Λύσεις των Ασκήσεων Φυλλαδίου 2Η

Ασκ. 1. Έχουμε: $f_x = f_u u_x + f_v v_x \Rightarrow$

$$\begin{aligned} f_{x^2} &= (f_{u^2} u_x + f_{uv} v_x) u_x + f_u u_{x^2} + (f_{vu} u_x + f_{v^2} v_x) v_x + f_v v_{x^2} \\ &= f_{u^2} u_x^2 + 2 f_{uv} u_x v_x + f_{v^2} v_x^2 + f_u u_{x^2} + f_v v_{x^2} \end{aligned} \tag{1}$$

Ομοίως (λόγω συμμετρίας)

$$f_{y^2} = f_{u^2} u_y^2 + 2 f_{uv} u_y v_y + f_{v^2} v_y^2 + f_u u_{y^2} + f_v v_{y^2} \tag{2}$$

οπότε χρησιμοποιώντας και τις σχέσεις: $u_x = v_y$, $u_y = -v_x$ έχουμε:

$$\begin{aligned} (1) + (2) \Rightarrow f_{x^2} + f_{y^2} =& f_{u^2}(u_x^2 + u_y^2) + 2 f_{uv}(u_x v_x + u_y v_y) + f_{v^2}(u_x^2 + v_x^2) \\ &+ f_u (u_{x^2} + u_{y^2}) + f_v (v_{x^2} + v_{y^2}) \\ =& (f_{u^2} + f_{v^2})(u_x^2 + u_y^2) + f_u (v_{yx} - v_{xy}) + f_v(-u_{xy} + u_{yx}) \\ =& (f_{u^2} + f_{v^2})(u_x u_x + v_x v_x) = (f_{u^2} + f_{v^2})(u_x v_y - u_y v_x). \end{aligned}$$

Ασκ. 2. Από $u_x = 1/t, v_x = -\frac{x}{2t}, w_x = 0, u_y = w_y = 0, v_y = 1$ και $g_x = t^{-1/2}(f_u u_x + f_v v_x + f_w w_x) = \cdots = t^{-3/2}(f_u - \frac{x}{2} f_v)$ έχουμε:

$$\begin{aligned} \bullet g_{x^2} =& t^{-3/2}(f_{u^2} u_x + f_{uv} v_x + f_{uw} w_x) - 1/2 t^{-3/2} f_v \\ &- \frac{x}{2} t^{-3/2}(f_{vu} u_x + f_{v^2} v_x + f_{vw} w_x) = \\ =& \cdots = t^{-5/2} f_{u^2} - x t^{-5/2} f_{uv} - \frac{1}{2} t^{-3/2} f_v + \frac{1}{4} x^2 t^{-5/2} f_{v^2}. \end{aligned} \tag{3}$$

Ομοίως $g_y = t^{-1/2}(f_u u_y + f_v v_y + f_w w_y) = t^{-1/2} f_v$ και

$$\begin{aligned} \bullet g_{yt} =& \frac{1}{2} t^{-3/2} f_v + t^{-1/2}(f_{vu} u_t + f_{v^2} v_t + f_{vw} w_t) = \cdots \\ =& -\frac{1}{2} t^{-3/2} f_v - x t^{-5/2} f_{vu} + \frac{1}{4} x^2 t^{-5/2} f_{v^2} + t^{-5/2} f_{vw}. \end{aligned} \tag{4}$$

(3)-(4) $\Rightarrow \quad g_{x^2} - g_{yt} = t^{-5/2}(f_{u^2} - f_{vw}) = 0$, αφού $f_{u^2} - f_{vw} = 0$ από υπόθεση.

Ασκ. 3. Θέτουμε $u = x + g(x-y), v = x - y$. Τότε: $\Phi(x,y) = h(u,x)$ και $\Phi_x = h_u u_x + h_x = h_u(1 + g') + h_x, \Phi_y = h_u u_y = h_u(-g')$ οπότε $\Phi_x + \Phi_y = h_u + h_u g' + h_x - h_u g' = h_u + h_x = 0$, αφού $h(u,x) \Rightarrow h_u + h_x = 0$ από υπόθεση.

Ασκ. 4. Θέτουμε $u = tx, v = ty, w = tz$. Παραγωγίζουμε την $F_i(u,v,w) = tF_i(x,y,z)$ ως προς t: $\frac{\partial F_i}{\partial t} = F_i$ (**1**). Όμως
$\frac{\partial F_i}{\partial t} = \frac{\partial F_i}{\partial u} u_t + \frac{\partial F_i}{\partial v} v_t + \frac{\partial F_i}{\partial w} w_x = x \frac{\partial F_i}{\partial u} + y \frac{\partial F_i}{\partial v} + z \frac{\partial F_i}{\partial w} \overset{(1)}{=} F_i \overset{t=1}{\Rightarrow} x \frac{\partial F_i}{\partial x} + y \frac{\partial F_i}{\partial y} + z \frac{\partial F_i}{\partial z} = F_i,\ i = 1,2,3$
(βλέπε σχετικά και Παράδ. 5.6/σελ. 142 βιβλίου).
Από $\nabla \times \mathbf{F} = \mathbf{0} \Rightarrow \{\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}\}$ (**2**), οπότε:

$$\begin{aligned} \nabla f =& \frac{1}{2}\left[\nabla(xF_1 + yF_2 + zF_3)\right] \\ =& \frac{1}{2}[(f_1 + x\frac{\partial F_1}{\partial x} + y\frac{\partial F_2}{\partial x} + z\frac{\partial F_3}{\partial x})\mathbf{i} + (F_2 + x\frac{\partial F_1}{\partial y} + y\frac{\partial F_2}{\partial y} + z\frac{\partial F_3}{\partial y})\mathbf{j} + (F_3 + x\frac{\partial F_1}{\partial z} + y\frac{\partial F_2}{\partial z} + z\frac{\partial F_3}{\partial z})\mathbf{k}] \\ \overset{(2)}{=}& \frac{1}{2}[(F_1 + \underbrace{x\frac{\partial F_1}{\partial x} + y\frac{\partial F_1}{\partial y} + z\frac{\partial F_1}{\partial z}}_{F_1})\mathbf{i} + (F_2 + \underbrace{x\frac{\partial F_2}{\partial x} + y\frac{\partial F_2}{\partial y} + z\frac{\partial F_2}{\partial z}}_{F_2})\mathbf{j} + (F_3 + \underbrace{x\frac{\partial F_3}{\partial x} + y\frac{\partial F_3}{\partial y} + z\frac{\partial F_3}{\partial z}}_{F_3})\mathbf{k}] \\ =& \frac{1}{2}(2F_1\mathbf{i} + 2F_2\mathbf{j} + 2F_3\mathbf{k}) = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} = \mathbf{F}. \end{aligned}$$

Ασκ. 5.(ι) Είναι $\dfrac{1}{R}=\dfrac{R_1R_2+R_2R_3+R_3R_1}{R_1R_2R_3}\Rightarrow R=\dfrac{R_1R_2R_3}{R_1R_2+R_2R_3+R_3R_1}\Rightarrow$

$\dfrac{\partial R}{\partial R_1}=\dfrac{R_2R_3(R_1R_2+R_2R_3+R_3R_1)-R_1R_2R_3(R_2+R_3)}{(R_1R_2+R_2R_3+R_3R_1)^2}=\dfrac{R_2^2R_3^2}{(R_1R_2+R_2R_3+R_3R_1)^2}$, οπότε (λόγω συμμετρίας)

$$\frac{\partial R}{\partial R_1}+\frac{\partial R}{\partial R_2}+\frac{\partial R}{\partial R_3}=\frac{R_2^2R_3^2+R_1^2R_3^2+R_1^2R_2^2}{(R_1R_2+R_2R_3+R_3R_1)^2}=\frac{\frac{1}{R_1^2}+\frac{1}{R_2^2}+\frac{1}{R_3^2}}{(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3})^2}$$

(ιι) Έστω P $=(100,100,200)$, τότε $\triangle R=R(100+1,100+1,200-2)-R(100,100,200)\cong \dfrac{\partial R}{\partial R_1}(\mathrm{P})\triangle R_1+\dfrac{\partial(\mathrm{R})}{\partial R_2}(\mathrm{P})\triangle R_2+\dfrac{\partial(\mathrm{R})}{\partial R_3}(\mathrm{P})\triangle R_3=\dfrac{100^2\cdot 200^2}{(20000+20000+100000)^2}\cdot 1+\dfrac{100^2\cdot 200^2}{50.000^2}\cdot 1+\dfrac{100^2\cdot 100^2}{50.000^2}\cdot(-2)=\dfrac{4}{25}+\dfrac{4}{25}-\dfrac{2}{25}=\dfrac{6}{25}\frac{\Omega}{sec}$.

Ασκ. 6. Επειδή f διαφορίσιμη θα είναι $D_{\mathbf{u}}f(P)=\nabla f(P)\cdot\mathbf{u}$, $\mathbf{u}=\dfrac{\mathbf{n}}{|\mathbf{n}|}$, όπου $\mathbf{n}$ το κάθετο διάνυσμα στην επιφάνεια στο P. Το σημείο P επαληθεύει την εξίσωση $x^2+y^2+z^2=3$. Άρα $\mathbf{n}=\nabla\varphi(\mathrm{P})$, όπου $\varphi(x,y,z)=x^2+y^2+z^2-3$. Είναι: $\dfrac{\partial\varphi}{\partial x}=2x$, $\dfrac{\partial\varphi}{\partial y}=2y$, $\dfrac{\partial\varphi}{\partial z}=2z$, οπότε $\dfrac{\partial\varphi}{\partial x}(\mathrm{P})=0$, $\dfrac{\partial\varphi}{\partial y}(\mathrm{P})=\sqrt{3}$, $\dfrac{\partial\varphi}{\partial z}(\mathrm{P})=3$, $\mathbf{n}=\sqrt{3}\mathbf{j}+3\mathbf{k}$, $\mathbf{u}=\dfrac{\mathbf{n}}{|\mathbf{n}|}=\frac{1}{2}\mathbf{j}+\frac{\sqrt{3}}{2}\mathbf{k}$. Επί πλέον $\dfrac{\partial f}{\partial x}=2x+2z$, $\dfrac{\partial f}{\partial y}=2y$, $\dfrac{\partial f}{\partial z}=2x$, $\Rightarrow\ \nabla f=(2x+2z)\mathbf{i}+2y\mathbf{j}+2x\mathbf{k}\Rightarrow\ \nabla f(\mathrm{P})=3\mathbf{i}+\sqrt{3}\mathbf{j}$. Επομένως

$$D_{\mathbf{u}}f(\mathrm{P})=\nabla f(P)\cdot\mathbf{u}=0\cdot 3+\tfrac{1}{2}\cdot\sqrt{3}+\tfrac{\sqrt{3}}{2}\cdot 0=\tfrac{\sqrt{3}}{2}.$$