12.2 Series

Infinite series is the mathematic terminology for the concept of the addition of infinite term, which is denoted by $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$

1. Definition Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$, let s_n denote its $\boldsymbol{n}^{\text{th}}$ **partial sum**

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n .$$

If the **sequence** of s_n is **convergent** and $\lim_{n\to\infty} s_n = s$ **exists as a real number**, then the series is called **convergent** and we write

$$a_1 + a_2 + \ldots + a_n + \ldots = s \qquad \text{or} \qquad \sum_{i=1}^{\infty} a_i = s .$$

The number s is called the **sum of the series**. Otherwise, the series is called **divergent**.

Precisely speaking, $\sum_{i=1}^{\infty} a_i = s$ means that by **adding sufficiently as many terms of the series we can get closer as we like to the number** s. Notice that $\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$

Example 1 Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum

SOLUTION The partial sum is $s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)}$. Notice that $\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$, we can rewrite the partial sum as

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

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And so $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$.

Therefore the series is convergent and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

2. The geometric series

The **geometric series** $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots$

- If $|r| < 1$, the geometric series is convergent and its sum is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$.
- If $|r| \geq 1$, the series is divergent.

Example 2 $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $a = 1/2$, $r = 1/2$ so it is convergent and its sum is $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1/2}{1 - 1/2} = 1$.

3. The *p*-series

The *p*- series $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$ is convergent if $p > 1$ and divergent if $p \le 1$.

Note: 1-series (p = 1) is called **harmonic series**.

4. Theorem If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

Bonus Assignment: Find a series $\sum_{n=1}^{\infty} a_n$ such that it is divergent, even though $\lim_{n\to\infty} a_n = 0$? Is it contradicted to the theorem 4?

5. The Test for Divergence

If $\lim_{n\to\infty} a_n = 0$ does not exist or if $\lim_{n\to\infty} a_n \ne 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent

Example 3 $\sum_{n=1}^{\infty} \sin n$ is divergent since $\lim_{n\to\infty} \sin n$ does not exist. So is $\sum_{n=1}^{\infty} \frac{n^2+5}{3n^2+4n-1}$ since $\lim_{n\to\infty} \frac{n^2+5}{3n^2+4n-1} = \lim_{n\to\infty} \frac{1 + 5/n^2}{3 + 4/n - 1/n^2} = \frac{1}{3} \ne 0$

6. Theorem If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum (a_n + b_n)$, and $\sum (a_n - b_n)$, and

$$\text{(i)} \quad \sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$

$$\text{(ii)} \quad \sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

Example 4 $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right) \overset{Theorem\ 4}{=} 3\sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n} \overset{ex.\ 1\ \&\ 2}{=} 3 \cdot 1 + 1 = 4$