $U^n(l,t)$.

(33)
$$\begin{aligned}
&\frac{\partial}{\partial t}U^1(l,t)+h\int_0^{1-l}\left(l+\xi-U^1(l,t)\right)w(\xi\,|\,l,t)d\xi+rl=rU^1(l,t)\\
&\frac{\partial}{\partial t}U^2(l,t)+h\int_0^{1-l}\left(U^1(l+\xi,t)-U^2(l,t)\right)w(\xi\,|\,l,t)d\xi=rU^2(l,t)\\
&\ldots\\
&\frac{\partial}{\partial t}U^n(l,t)+h\int_0^{1-l}\left(U^{n-1}(l+\xi,t)-U^n(l,t)\right)w(\xi\,|\,l,t)d\xi=rU^n(l,t)\,.
\end{aligned}$$

(33) uses the fact that a $n^{th}$-tD BDS reduces to a $(n-1)^{th}$ one with an additional default. Functions $h$ and $w$ follow those used in pricing the CDO. For a $m$-out-of-$n$ BDS, (33) may be modified to

(34)
$$\begin{aligned}
&\frac{\partial}{\partial t}U^1(l,t)+h\int_0^{1-l}\left(l+\xi-U^1(l,t)\right)w(\xi\,|\,l,t)d\xi+rl=rU^1(l,t)\\
&\frac{\partial}{\partial t}U^2(l,t)+h\int_0^{1-l}\left(U^1(l+\xi,t)-U^2(l,t)\right)w(\xi\,|\,l,t)d\xi+rl=rU^2(l,t)\\
&\ldots\\
&\frac{\partial}{\partial t}U^m(l,t)+h\int_0^{1-l}\left(U^{m-1}(l+\xi,t)-U^m(l,t)\right)w(\xi\,|\,l,t)d\xi+rl=rU^m(l,t)\\
&\frac{\partial}{\partial t}U^{m+1}(l,t)+h\int_0^{1-l}\left(U^m(l+\xi,t)-U^{m+1}(l,t)\right)w(\xi\,|\,l,t)d\xi=rU^{m+1}(l,t)\\
&\ldots\\
&\frac{\partial}{\partial t}U^n(l,t)+h\int_0^{1-l}\left(U^{n-1}(l+\xi,t)-U^n(l,t)\right)w(\xi\,|\,l,t)d\xi=rU^n(l,t)\,.
\end{aligned}$$

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