THE BOUNDED ROTATIONAL AND TRANSLATION MOTION OF A BODY (A MATERIAL POINT AS PHYSICAL POINT) ON A CIRCLE©

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This article is dedicated to the analysis of a forced motion of a rotor having one degree of freedom. Both rotational and translational motions are considered. The rotor motion is represented as a material point rotation around the fixed point assuming that a material point posses both a mass and a moment of inertia. Rotation and translation of the material point are treated as motion of the system "material point – fixed point" in an inertial reference system. An attention is drawn to the fact that in order to enable rotation and translation of a material point with a constant velocity, it is necessary to apply an additional torque for the purpose of overcoming of the inertial moment being created by a moment of inertia of the material point, which is produced due to a displacement of the material point with respect to the fixed point. The special attention is paid to the case of a simultaneous rotation and linear displacement of the material point by three virtual displacements with respect to the fixed point.

The first law of Newton says that if there is no effect of any other body to a body being under consideration, this body will maintain the quiescent state or keep a uniform rectilinear inertial motion. Such a motion is referred to as a free motion. The free-moving body consists of the material points being connected to each other creating the single whole without interacting as independent bodies do. Such an assumption is used in the classical mechanics. Therefore, one can say that the material points of the free-moving body are in the weightlessness state. The first law of Newton says nothing about free rotation of a body possessing the principal moment of inertia about the principal axis of inertia, although such motion exists. The reason is as follows: Newton has considered the planetary motion as a bounded motion of the material point possessing only the mass, while a size and a moment of inertia could be neglected. In fact, Newton considered the bounded translation of the material point along a curvilinear trajectory. Such a conclusion is evidently proven through the examples of the material point motion.

One of the examples provided in the physics is a bounded motion of the body under effect of parallel gravitational forces of the Earth. A thrown body performs a complex motion consisting of inertial translation with a constant linear velocity in the defined direction, and of the uniformly accelerated free fall. Because of these two motions, the body moves along a curvilinear trajectory. Gravitational forces can be referred to as mass forces due to the fact that they are applied in a continuous manner to each material point of the body. In this case, the material points as independent bodies are not interacting between themselves and, therefore, they are in the weightlessness state. The more complex example of the bounded motion is the planetary motion around the Sun. The motion of a planet as a material point is performed in the field of the gravitational forces converging to the center of the Sun. Due to such a feature of the field of the gravitational forces, the free motion of a planet is performed while changing the direction of the velocity. The gravitational field of the Earth affects in the same manner the moving satellites. It is considered that the motion of planets and satellites is performed along closed curvilinear trajectories. Despite a changed direction of the planet velocity, the principle of body's material point weightlessness is conserved also for that type of a motion.

In case of bounded translation, the body possessing the principal moment of inertia can perform a uniform rotation about principal axis of rotation as in the case when no forces are affecting the body. Such a fact demonstrates that the field of gravitational forces does not affect the body's rotation, because the gravitational forces are the mass forces not producing a torque. The free motion of the body possessing the principal moment of inertia is not left without due attention. The physics consider a rotation of the Earth, and it is shown that in the course of rotation the field of centrifugal accelerations is produced. It affects the body's material points and creates diverging centrifugal forces affecting each material point as gravitational forces do. There is another difference in addition to the path of the force application: the value of the centrifugal force depends on a radius of the body's rotation about the principal axis of the inertia. The body would be destroyed into separate material points if the forces of another nature will not exist. These forces prevent from such destruction. In case the material points being enabled to move inside the body are presented, then such material points are interacting between themselves and their interaction results in development of the ponderability state.

In the whole, the uniform rotation of the body as a motion has a principal difference with respect to an inertial rectilinear uniform motion of the body.

The bounded motion of a body along a curvilinear trajectory under the action of both the active force being generated by another body and the gravitational forces being generated by the Earth represents a new qualitative class of the motion. Under the action of a force, the body performs an accelerated motion in the direction of the force application. In this case the inertial properties of the body opposing its accelerated motion are manifesting themselves. The mass of the body being included into the definition of the active force accounts for the inertial properties of the body in an inertial reference system. Therefore, the inertia forces of the body are taken into consideration only in a non-inertial reference system. The distinctive feature of such translational motion of the body is an interaction of the material points of the body between themselves. Therefore the material points of the body are in ponderability state. A more complex motion of the body under action of an active force in case when the path of the force application does not go through the body's center of gravity might occur. In such a case, in addition to the rectilinear motion of the body occurs a complementary accelerated rotation of the body possessing the principal moment of inertia about the principal axis of inertia. Such a motion can be referred to as a bounded rotation under the action of the torque about the center of gravity resulted from the action of an active force. Hence, the bounded rotation of the body is performed about the principal axis of inertia. It is a uniformly accelerated motion, and it could occur only in presence of the torque-producing forces.

However there is also the third type of rotation of the body possessing the moment of inertia. Such type can be referred to as the bounded rotation of the body. Such a type of the body's rotation has been produced in an artificial way as a result of imposing the mechanical constraints onto the body. Such constraints force the body to perform an inertial rotation about previously chosen axis or a point under the action of the torque-producing forces. Due to that fact, a new type of the system has been established. It could be referred to as the system with the mechanical constraints or as the mechanical system. The distinctive feature of the system rotation is the rotation with the same velocity of all its points being situated at the different radial distances around the same fixed axis or point. In this case, any variation of the material points' relative position is finite and results from the deformation of the mechanical constraints. It is obvious that such systems in addition to the principal moment of inertia can possess a complementary one, because a determined axis of rotation may not coincide with the principal axis of inertia. If in the course of rotation a system allows changes of the material points' position as a function of the speed of rotation, then such a system became a system with variable moments of inertia. The common feature of all bodies and systems performing a rotation is the field of the centrifugal accelerations diverging from the axis of rotation. Such accelerations impel each material point to return to the uniform rectilinear motion. It is evident that in some cases a mechanical system can be represented as a material point possessing the mass and the principal moment of inertia, which is connected to a fixed axis in some particular way.

The mechanical system has been occurred as a result of the technological progress and they are the products of engineering activities. Usually the design foresees the coincidence of the body's center of gravity with the rotation axis. It results in the fact that the resultant of all centrifugal forces of the rotor’s material points is equal to zero. In practice, it is not always possible to locate the body's center of gravity on the rotation axis. As a result, the resultant of centrifugal forces emerges. It is applied at the rotor's center of gravity and is directed along the rotation radius. Usually the design approaches foresee superposition of the rotor's principal axis of inertia with the rotation axis. Such a system possesses the principal moment of inertia uniformly distributed around the rotation axis in the same manner as for the body performing a free rotation about the principal axis of inertia. Due to this fact the system is enabled to perform a uniform rotation during unlimited period of time if there are no applied torque-producing active forces. The rotating system having the rotation axis coinciding with the principal axis of inertia performs a uniformly accelerated rotary motion under the action of the torque-producing active force when no resistance forces are present. However, in practice, it is impossible to achieve a full coincidence between the principal axis of inertia of the system and the axis of rotation. The moment of inertia of such a system is greater than the principal moment of inertia. One could say that the system possesses a complementary moment of inertia because of not complete coincidence between the principal axis of inertia and the rotation axis. An important feature of a complementary moment of inertia is the fact that it introduces asymmetry into the system's moment of inertia. Therefore a complementary moment of inertia of the system could be considered as a disturbing factor that counteracts the system rotation. It is obvious that in order to overcome the negative effect of a complementary moment of inertia the torque should be applied to the system.

Exactly because of this type of the rotating system, disagreements in physics and engineering mechanics have been occurred. On the one side, it is considered that the mechanical system "solid body – fixed axis" possesses one degree of freedom, and a forced accelerated rotation of the body about fixed axis (in accordance with the energy conservation law) is performed under the action of forces being able to produce a torque. In case such forces are not present, a mechanical system performs a uniform rotation about a rotation axis if the system possesses only the principal moment of inertia. However, while analyzing a system "material point-fixed axis" it is considered that the material point being included into the system possesses two degrees of freedom, and a rotation of the material point about fixed axis is performed under the action of a centripetal force, which is not able to produce a torque. In this case, it is not considered a fact that such a system possesses a certain moment of inertia, which includes the principal moment of a material point. The law of the kinetic energy conservation leaves a room for a real centrifugal force, while rotation of the material point about fixed axis requires considering a centrifugal force as an imaginary one. The discrepancy between the model of the material point rotation about fixed axis and the law of the kinetic energy conservation can be explained by a simple carryover of the planetary rotation physics provisions to the mechanical system rotation.

Considering above-written comments, we can analyze a forced rotation of the material point C about fixed point under the action of the active force F_a in the horizontal plane (see Fig. 1). This material point possesses the mass m and the moment of inertia I_z.

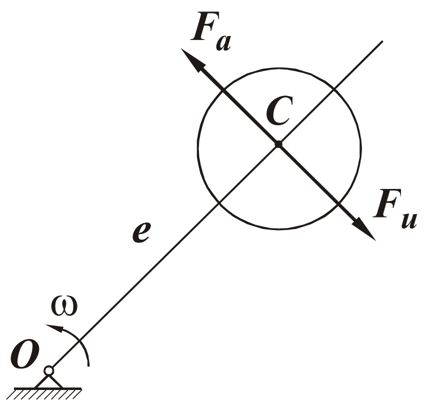

Fig. 1

We assume that the material point C is put off to the distance e and is mechanically connected with the fixed point O. The mechanical connection is ideal. Such a system possesses one degree of freedom. In case of sufficiently big distance between the material point and the fixed point, the motion of the material point with the rotation speed ω can be considered as the one being close to the rectilinear motion. Therefore, at first glance, the law of the uniformly variable curvilinear motion of the material point would look like the law of the uniformly variable rectilinear motion of the material point. The only difference is the following: in case of curvilinear motion of the material point the linear velocity and tangential acceleration are taken into account [2]

$$S = S_0 + V_0^{\tau} t + \frac{a^{\tau} t^2}{2}, \tag{1}$$

where S is the pathway traversed by the material point,
S_0 - is the pathway traversed by the material point by the time of initiation of the uniformly accelerated motion
V_0^{τ} - tangential velocity of the material point prior to initiation of the uniformly accelerated motion,
t - time of the material point motion,
a^{τ} - tangential acceleration of the rotating material point.

The expression (1) can be also represented in the following form:

$$S = S_0 + e\omega t + \frac{e\xi t^2}{2}, \tag{2}$$

where ω - is the angular velocity of the material point,
ξ - angular acceleration of the material point.

However, the material point makes part of the system, and a rotation of the material point is performed about an axis, which is parallel to the principal axis of inertia of the material point at the distance e. In this case, the system's moment of inertia consists of principal moment of inertia of the material point and the complementary moment of inertia of the material point about the fixed point

$$I_0 = I_z + me^2. \tag{3}$$

General moment of inertia of the system is an asymmetrical moment of inertia. Therefore, so called inertial moment is produced while material point is rotated about the fixed point. Such inertial moment counteracts the system's rotation [3]

$$M_u = m\omega^2 e^2. \tag{4}$$

The inertial moment of the system acts around the fixed point opposing a direction of the system's rotation. Then, it can be represented as the force F_u, which is acting onto the material point in the direction opposite to the direction of the force F_a application

$$F_u = m\omega^2 e. \tag{5}$$

In this case, for the time point t of the rotor's spinning, the variable resultant force whose action enables the material point motion can be derived

$$R = F_a - F_u = ma^* - m\omega^2 e = ma_*, \tag{6}$$

where a^* - is the acceleration determining the value of the defined force,
a_* - an instantaneous acceleration at the velocity value ω at the given time point.

It is evident that the value of F_u depends on the system speed of rotation, while the speed of rotation in turn depends on duration of the force F_a application.

In this case a general equation of the rotary motion of the system may be derived in the form shown below basing on the law of the kinetic energy conservation and taking into account the known general equation of the rotary motion [2]:

$$I_0 \frac{d\omega}{dt} = ma * e - m\omega^2 e^2 . \tag{7}$$

The equation shows that under the action of the force having a constant value, the system's rotation is passes with variable acceleration. In order to ensure the system's rotation with a constant value of acceleration, which is characterized with the equation (1), it is necessary to apply a variable force, whose value is varied in accordance with a certain law.

Separating the variables and resolving the differential equation, we can derive a dependence between the duration of the force F_a and the material point rotation speed ω

$$t = \frac{I_0}{2me\sqrt{a*e}} + \ln \frac{\sqrt{a*e} + e\omega}{\sqrt{a*e} - e\omega} . \tag{8}$$

The maximum value of the system rotation speed $\omega_{\max}$ is achieved at $\frac{d\omega}{dt} = 0$.

In this case, from the equation (7) we can derive:

$$ma * -m\omega_{\max}^2 e = 0 , \tag{9}$$

$$\omega_{\max} = \sqrt{\frac{a*}{e}} , \tag{10}$$

$$a* = \omega_{\max}^2 e , \tag{11}$$

$$\xi* = \omega_{\max}^2 , \tag{12}$$

where $\xi*$ - is an initial angular acceleration at the beginning of the material point motion.

Setting various values to ω the time of necessary to reach the required level of the material point rotation speed can be evaluated using the equation (8). The expression (9) shows the condition for the uniform rotation of the system too.

It follows from the condition that the uniform rotation of the system can be achieved only in the case when the active force F_a is applied, or in case the torque is applied to the system in order to counteract its inertial moment.

The equation (9) shows that in case the value of both the force F_a and the distance e are constant, the material point initially performs a circular motion with acceleration, while after a time it starts a motion about the fixed point with a finite rotation speed $\omega_{\max}$. The tangential velocity of the material point, which we will refer to as V^{τ} will be upon a certain time the finite one too.

In the process of the material point circular motion, the existing mechanical tie forces the material point to change the rotation speed direction while keeping on the distance between the material point and the fixed point.

Let us assume that during a small period Δt the direction of the tangential velocity has been changed by a small angle $\Delta\varphi$. It is evident that the tangential velocity will obtain an increment

$\triangle V^{\tau} = V^{\tau} \triangle\varphi$ if $\triangle\varphi \to 0$. In such a case, the variation of the tangential velocity in small interval of time defines in the limit an acceleration of the material point under the action of the reaction force F_R being applied from the fixed point by means of the mechanical connection, which is sufficient for changing the material point velocity direction

$$a = \frac{V_{\tau} d\varphi}{dt} = V^{\tau}\omega = \omega^2 e. \tag{13}$$

The reaction force F_R of the fixed point (see Fig. 2) occurs as a response to attempts of the material point to perform its inertial rectilinear.

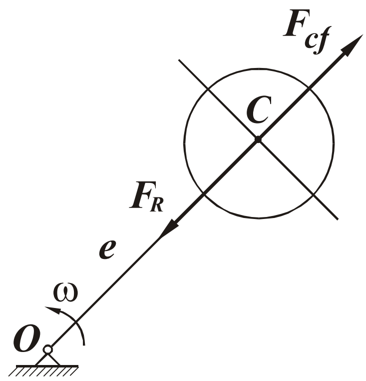

Fig. 2

It is obvious that the material point applies to the fixed point the same force trying to impart the same acceleration to the fixed point. By this is meant that the inertia of the material point produces initially a centrifugal force, and as the response to its action (in accordance with the third law of Newton), the reaction force of the fixed point is occurred

$$F_{cf} = m\omega^2 e = F_R. \tag{14}$$

The expression (13) also shows that the occurring acceleration is equal to the tangent acceleration in terms of magnitude at the beginning of the material point motion under the action of the active force.

Usually the mechanics deal with a motion of more complex systems under the action of several forces of different nature including also the weight. Therefore the reaction force depends on all forces being applied to the material point. Due to this fact, the concept of the centrifugal force is extensively used in the rotor dynamics. This force could be easily considered in the analysis together with other forces being applied to the body.

Therefore, a particular feature of the forced rotation of the material point in the selected coordinate system (as distinct from a rectilinear motion of the material point under the action of a force) is an origination of two additional forces: the centrifugal force and the support reaction force.

There were some attempts in the rotor dynamics theory being based on the oscillation theory to introduce an effect of Coriolis forces onto the rotor spinning. It is pointed-out that consideration of Coriolis forces increases the calculation accuracy [1]. In this connection, an analysis of the rotary motion features of the system "material point – fixed point" plays a key role in the case when the material point is able to perform not only a rotation, but also a translation with some linear velocity relative to the fixed point (e.g. in the radial direction).

In the beginning, we shall consider uniform circular motion of the material point about the fixed point O with a radius e (see Fig. 3).

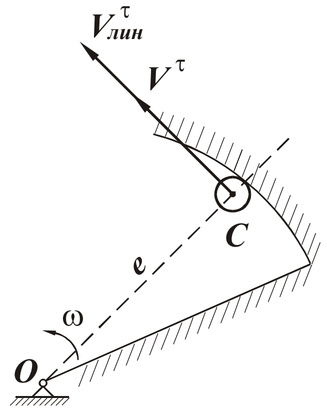

Fig. 3

We assume that the material point performs a rotary motion along a circle with a constant tangential velocity V^{τ} and in addition it moves with a linear velocity $V_{лин}^{\tau}$ along the same circle.
We assume that the tangential and the linear velocity of the material point have the same direction.
An absolute velocity of the material point can be defined as

$$V_a = V^{\tau} + V_{лин}^{\tau}. \tag{15}$$

An acceleration acting on the material point during its rotation and changing the velocity direction as is well known, can be defined as

$$a = \frac{V_a^2}{e}. \tag{16}$$

Therefore an absolute centrifugal acceleration of the material point will be equal to

$$a_a = \frac{(V^{\tau})^2}{e} + \frac{(V_{лин}^{\tau})^2}{e} + \frac{2V^{\tau}V_{лин}^{\tau}}{e} \qquad \text{or to}$$

$$a_a = \omega^2 e + \frac{(V_{лин}^{\tau})^2}{e} + 2\omega V_{лин}^{\tau}. \tag{17}$$

An analysis of the relationship (17) shows that acceleration of the material point is composed of the following parts: the acceleration, which could have a material point in case it would perform only rotary motion; the acceleration, which could have a material point in case it would perform only a circular motion with a linear velocity; the complementary acceleration, which is referred to as the Coriolis acceleration. Of three above mentioned accelerations the two former ones have the same directions. The direction of the Coriolis acceleration is orthogonal to them [2].

That means that for the rotation of such a system it is necessary to increase an additional torque. It is greater than the torque $M_{кр}$ required for rotation of a fixed material point

$$M_{кр} = m(\omega^2 e + \frac{(V_{лин}^{\tau})^2}{e} + 2\omega V_{лин}^{\tau})e. \tag{18}$$

If the linear velocity $V_{лин}^{\tau}$ of the material point is oppositely directed, then the absolute centrifugal acceleration can be defined as

$$a_a = \omega^2 e + \frac{(V^{\tau}_{лин})^2}{e} - 2\omega V^{\tau}_{лин}. \quad (19)$$

If the material point moves with some small angle α with respect to a meridian circle, then the absolute centrifugal acceleration can be defined in the following way (depending on the direction of a motion)

$$a_a = [\omega^2 e + \frac{(V^{\tau}_{лин})^2}{e} \pm 2\omega V^{\tau}_{лин}]\cos\alpha. \quad (20)$$

In case when the material point motion is directed at a great angle α, it is necessary to take into account a displacement of the material point along the radius while defining the absolute centrifugal acceleration.

Let us assuming that the material point moves with a constant linear velocity along the rotation radius from the point N to the point M. Such a velocity will be referred to as $V^{e}_{лин}$ (see Fig. 4).

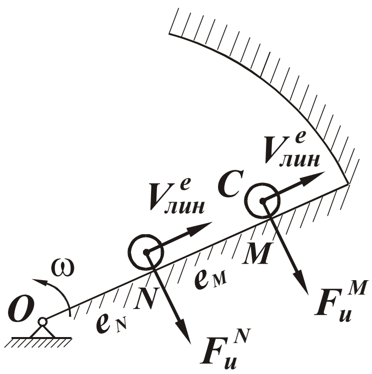

Fig. 4

The force F_u^M acts onto the material point at the point M being produced by an inertial moment, while at the point N the force F_u^N acts, which is also produced by an inertial moment. We shall make an assumption that $e_M - e_N \to 0$. The tangential velocity of the material point at the point N is equal to

$$V^{\tau}_N = \omega e_N. \quad (21)$$

The linear velocity of the material point can be defined in the following way:

$$V^{e}_{лин} = V^{\tau}_M - V^{\tau}_N. \quad (22)$$

The tangential velocity of the material point at the point M being defined from the relationship (20) is equal to

$$V^{\tau}_M = V^{e}_{лин} - V^{\tau}_N. \quad (23)$$

Basing on the equation (23), we can define an absolute centrifugal acceleration at any point M

$$a_a^M = \omega^2 e + \frac{(V_{лин}^e)^2}{e} + 2\omega V_{лин}^e . \quad (24)$$

If the linear velocity of the material point is directed towards the fixed point O, then an absolute centrifugal acceleration at any point can be defined in accordance with the following relationship

$$a_a^M = \omega^2 e + \frac{(V_{лин}^e)^2}{e} - 2\omega V_{лин}^e . \quad (25)$$

The driving torque $M_{кр}^M$ required for a uniform rotation of a movable material point can be derived from the following relationship:

$$M_{кр}^M = m(\omega^2 e + \frac{(V_{лин}^e)^2}{e} + 2\omega V_{лин}^e)e . \quad (26)$$

If a uniformly rotating material point performs at the same time a motion along both the radius and the circular trajectory with constant linear velocities, then, obviously, an absolute acceleration of the material point will be defined in terms of absolute centrifugal accelerations of the material point in considered directions.

A motion of the material point can also be performed along a generatrix, which is perpendicular to the rotation plane. This is the third virtual displacement of the material point, which does not make any influence on the value of a centrifugal acceleration because a tangential velocity of the material point does not change its magnitude in case of such displacement. For most rotors the geometry and mass characteristics are constant values. However, it is typical of the rotors of separation centrifuges to change their masses. Therefore, allowing for the absolute centrifugal acceleration and with it an effect of the Coriolis forces to the rotor dynamics are applicable only to the limited class of variable-mass rotors.

In addition to the mechanical systems performing a rotation about some defined axis, there are mechanical systems enabling translational motion of a body around defined axis along a circle. At real conditions, a mechanical system usually includes bodies, which perform their motion, and bodies performing a translational motion along a circle. These types of body's motion have some similarities; however they have some qualitative differences and distinct physical essence. As distinct from a rotary motion of the mechanical system, in case of a translational motion of a system along a circle, the principal moment of inertia of the system can be disregarded. For what concerns the rotor dynamics, a translational motion of a body along a circle has been erroneously considered as a rotary motion. However a translational motion of particular elements of the rotating mechanical system plays also a significant role in the rotor dynamics. The equation for the material point translational motion along a circle can represented as

$$e\frac{d\omega}{dt} = a^* - \omega^2 e . \quad (27)$$

The material point rotation speed and duration of a force application are bounded

$$t = \frac{e}{2\sqrt{a^* e}} + \ln\frac{\sqrt{a^* e} + e\omega}{\sqrt{a^* e} - e\omega} . \quad (28)$$

The studies has been performed show that the system "material point – fixed point", in the same way as the system "rotor – supports" is a system with asymmetrical moment of inertia, which does not depend on the rotor's rotation speed. Correspondingly, an inertial moment emerges in the system. It depends on velocity and counteracts the rotation of the system. Obtained equations and relations enable evaluating of the rotary motion specialties, moments of inertia of the system, inertial moments of the system, and a running torque of a drive for each particular case.

The studies are performed following principal laws of physics and engineering mechanics. They are also verified by the experiment. The comparative analysis of the obtained data with known experimental data [4] confirms the appropriateness of the study has been performed. The results obtained from the study make part of the further activities aimed at an establishment of a new model of rotation of the rotors based on a flexible shaft.

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