Investigations of Pascal's Triangle

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Abstract

This report will explore and prove several patterns in the famous Pascal's Triangle. First, it will be proven that the ratios within each vertical column in the triangle converge to a power of 2 dependant on the number of rows between the terms. We will also prove that the horizontal and diagonal ratios converge to 1 and thus are growing faster than their respective differences. Next, we will introduce and derive the *Combinatorial Derivative* and its respective *Combinatorial Polynomial Derivatives* and *Combinatorial Polynomials*. We will then discuss their relationship to Pascal's Triangle, develop a method for finding the polynomial functions, and provide some examples. Finally, we will investigate certain patterns arising in the coefficients of the Combinatorial Polynomials and investigate them in a number triangle.

Background

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Figure 1-1. Pascal’s Triangle to the 10^{th} row.

Pascal’s Triangle has been one of the most widely investigated topics in number theory and it remains to be so today. Each number in the triangle is the sum of the two numbers above it. It has been known and proven that the combinatorial numbers, the total number of unordered permutations of *r* members of a set of size *n,* produce the numbers in Pascal’s Triangle in a given row *n* and term index *r*. It has also been known that the numbers in each row correspond to the respective coefficients of each term in the expansion of binomials raised to a power *n*. It is because of this relationship that the formula for computing the combinatorial numbers has been known as the *Binomial Theorem.* This formula is crucial in the proofs that will be demonstrated in this report and is stated here without proof:

$$ {}_{n}C_{r} = \binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{1} $$

Convergence Theorems

Theorem 1-1. (Column Convergence Theorem)

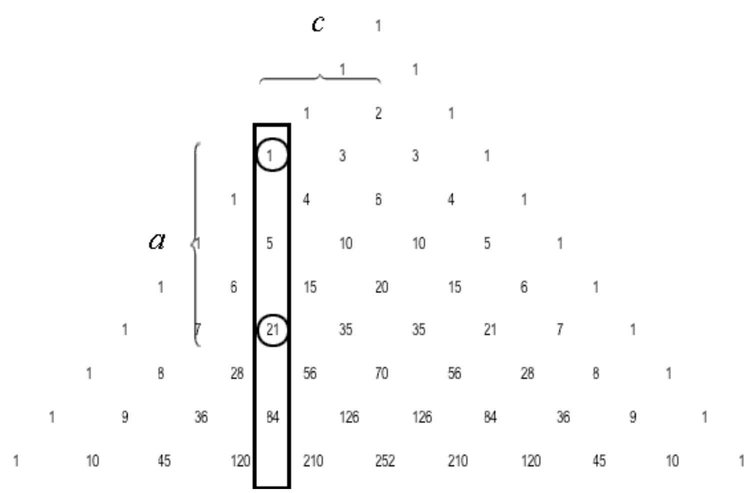

Figure 1-2. Graphical representation of the column ratios.

We start by finding the middle term of a given row using the Binomial Theorem (1). To find the middle term in the *nth* row of Pascal's Triangle, we can use $\frac{n}{2}$ as our term index. However, if *n* is not a multiple of 2, we end up with a non-integer for the term index and that is undesirable because of the factorials in (1). If instead we double the row *n*, then the term index would simply be *n*. This takes the ratio of the terms that 'line up.'

$$\binom{2n}{n}$$

To prove that there is convergence no matter how many rows are in between the column numbers, we will vary the spacing. To increase the number of rows in between the column numbers (in this case, by multiples of 2 because we doubled the row and term index), we multiply a number *a* into our row number *2n* and our term *n*.

$$\binom{2an}{an}$$

Finally, if we want to offset the column from the center to prove the convergence of all columns, we increase the row $2an$ by a number *c* and keep *an* as our term index.

$$\binom{2an+c}{an} \tag{2}$$

Using (2), we can set up a ratio to represent the column ratios by increasing *n* by 1 in the numerator. We then expand using the Binomial Theorem (1), and we have

$$\frac{\binom{2a(n+1)+c}{a(n+1)}}{\binom{2an+c}{an}} = \frac{\frac{(2an+2a+c)!}{(an+a)!(2an+2a+c-(an+a))!}}{\frac{(2an+c)!}{(an)!(2an+c-an)!}} = \left(\frac{(2an+2a+c)!}{(an+a)!(an+a+c)!}\right)\left(\frac{(an)!(an+c)!}{(2an+c)!}\right) =$$

$$\frac{(2an+c+2a)(2an+c+2a-1)\ldots(2an+c+1)(2an+c)!}{\left[(an+a)(an+a-1)\ldots(an+1)(an)!\right]\left[(an+c+a)(an+c+a-1)\ldots(an+c+1)(an+c)!\right]} \times \frac{(an)!(an+c)!}{(2an+c)!} =$$

$$\frac{(2an+c+2a)(2an+c+2a-1)\ldots(2an+c+1)}{\left[(an+a)(an+a-1)\ldots(an+1)\right]\left[(an+c+a)(an+c+a-1)\ldots(an+c+1)\right]}$$

As we take the limit as *n* approaches infinity, only the terms of *n* will become significant, so all the constants can be ignored. We end up with

$$\lim_{n\to\infty}\frac{(2an+c+2a)(2an+c+2a-1)\ldots(2an+c+1)}{\left[(an+a)(an+a-1)\ldots(an+1)\right]\left[(an+c+a)(an+c+a-1)\ldots(an+c+1)\right]}$$

$$= \frac{(2an)_1(2an)_2\ldots(2an)_{2a}}{\left[(an)_1(an)_2\ldots(an)_a\right]\left[(an)_1(an)_2\ldots(an)_a\right]} = \frac{(2an)^{2a}}{(an)^a(an)^a} = \frac{2^{2a}(an)^{2a}}{(an)^{2a}} = 2^{2a}$$

Since this expression finds the convergence of the *2ath* terms in the columns, then $a-1$ must be the total number of rows in between, since we jumped to every 2nd row in order to 'line up' the columns. This means that the ratios of the columns that do not 'line up' can be found by using the closest number to the left or right (since they are equal) and will thus converge.

$$\text{Convergance of ratio of the } a^{th} \text{ terms of the vertical columns} = 2^a \tag{3}$$

This estimation is a lower bound.

Relation to Row Sums

There is an interesting connection between the column convergence and the row sum convergence. It has been known that the sum of a row *n* is equal to 2 to the power of the row number *n*.

1	$\Sigma = 2^0$
1 1	$\Sigma = 2^1$
1 2 1	$\Sigma = 2^2$
1 3 3 1	$\Sigma = 2^3$
1 4 6 4 1	$\Sigma = 2^4$
1 5 10 10 5 1	$\Sigma = 2^5$
1 6 15 20 15 6 1	$\Sigma = 2^6$
1 7 21 35 35 21 7 1	$\Sigma = 2^7$
1 8 28 56 70 56 28 8 1	$\Sigma = 2^8$
1 9 36 84 126 126 84 36 9 1	$\Sigma = 2^9$
1 10 45 120 210 252 210 120 45 10 1	$\Sigma = 2^{10}$

Figure 1-3. Sums of the rows of Pascal's Triangle.

The connection becomes apparent when you take the ratios of the row sums. If we take the ratios of the sums of every *ath* row, then we have

$$\frac{2^{n+a}}{2^n} = 2^{n+a-n} = 2^a$$

This is similar to our results in (3). From this, we can infer that the growth of the individual terms in the same column is convergent to the growth of their entire rows. However, since it is asymptotic to the growth of the rows as a whole, the growth of the terms will never equal, let alone fall below, the growth of the row sum.

Theorem 1-2. (Row Convergence Theorem)

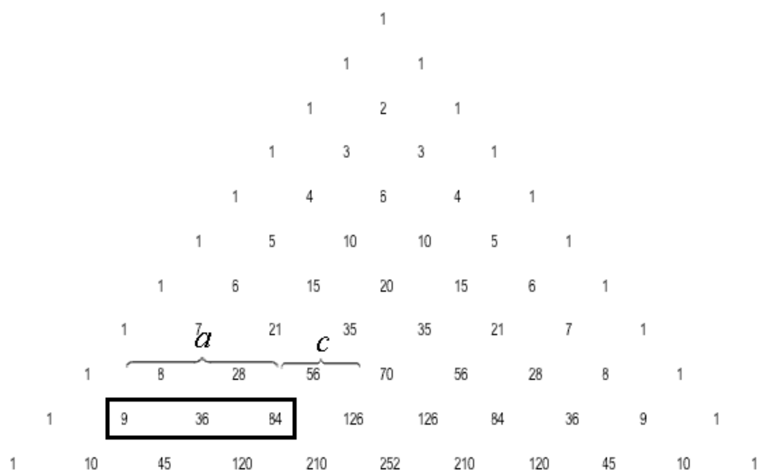

Figure 1-4. Graphical representation of the horizontal ratios.

We will now take the horizontal ratios of the terms in a row of Pascal's Triangle. We will use the middle term as our denominator and the numbers from a column a constant number of terms to the left or right as our numerator. Once again, we will double the row number in order to have an integer middle term. In order to prove that the horizontal ratios converge regardless of where the term is located, we will add a number c to the row, as done previously, to offset the middle term. We end up with an expression similar to (2).

$$\binom{2n+c}{n} \tag{4}$$

Finally, since we will take the ratio of the term a columns to the right or left of our middle term, we will add a number a to our term index in (4). It is not important whether the term is taken from the left or right, because as we will see later on, they will still converge to the same number, just from different directions. With this, we end up with the following:

$$\binom{2n+c}{n+a} \tag{5}$$

To set up our ratio, we will use (5) for our numerator and (4) for our denominator. Expanding, we end up with

$$\frac{\binom{2n+c}{n+a}}{\binom{2n+c}{n}} = \frac{\dfrac{(2n+c)!}{(n+a)!(2n+c-(n+a))!}}{\dfrac{(2n+c)!}{(n)!(2n+c-n)!}} = \left(\frac{(2n+c)!}{(n+a)!(n+c-a)!}\right)\left(\frac{(n)!(n+c)!}{(2n+c)!}\right) =$$

$$\frac{(2n+c)!}{\left[(n+a)(n+a-1)\ldots(n+1)(n)!\right](n+c-a)!} \times \frac{(n)!(n+c)(n+c-1)\ldots(n+c-a+1)(n+c-a)!}{(2n+c)!} =$$

$$\frac{(n+c)(n+c-1)\ldots(n+c-a+1)}{(n+a)(n+a-1)\ldots(n+1)}$$

Taking the limit as n approaches infinity, only the terms of n will be significant; therefore, the constants can be ignored.

$$\lim_{n\to\infty}\frac{(n+c)(n+c-1)\ldots(n+c-a+1)}{(n+a)(n+a-1)\ldots(n+1)} = \frac{(n)_1(n)_2\ldots(n)_a}{(n)_1(n)_2\ldots(n)_a} = 1$$

This result proves that the horizontal ratios of two terms within a row of Pascal's Triangle with any constant number of terms in between will converge to 1 as Pascal's Triangle approaches the infinite row. From this, we can infer that as the rows of the triangle grow increasingly large, the numbers in that row will grow faster than their differences with respect to the terms immediately next to them. Even though the numbers (as well as their numerical differences) are growing infinitely large, the amount of deviation between immediate terms in a row are becoming smaller with respect to the growth of the terms themselves, regardless of how large the constant gap between them is. This convergence can be an upper or a lower bound depending on which expression is in the numerator and which is in the denominator, as well as, whether the ratios are taken from the left or right.

Theorem 1-3. (Diagonal Convergence Theorem)

Now we will take the ratios of the diagonals in Pascal's Triangle. If we start with a certain row and continue to increase both the row and the term at the same rate, we will progress linearly from the left to right in a diagonal. Also, if we multiply both the row and the term by a number *a* then we will skip $a-1$ numbers in the diagonal. Thus, if we start at row *c*, term 0, then increase the row and term by a number *n* multiplied by *a*, we end up with

$$\binom{an+c}{an} \tag{6}$$

We can then set up a ratio using (6) by increasing *n* by 1 in the numerator. Expanding, we end up with

$$\frac{\binom{a(n+1)+c}{a(n+1)}}{\binom{an+c}{an}} = \frac{\dfrac{(an+a+c)!}{(an+a)!(an+a+c-(an+a))!}}{\dfrac{(an+c)!}{(an)!(an+c-an)!}} = \left(\frac{(an+a+c)!}{(an+a)!(c)!}\right)\left(\frac{(an)!(c)!}{(an+c)!}\right)$$

$$\frac{(an+c+a)(an+c+a-1)\dots(an+c+1)(an+c)!}{\left[(an+a)(an+a-1)\dots(an+1)(an)!\right](c)!} \times \frac{(an)!(c)!}{(an+c)!} = \frac{(an+c+a)(an+c+a-1)\dots(an+c+1)}{(an+a)(an+a-1)\dots(an+1)}$$

Once again taking the limit as *n* approaches infinity, the constants can be ignored, giving us

$$\lim_{n\to\infty}\frac{(an+c+a)(an+c+a-1)\dots(an+c+1)}{(an+a)(an+a-1)\dots(an+1)} = \frac{(an)_1(an)_2\dots(an)_a}{(an)_1(an)_2\dots(an)_a} = 1$$

This limit is a lower bound. This will be proved later in the text using Combinatorial Derivative Polynomials, which will be introduced next.

The Combinatorial Derivative

Here we will derive a formula for calculating the derivative of the Binomial Theorem by expressing the factorials in terms of the Gamma Function and then differentiating.

The Gamma Function

The Gamma Function, developed by Leonhard Euler, is an extension of the factorial to all Real Numbers. Its formal definition is not important in this text, since we will only be dealing with its derivative, but is provided in the Appendix. The following relationship holds true:

$$\Gamma(n) = (n-1)! \qquad \text{or} \qquad n! = \Gamma(n+1) \tag{7}$$

The derivative of the Gamma Function is obtained through logarithmic differentiation and its derivation is beyond the scope of this report, but is provided in the Appendix. Therefore, it is stated here without proof [**1**, formula (24)]:

$$\Gamma'(n) = -(n-1)!\left(\frac{1}{n} + \gamma - \sum_{k=1}^{n}\frac{1}{k}\right) \tag{8}$$

where γ is the Euler-Mascheroni Constant ($\gamma \approx 0.577215664901\ldots$). Using the relationship in (7) and (8), we can infer the derivative of the factorial of an integer n.

$$\frac{d}{dn}n! = \Gamma'(n+1) = -(n+1-1)!\left(\frac{1}{n+1} + \gamma - \sum_{k=1}^{n+1}\frac{1}{k}\right) = -(n)!\left(\gamma - \sum_{k=1}^{n}\frac{1}{k}\right) \tag{9}$$

Differentiating the Binomial Theorem

We can now use (9) to differentiate the Binomial Theorem. We will start by taking the derivative with respect to n. We expand using the derivative quotient rule, differentiate with (9), and we have

$$\frac{\partial}{\partial n}\binom{n}{r} = \frac{\partial}{\partial n}\left(\frac{n!}{r!(n-r)!}\right) = \frac{r!(n-r)\frac{d}{dn}(n!) - n!\frac{d}{dn}(r!(n-r)!)}{\left[r!(n-r)!\right]^2} = \frac{r!(n-r)!\Gamma'(n+1) - n!r!\Gamma'(n-r+1)}{\left[r!(n-r)!\right]^2} =$$

$$\frac{r!(n-r)!\left[(-n!)\left(\gamma - \sum_{k=1}^{n}\frac{1}{k}\right)\right] - r!n!\left[(-(n-r)!)\left(\gamma - \sum_{k=1}^{n-r}\frac{1}{k}\right)\right]}{\left[r!(n-r)!\right]^2} = \frac{r!(n-r)!n!\left[-\left(\gamma - \sum_{k=1}^{n}\frac{1}{k}\right) + \left(\gamma - \sum_{k=1}^{n-r}\frac{1}{k}\right)\right]}{r!(n-r)!r!(n-r)!} =$$

$$\frac{n!\left(\sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{n-r}\frac{1}{k}\right)}{r!(n-r)!} = \frac{n!}{r!(n-r)!}\left[\left(\frac{1}{1}+\frac{1}{2}+\cdots\frac{1}{n}\right) - \left(\frac{1}{1}+\frac{1}{2}+\ldots\frac{1}{n-r}\right)\right] = \binom{n}{r}\left(\frac{1}{n-r+1} + \frac{1}{n-r+2} + \cdots\frac{1}{n}\right)$$

$$= \binom{n}{r}\sum_{k=n-r+1}^{n}\frac{1}{k} = \binom{n}{r}\sum_{k=0}^{r-1}\frac{1}{n-k} \tag{10}$$

Since this is the derivative of the combinatorial formula (1), we will call it the *Combinatorial Derivative*. This formula assumes r is a constant, which means that there will be an infinite number of functions of n, one for each term index r. Because our term index is constant and our row number is variable, the function progresses in a diagonal from right to left as n is increased. We can also think of this as a vertical progression in the columns of a flattened Pascal's Triangle. The combinatorial does not make sense if the row number n is less than the term index r, since there are n terms in the *nth* row. Therefore, we must make sure that $n \geq r$, meaning that we set n equal to r for the first term in the diagonal and each successive value to $r + a$. This formula yields the rate of change of the diagonals starting with the *rth* row in Pascal's Triangle.

Combinatorial Derivative Polynomials

Plotting the Combinatorial Derivative yields graphs similar to polynomials of increasing degrees as r is increased.

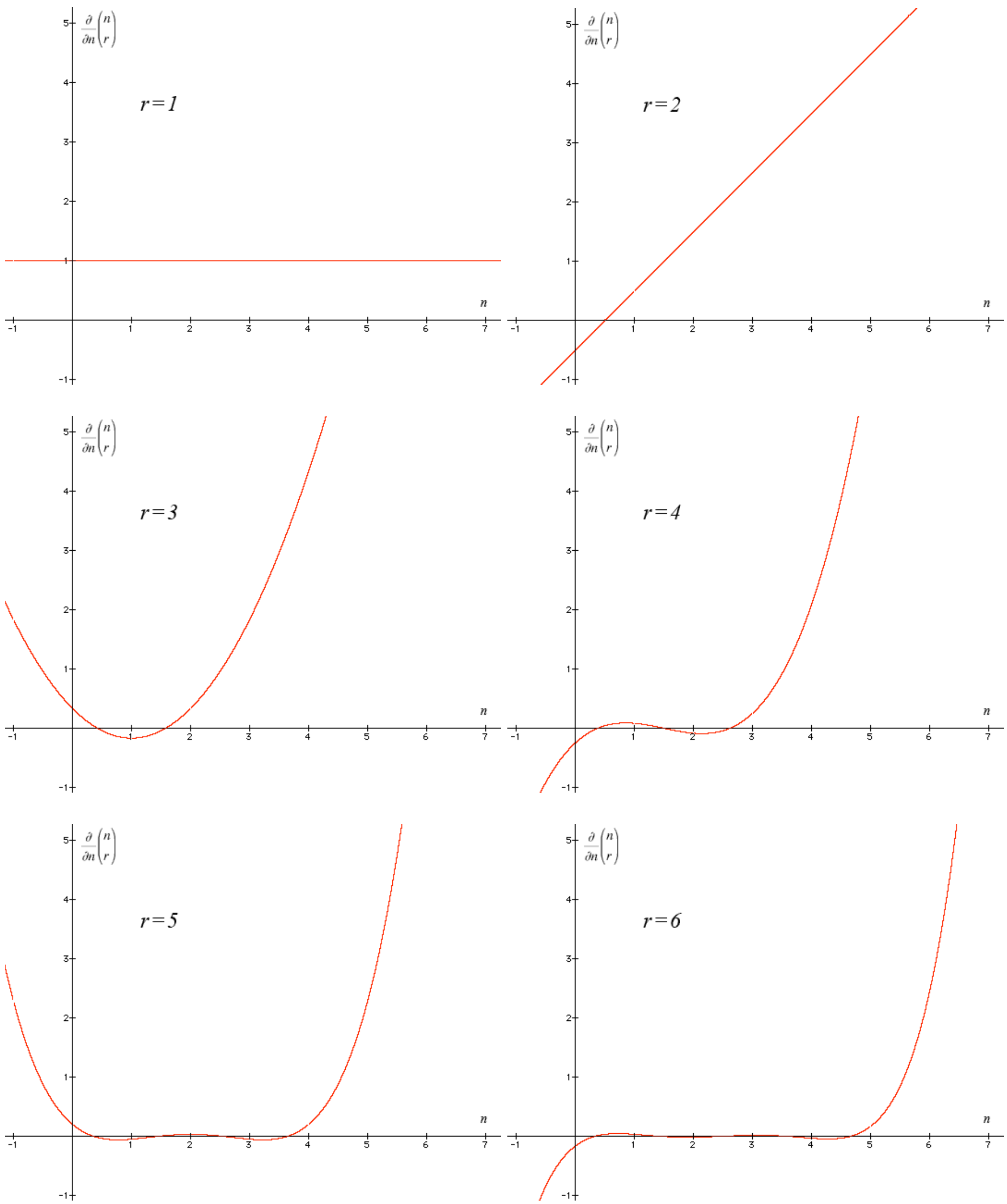

Figure 2-1. The *rth* order Combinatorial Derivative vs. n.

From this, we can conclude that the diagonals of the Pascal's Triangle increase in a polynomial fashion. We will therefore call these polynomials *Combinatorial Polynomial Derivatives*. The *rth* order Combinatorial Polynomial Derivative $p_r(n)$ will always have a degree of $r-1$. This can be proved through expansion of (10).

$$\binom{n}{r}\sum_{k=0}^{r-1}\frac{1}{n-k}=\frac{n!}{r!(n-r)!}\sum_{k=0}^{r-1}\frac{1}{n-k}=\frac{(n)(n-1)...(n-r+1)(n-r)!}{r!(n-r)!}\sum_{k=0}^{r-1}\frac{1}{n-k}=$$

$$\frac{(n)(n-1)...(n-r+1)}{r!}\left(\frac{1}{n}+\frac{1}{n-1}+\cdots\frac{1}{n-r+1}\right) \tag{11}$$

Note that this leads us to an alternate definition of the Combinatorial Derivative.

$$\frac{\partial}{\partial n}\binom{n}{r} = \left(\frac{1}{r!}\right)\left(\prod_{j=0}^{r-1}(n-j)\right)\left(\sum_{k=0}^{r-1}\frac{1}{n-k}\right) \tag{12}$$

Now we will distribute the product in the numerator of the fraction on the left in (11) with the summation of reciprocals on the right.

$$\frac{1}{r!}\left(\frac{(n)(n-1)\ldots(n-r+1)}{n}+\frac{(n)(n-1)\ldots(n-r+1)}{n-1}+\cdots\frac{(n)(n-1)\ldots(n-r+1)}{n-r+1}\right) \tag{13}$$

Exactly one $(n-j)$ factor from the product in each numerator will cancel out its denominator. Since there are r factors in each of the numerators, the summation after the cancellations becomes the sum of several polynomials of degree $r-1$. An interesting property of this expansion is that when the $(n-j)$ terms are cancelled out, the resulting summation is actually the sum of the products of all possible unordered permutations of the set containing the factors of the original product. To better demonstrate this, we will take the Combinatorial Derivative when $r=3$ and expand it using (13).

$$\frac{\partial}{\partial n}\binom{n}{r}=\frac{1}{3!}\left(\frac{(n)(n-1)(n-2)}{n}+\frac{(n)(n-1)(n-2)}{n-1}+\frac{(n)(n-1)(n-2)}{n-2}\right)=$$

$$\tfrac{1}{6}\Big((n-1)(n-2)+(n)(n-2)+(n)(n-1)\Big) \tag{14}$$

Now we create a set S containing the factors of the original product.

$$S=\{(n),(n-1),(n-2)\}$$

Since each $(n-j)$ reciprocal cancels out exactly 1 of the original product factors, we want to find the total number of unordered ways to arrange $r-1$ (in this case 2) members of S, which has r (in this case 3) elements. As previously stated, the combinatorial numbers, which describe the number of unordered permutations, can be found using the Binomial Theorem (1). The number of unordered permutations is therefore

$$_3C_2=\binom{3}{2}=\frac{3!}{2!(3-2)!}=\frac{6}{2(1)!}=3$$

Notice that there are 3, or $_3C_2$, products in the sum after the cancellations in (14). These products happen to be the individual products of the three unordered permutations of 2 members of S.

$$\{(n)(n-1),\ (n)(n-2),\ (n-1)(n-2)\}$$

Finding the Combinatorial Polynomial Derivatives

Now that we've come up with an alternate definition for the Combinatorial Derivative in (12) and (13), we can use it to find the Combinatorial Polynomial Derivatives. As an example, we will find the 3rd order Combinatorial Polynomial Derivative. The derivative has already been expanded for us in (14), so we will continue from there.

$$\begin{aligned}p_3(n)&=\tfrac{1}{6}\Big((n-1)(n-2)+(n)(n-2)+(n)(n-1)\Big)=\tfrac{1}{6}\Big((n^2-3n+2)+(n^2-2n)+(n^2-n)\Big)\\&=\tfrac{1}{6}\Big(3n^2-6n+2\Big)=\tfrac{1}{2}n^2-n+\tfrac{1}{3}\end{aligned}$$

This method can be used to find any order Combinatorial Derivative Polynomial. We will list the first 8.

$$p_0(n) = 0$$
$$p_1(n) = 1$$
$$p_2(n) = n - \tfrac{1}{2}$$
$$p_3(n) = \tfrac{1}{2}n^2 - n + \tfrac{1}{3}$$
$$p_4(n) = \tfrac{1}{6}n^3 - \tfrac{3}{4}n^2 + \tfrac{11}{12}n - \tfrac{1}{4}$$
$$p_5(n) = \tfrac{1}{24}n^4 - \tfrac{1}{3}n^3 + \tfrac{7}{8}n^2 - \tfrac{5}{6}n + \tfrac{1}{5}$$
$$p_6(n) = \tfrac{1}{120}n^5 - \tfrac{5}{48}n^4 + \tfrac{17}{36}n^3 - \tfrac{15}{16}n^2 + \tfrac{137}{180}n - \tfrac{1}{6}$$
$$p_7(n) = \tfrac{1}{720}n^6 - \tfrac{1}{40}n^5 + \tfrac{25}{144}n^4 - \tfrac{7}{12}n^3 + \tfrac{29}{30}n^2 - \tfrac{7}{10}n + \tfrac{1}{7}$$
$$\vdots$$

Notice that all Combinatorial Derivative Polynomials are of the order (i.e. asymptotic to)

$$p_r(n) \sim \frac{1}{(r-1)!}n^{r-1}$$

Combinatorial Polynomials

If we integrate the Combinatorial Polynomial Derivatives, we should get functions that will yield the progressive values along the diagonals of Pascal's Triangle. We will therefore call these functions *Combinatorial Polynomials* and define them as

$$P_r(n) = \int_0^n p_r(n)dn \qquad\qquad P_0(n) = 1$$

The *rth* order Combinatorial Polynomial $P_r(n)$ will always be of degree *r*. Thus, the diagonal values of Pascal's Triangle starting at the *rth* row grow at a polynomial degree *r*. As stated before with the Combinatorial Derivative Polynomials, we must start with $n = r$ for the first term. As an example, we will integrate the 2nd order Combinatorial Derivative Polynomial with respect to *n* to obtain its respective Combinatorial Polynomial.

$$P_2(n) = \int_0^n p_2(n)dn = \int_0^n \left(n - \tfrac{1}{2}\right)dn = \tfrac{1}{2}n^2 - \tfrac{1}{2}n$$

As shown, the values of $P_2(n)$ match the values in Pascal's Triangle nicely (note that we start with *r*, or 2).

n	$P_2(n)$
2	1
3	3
4	6
5	10
6	15
7	21
8	28
9	36
10	45

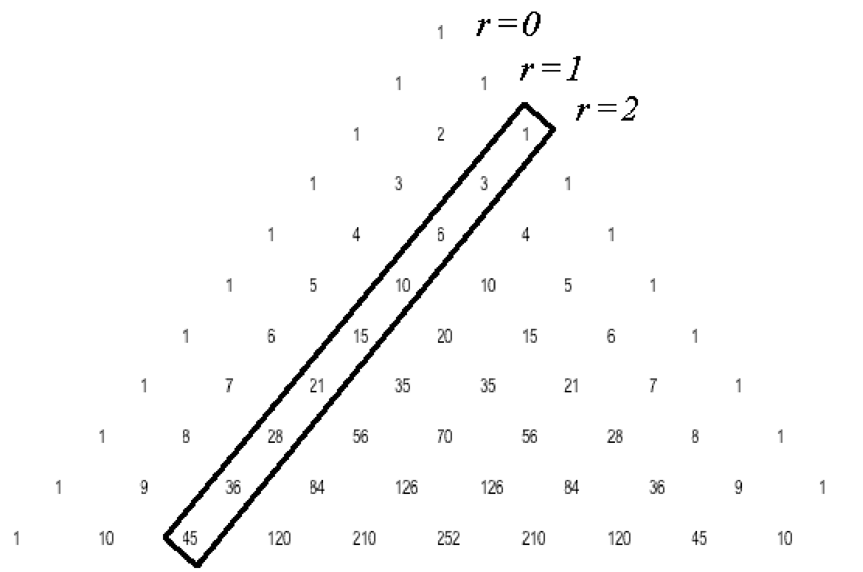

Figure 2-2. The diagonal values of Pascal's Triangle.

We can find any Combinatorial Polynomial—thus, the function modeling the values of the diagonals in Pascal's Triangle—by integrating its respective Combinatorial Derivative Polynomial. The first 8 are given here:

$$
\begin{aligned}
&P_0(n) = 1\\
&P_1(n) = n\\
&P_2(n) = \tfrac{1}{2}n^2 - \tfrac{1}{2}n\\
&P_3(n) = \tfrac{1}{6}n^3 - \tfrac{1}{2}n^2 + \tfrac{1}{3}n\\
&P_4(n) = \tfrac{1}{24}n^4 - \tfrac{1}{4}n^3 + \tfrac{11}{24}n^2 - \tfrac{1}{4}n\\
&P_5(n) = \tfrac{1}{120}n^5 - \tfrac{1}{12}n^4 + \tfrac{7}{24}n^3 - \tfrac{5}{12}n^2 + \tfrac{1}{5}n\\
&P_6(n) = \tfrac{1}{720}n^6 - \tfrac{1}{48}n^5 + \tfrac{17}{144}n^4 - \tfrac{5}{16}n^3 + \tfrac{137}{360}n^2 - \tfrac{1}{6}n\\
&P_7(n) = \tfrac{1}{5040}n^7 - \tfrac{1}{240}n^6 + \tfrac{5}{144}n^5 - \tfrac{7}{48}n^4 + \tfrac{29}{90}n^3 - \tfrac{7}{20}n^2 + \tfrac{1}{7}n\\
&\vdots
\end{aligned}
$$

Notice that all the Combinatorial Polynomials are of the order (i.e. asymptotic to)

$$P_r(n) \sim \frac{1}{r!}n^r$$

Using the Combinatorial Polynomials to Prove Convergence Theorems

The Combinatorial Polynomials can also give further proof to the assertions made about the growth of the values in Pascal's Triangle. We will now use them to prove the Diagonal Convergence Theorem (Theorem 1-3). If we express the values of the diagonals as an *rth* order Combinatorial Polynomial, we can set up the ratio of a number *c* terms ahead of a number *n*, take the limit as *n* approaches infinity, and see that the ratio converges to 1.

$$\frac{P_r(n+c)}{P_r(n)} \sim \frac{\frac{1}{r!}(n+c)^r}{\frac{1}{r!}(n)^r}$$

$$\lim_{n \to \infty} \frac{(n+c)^r}{(n)^r} = \frac{n^r}{n^r} = 1$$

The numerator is obviously larger than the denominator, thus proving our previous assertion that 1 is a lower bound for the ratio.

Combinatorial Polynomial Coefficients

There are several interesting patterns with the coefficients of the Combinatorial Polynomials. The first term is always positive and the sign alternates with each successive term and, curiously, the coefficients add up to 0. Now, because all of the denominators are factors of *r!*, we can set up a triangle with the *unsimplified* numerators of each coefficient—that is, the numerators after we set each coefficient to a common denominator *r!*.

$^{1}1$

$^{2}1 \quad 1$

$^{3}1 \quad 3 \quad 2$

$^{4}1 \quad 6 \quad 11 \quad 6$

$^{5}1 \quad 10 \quad 35 \quad 50 \quad 24$

$^{6}1 \quad 15 \quad 85 \quad 225 \quad 274 \quad 120$

$^{7}1 \quad 21 \quad 175 \quad 735 \quad 1624 \quad 1764 \quad 720$

There is a very interesting pattern in this triangle that is not easily seen at first. The first column is filled with 1's. Every other number in the triangle is equal to the number diagonally to the left times its degree, or $r-1$, plus the number directly above it. From this, we can set up a recursive formula for each term in the triangle at any row n and column r.

$$Q_{n,r} = (n-1)Q_{n-1,r-1} + Q_{n-1,r}$$
$$Q_{n,0} = 1 \quad and \quad Q_{n,n} = 0$$

As an example, lets find $Q_{7,3}$.

$$\begin{aligned}
Q_{7,3} &= 6Q_{6,2} + Q_{6,3} \\
&= 6(5Q_{5,1} + Q_{5,2}) + 5Q_{5,2} + Q_{5,3} \\
&= 6\left[5(4Q_{4,0} + Q_{4,1}) + (4Q_{4,1} + Q_{4,2})\right] + 5(4Q_{4,1} + Q_{4,2}) + (4Q_{4,2} + Q_{4,3}) \\
&= 6\left\{5\left[4 + (3Q_{3,0} + Q_{3,1})\right] + \left[4(3Q_{3,0} + Q_{3,1}) + (3Q_{3,1} + Q_{3,2})\right]\right\} \\
&\quad + 5\left[4(3Q_{3,0} + Q_{3,1}) + (3Q_{3,1} + Q_{3,2})\right] + \left[4(3Q_{3,1} + Q_{3,2}) + 3Q_{3,2}\right] \\
&= 6\left(5\left\{4 + \left[3 + (2Q_{2,0} + Q_{2,1})\right]\right\} + \left\{4\left[3 + (2Q_{2,0} + Q_{2,1})\right] + \left[\left(3(2Q_{2,0} + Q_{2,1}) + 2Q_{2,1}\right)\right]\right\}\right) \\
&\quad + 5\left\{4\left[3 + (2Q_{2,0} + Q_{2,1})\right] + \left[3(2Q_{2,0} + Q_{2,1}) + 2Q_{2,1}\right]\right\} + \left\{4\left[3(2Q_{2,0} + Q_{2,1}) + 2Q_{2,1}\right] + 3(2Q_{2,1})\right\} \\
&= 6\left[5(4+3+2+1) + 4(3+2+1) + 3(2+1) + 2\right] + 5\left[4(3+2+1) + 3(2+1) + 2\right] + 4\left[3(2+1) + 2\right] + 3\left[2\right] \\
&= 735
\end{aligned}$$

We can express this as a composition of sums.

$$Q_{7,3} = \sum_{i=3}^{6}\left(\sum_{j=2}^{i-1}\left(\sum_{k=1}^{j-1} kj\right)i\right)$$

From this, we can develop a general summation formula.

$$Q_{n,r} = \sum_{j_1=r}^{n-1}\left(\sum_{j_2=r-1}^{j_1-1}\left(\sum_{j_3=r-2}^{j_2-1}\left(\cdots\sum_{j_r=1}^{j_{r-1}-1} j_r j_{r-1}\cdots\right)j_2\right)j_1\right)$$

Now that we understand $Q_{n,r}$, we can provide another definition for the Combinatorial Polynomials.

$$P_r(n) = \sum_{k=0}^{r-1}(-1)^k \frac{Q_{r,k}n^{r-k}}{r!}$$

References

[1] *Gamma Function -- From MathWorld*. (n.d.). Retrieved November 01, 2005, from http://mathworld.wolfram.com/GammaFunction.html

Appendix

Formal Definition of the Gamma Function and its Differentiation

Definition

The Gamma Function is an extension of the factorial developed by Leonhard Euler. It is most commonly expressed in integral form as [**1**, formula (2)]

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$$

The relationship between the Gamma Function and the factorial can be seen when we use integration by parts [**1**, formulas (5)-(8)].

$$\begin{aligned}\Gamma(z) &= \int_0^\infty t^{z-1}e^{-t}\,dt = \left[-t^{z-1}e^{-t}\right]_0^\infty + \int_0^\infty (z-1)t^{z-2}e^{-t}\,dt \\ &= (z-1)\int_0^\infty t^{z-2}e^{-t}\,dt = (z-1)\Gamma(z-1)\end{aligned}$$

If z is an integer $n = 1,2,3,\ldots$, then [**1**, formulas (9)-(10)]

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = (n-1)(n-2)\ldots 1 = (n-1)!$$

Differentiation

The Gamma Function can be defined in infinite product form (Weierstrass Form) [**1**, formula (12)] as

$$\Gamma(z) = \left[ze^{\gamma z}\prod_{r=1}^{\infty}\left(1+\frac{z}{r}\right)e^{-z/r}\right]^{-1},$$

where γ is the Euler-Mascheroni Constant ($\gamma \approx 0.577215664901\ldots$). We can differentiate after we take the natural logarithm of both sides [**1**, formula (16)].

$$-\ln\left[\Gamma(z)\right] = \ln z + \gamma z + \sum_{r=1}^{\infty}\left[\ln\left(1+\frac{z}{r}\right) - \frac{z}{r}\right]$$

Now we now take the derivative of both sides with respect to z [**1**, formulas (17)-(18)].

$$
\begin{aligned}
-\frac{\Gamma'(z)}{\Gamma(z)} &= \frac{1}{z} + \gamma + \sum_{r=1}^{\infty}\left(\frac{\frac{1}{r}}{1+\frac{z}{r}} - \frac{1}{r}\right) \\
&= \frac{1}{z} + \gamma + \sum_{r=1}^{\infty}\left(\frac{1}{r+z} - \frac{1}{r}\right)
\end{aligned}
$$

If n is an integer, then

$$
\begin{aligned}
\Gamma'(n) &= -\Gamma(n)\left\{\frac{1}{n} + \gamma + \left[\left(\frac{1}{1+n} - 1\right) + \left(\frac{1}{2+n} + \frac{1}{2}\right) + \left(\frac{1}{3+n} + \frac{1}{3}\right) + \ldots\right]\right\} \\
&= -(n-1)!\left(\frac{1}{n} + \gamma - \sum_{k=1}^{n}\frac{1}{k}\right)
\end{aligned}
$$