ON THE FLOW OF A SECOND ORDER FLUID IN AN ORTHOGONAL RHEOMETER

BY

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Abstract. In this paper we study the flow of a special second order fluid (which well approximates the first normal stress difference for a solution of polyisobuthylene) in an orthogonal rheometer. We prove the existence and uniqueness of the classical solutions for the set of the approximate flow problems in an asymptotic development in respect to the Weissemberg number.

Key words: second order fluid, first normal stress difference, orthogonal rheometer, boundary value problem, traction force.

1. Introduction

We consider a second order fluid which has a constitutive law described by

(1) $$T = -pI + mA_1 + a_1 A_2 + a_2 A_1^2,$$

where p is the hydrostatic pressure and m, a_1 and a_2 are the constitutive moduli which can depend on J

(2) $$J = \{trA_1, trA_1^2, trA_1^3, trA_2, trA_2^2, trA_2^3, tr(A_1A_2 + A_2A_1)\}$$

In (1) and (2), A_1, A_2 are the well-known Rivlin-Ericksen tensors.

The constitutive law (1) has been analysed from thermodynamic point of view and in connection with the asymptotic stability of the rest state questions in [1]-[3]. For particular cases, some flow problems have been discussed in [4]-[6].

In [2], Dunn proves that a sufficient condition in order to obtain the asymptotic stability of the rest state is $a_1 \equiv a_1(trA_1^2)$. This conclusion together with the experimental curves giving the first and second normal stress differences for a polyisobuthylene (see Larson [7]), have led to the following formula for $\boldsymbol{a}_1$ (in a first approximation):

(3) $$\boldsymbol{a}_1 = \frac{btrA_1^{\;2}}{1 + a(trA_1^{\;2})^2}.$$

As a consequence, we write the constitutive equation as

(4) $$T = pI + T_E(A_1, A_2),$$

$$T_E(A_1, A_2) = mA_1 + \frac{btrA_1^{\;2}}{1 + a(trA_1^{\;2})^2} A_2 + \boldsymbol{a}_2 A_1^{\;2}$$

where the constitutive moduli m and $\boldsymbol{a}_2$ are supposed to be constant.

For such a fluid we solve the flow problem in an orthogonal rheometer (which has been treated for a BKZ-fluid in [8]), employing a kinematical admissible velocity field introduced in [9].

2. The Flow Problem

An orthogonal rheometer is sketched in Fig.1. In the figure, d is the distance between the two parallel discs and r is the distance between the two rotational axes. The two discs which, for mathematical reasons, are supposed to be of infinite radius, have an uniform rotation with the constant angular velocity Ω. The velocity field employed is (see [9])

(5) $v_1 = -\Omega[y - g(z)], v_2 = \Omega[x - f(z)], v_3 = 0$,

where v_1, v_2, v_3 are the assumed components in x, y and z directions, respectively. It is easy to prove that the corresponding motion is with constant stretch history and then, the Cauchy stress tensor T is expressed like in $(4)_1$. Assuming that the specific body force is conservative, the equations of motion are

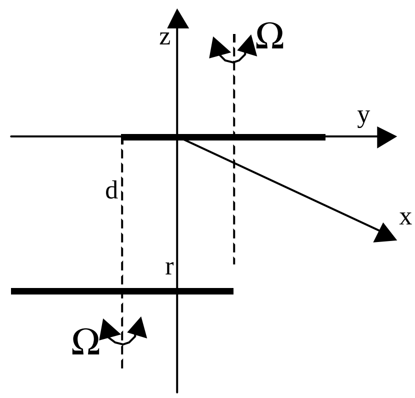

Fig. 1

(6) $$\frac{dT_{E_{13}}}{dz} = r\Omega^2 g(z); \quad \frac{dT_{E_{23}}}{dz} = r\Omega^2 f(z).$$

These equations have to be solved with standard no-slip boundary conditions. We denote with

$$A(f, g) = 4a\Omega^4(g'^2 + f'^2)^2.$$

We introduce (4) and (5) into equations (6), we employ also the above mentioned notation and after some long but straightforward calculi we arrive to the following system

$$(7)\quad \begin{cases} m\Omega g'' - \dfrac{b[1-A(f,g)]}{[1+A(f,g)]^2}4\Omega^4(g'g''+f'f'')f' - \\ -\dfrac{2b\Omega^2(g'^2+f'^2)}{A(f,g)}\Omega^2 f'' = r\Omega^2 f, \\ -m\Omega f'' - \dfrac{b[1-A(f,g)]}{[1+A(f,g)]^2}4\Omega^4(g'g''+f'f'')g' - \\ -\dfrac{2b\Omega^2(g'^2+f'^2)}{A(f,g)}\Omega^2 g'' = r\Omega^2 g, \end{cases}$$

and the corresponding boundary value problem is

$$(8)\quad f(0) = f(d) = 0;\quad g(0) = -r/2;\quad g(d) = r/2.$$

3. The Asymptotic Analysis of the Flow Problem

We introduce the non-dimensional variables and functions by

$$(9)\quad x = r\overline{x},\ y = r\overline{y},\ z = d\overline{z};\ f = r\overline{f},\ g = r\overline{g}.$$

Use of (9) in the system (7) lead us to the non-dimensional problem

$$(10)\quad \begin{cases} g'' = \mathrm{Re}\, f + 2Waf''\dfrac{g'^2+f'^2}{1+A(f,g)} + 4Waf'\dfrac{(g'g''+f'f'')[1-A(f,g)]}{[1+A(f,g)]^2}, \\ f'' = -\mathrm{Re}\, g - 2Wag''\dfrac{g'^2+f'^2}{1+A(f,g)} - 4Wag'\dfrac{(g'g''+f'f'')[1-A(f,g)]}{[1+A(f,g)]^2}, \\ \\ f(0) = f(1) = 0,\ g(0) = -1/2,\ g(1) = 1/2, \end{cases}$$

where we have dropped the overlines. In (10) $\mathrm{Re} = \dfrac{r\Omega d^2}{m}$, $Wa = \dfrac{b\Omega^3}{m}(\dfrac{r}{d})^2$ are the Reynolds number and the Weissemberg number respectively.

We mention that Wa is classically defined as: $Wa \equiv l_1 U_0 / L_0$, where l_1 is a microscopic relaxation time and L_0 / U_0 can be interpreted as a macroscopic relaxation time. We suppose, in what follows, that $Wa << 1$ which means that the considered melt has short macromolecular chains only (see for instance [7], [10]). Consequently, we look to the solution of the problem (10) in an asymptotic development in respect to Wa.

$$(11)\quad f = \sum_{n=0}^{\infty} Wa^n f_n\ ;\ g = \sum_{n=0}^{\infty} Wa^n g_n\ .$$

We introduce (11) in (10) and after some calculi we arrive to the following set of approximate problems

(12) $$\begin{cases} g_0''=\mathrm{Re}\, f_0; \quad f_0''=-\mathrm{Re}\, g_0; \\ f_0(0)=f_0(1)=0; \quad g_0(0)=-1/2; \quad g_0(1)=1/2, \end{cases}$$

(13) $$\begin{cases} g_1''=\mathrm{Re}\, f_1+\dfrac{2f_0''(g_0'^2+f_0'^2)}{1+A(f_0,g_0)}+\dfrac{4(g_0'g_0''+f_0'f_0'')f_0'[1-A(f_0,g_0)]}{[1+A(f_0,g_0)]^2}, \\ f_1''=-\mathrm{Re}\, g_1-\dfrac{2g''(g_0'^2+f_0'^2)}{1+A(f_0,g_0)}+\dfrac{4(g_0'g_0''+f_0'f_0'')g_0'[1-A(f_0,g_0)]}{[1+A(f_0,g_0)]^2}, \\ \\ f_0(0)=f_0(1)=g_0(0)=g_0(1)=0, \end{cases}$$

where $A(f_0,g_0)$ is $4a\Omega^4(r/d)^4(g_0'^2+f_0'^2)^2$.

We simply remark that the general form of the nth approximate problem is given in compact form by

(14) $$-Y_n''+AY_n=-F_n; \quad Y_n(0)=Y_n(1)=0, \quad n\geq 1,$$

where $A=\begin{pmatrix} 0 & \mathrm{Re} \\ -\mathrm{Re} & 0 \end{pmatrix}$, $Y_n=\begin{pmatrix} g_n \\ f_n \end{pmatrix}$ and F_n is a known function depending on the first n-1 approximation of f and g.

We are ready now to state the following theorems:

Theorem 1. The problem (12) has a unique solution given by

(15) $$\begin{cases} f(z)=e^q(-c_1\sin q+c_2\cos q)+e^{-q}(c_3\sin q+c_4\cos q), \\ g(z)=e^q(c_1\cos q+c_2\sin q)+e^{-q}(c_3\cos q-c_4\sin q), \end{cases}$$

where $q=\sqrt{\mathrm{Re}/2}z$ and the constants $c_i, i=\overline{1,4}$ are to be determined from the conditions $(12)_2$.

The proof is immediate.

Theorem 2. The general problem (14) has a unique classical solution.

For the proof we denote by L the corresponding differential operator given by $LY\equiv-\dfrac{d}{dz}\mathbf{(}Y'\mathbf{)}+AY$. We simply remark that L is self adjoint and

(16) $$(LY,Y)\geq p^2\|Y\|^2$$

where we have employed the Friedrichs inequality. That means, by a standard reasoning, that there is a unique solution of the problem (14) and this coincides with the classical one: $Y\in C^2((0,1))\cap C^1[0,1]$.

Remark. We easily evaluate the L^2 norms of Y_n and Y_n' employing Friedrichs inequality and $(14)_1$ and we arrive to

(17) $$\|Y_n\|^2\leq\frac{1}{2p^2-1}\|F_n\|^2; \quad \|Y_n'\|^2\leq\frac{p^2}{2p^2-1}\|F_n\|^2.$$

Final remarks:

a. In order to compute the traction on the inferior disc, for instance, $T\vec{n}\cdot\vec{t} = (t_x{}^2 + t_y{}^2)^{1/2}$ we evaluate x and y components, t_x, t_y of the corresponding force, which in non-dimensional variables give

$$T\vec{n}\cdot\hat{o} = (g_0(0)'^2 + f_0(0)'^2)\{1 + \frac{\frac{4b^2\Omega^6}{m^2}(\frac{r}{d})^4((g_0(0)'^2 + f_0(0)'^2)^2}{[1 + 4a\Omega^4(g_0(0)'^2 + f_0(0)'^2)^2]^2}\},$$

where, $g_0{}'(0) = \sqrt{\mathrm{Re}/2}(2c_1 + 2c_2 + 0.5)$, $f_0{}'(0) = \sqrt{\mathrm{Re}/2}(-2c_1 - 0.5)$.

b. In Fig.2 we give a plot for the first approximation of the nondimensional function g/r, for the Reynolds number Re = 4, where g is given by:

$$g_0(z) = e^{\sqrt{2}z}(-0.0107768\cos(\sqrt{2}z) + 0.122113\sin(\sqrt{2}z)) +$$
$$+ e^{-\sqrt{2}z}(-0.489223\cos(\sqrt{2}z) + 0.122113\sin(\sqrt{2}z)),$$

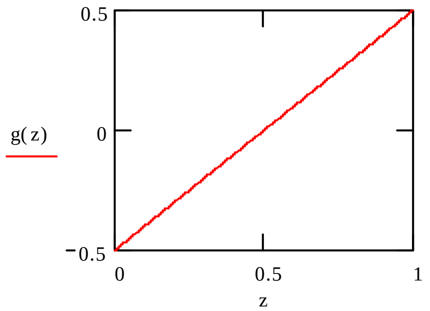

Fig. 2.

Like in [8], we obtain an apparently linear graph for the plotted function (actually the above formulae for g is essentially non linear). This is not surprising, if we have in mind the boundary condition and the fact that the Reynolds number is in the neighbourhood of 1.

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ASUPRA MISCARII UNUI FLUID DE ORDINUL DOI INTR-UN REOMETRU ORTOGONAL

(Rezumat)

In aceasta lucrare studiem miscarea unui fluid particular de ordinul doi (care aproximeaza multumitor prima diferenta a tensiunilor normale pentru o solutie de poliizobutilena) intr-un reometru ortogonal. Demonstram existenta si unicitatea solutiei clasice pentru aproximarile problemei de miscare cu conditii la frontiera intr-o dezvoltare asimptotica in raport cu numarul lui Weissemberg.