# Input-Output Analysis

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Session 1.2

# Impact analysis & Additional concepts

# Impact Analysis



## So far …

- All information regarding the structure of production input and output have been placed in a relatively complete table
- That table is a portrait of an economy in a particular point of time – many analysis possible at this stage
- Now, more advanced analysis

## Input-output coefficient

- Other names: *direct input coefficient, technical coefficient*

$$a_{ij} = \frac{z_{ij}}{X_j}$$

$a_{32}$ = 0,3 means: to produce $1 output, sector 2 needs $0.3 intermediate input from sector 3

| | | Production Sector | | Final Demand | | | | Total Output |
|---|---|---|---|---|---|---|---|---|
| | | 1 | 2 | C | I | G | E | X |
| Production Sector | 1 | $z_{11}$ | $z_{12}$ | $C_1$ | $I_1$ | $G_1$ | $E_1$ | $X_1$ |
| | 2 | $z_{21}$ | $z_{22}$ | $C_2$ | $I_2$ | $G_2$ | $E_2$ | $X_2$ |
| Value Added | L | $L_1$ | $L_2$ | | | | | L |
| | N | $N_1$ | $N_2$ | | | | | N |
| Import | M | $M_1$ | $M_2$ | | | | | M |
| Total Input | X | $X_1$ | $X_2$ | C | I | G | E | X |



## Technology matrix

- For n sectors, there would be as many as *nxn* input-output coefficients $a_{ij}$.
- All of those coefficients can be presented in a matrix, conventionally called A, as shown
- (**I – A**) is called technology matrix
- One consequence of the input-output coefficient is the following:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$a_{ij} = \frac{z_{ij}}{X_j} \Leftrightarrow z_{ij} = a_{ij} X_j$$

## With a few algebraic manipulations

- By stating

  $z_{ij} = a_{ij} \, . \, X_j$

  then the previous system of equations can be rewritten in the following form

$$\begin{cases} X_1 = z_{11} + z_{12} + z_{13} + \ldots + z_{1n} + Y_1 \\ X_2 = z_{21} + z_{22} + z_{23} + \ldots + z_{2n} + Y_2 \\ \vdots \\ X_n = z_{n1} + z_{n2} + z_{n3} + \ldots + z_{nn} + Y_n . \end{cases}$$

$$\begin{cases} X_1 = a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n + Y_1 \\ X_2 = a_{21}X_1 + a_{22}X_2 + \ldots + a_{2n}X_n + Y_2 \\ \vdots \\ X_n = a_{n1}X_1 + a_{n2}X_2 + \ldots + a_{nn}X_n + Y_n . \end{cases}$$



## A few algebra further,

$$\begin{cases} X_1 - a_{11}X_1 - a_{12}X_2 - \ldots - a_{1n}X_n = Y_1 \\ X_2 - a_{21}X_1 - a_{22}X_2 - \ldots - a_{2n}X_n = Y_2 \\ \vdots \\ X_n - a_{n1}X_1 - a_{n2}X_2 - \ldots - a_{nn}X_n = Y_n \end{cases}$$

$$\begin{cases} (1-a_{11})X_1 - a_{12}X_2 - \ldots - a_{1n}X_n = Y_1 \\ -a_{21}X_1 + (1-a_{22})X_2 - \ldots - a_{2n}X_n = Y_2 \\ \vdots \\ -a_{n1}X_1 - a_{n2}X_2 - \ldots + (1-a_{nn})X_n = Y_n . \end{cases}$$

$$\begin{bmatrix} 1-a_{11} & -a_{12} & \ldots & -a_{1n} \\ -a_{21} & 1-a_{22} & \ldots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \ldots & 1-a_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}$$

$$\mathbf{(I - A)X = Y}$$

## Therefore, if we ask

- What is the effect of an exogenous change (or shock) **Y**, i.e., in the final demand, on the output **X**?

We know that (I – A)X = Y. Therefore,

$$\mathbf{X} = \underbrace{(\mathbf{I}-\mathbf{A})^{-1}}_{\text{Leontief Inverse}} \mathbf{Y}$$



## Leontief Inverse & Keynesian multiplier

$$\mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1} \mathbf{Y}$$

$$(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$Y = \frac{1}{(1-c)}(C_0 + I_0 + G_0)$$

## Hypothetical example – Transaction table of an economy

| | | Production Sectors | | Final Demand | | Total Output |
|---|---|---|---|---|---|---|
| | | 1 | 2 | C | I | X |
| Production Sector | 1 | 100 | 400 | 300 | 200 | 1000 |
| | 2 | 300 | 600 | 500 | 600 | 2000 |
| Primary Input | L | 200 | 700 | | | |
| | N | 400 | 300 | | | |
| Total Input | | 1000 | 2000 | | | |

$$\mathbf{A} = \mathbf{Z}(\hat{\mathbf{X}})^{-1} = \begin{bmatrix} 100 & 400 \\ 300 & 600 \end{bmatrix} \begin{bmatrix} 1/1000 & 0 \\ 0 & 1/2000 \end{bmatrix}$$

$$= \begin{bmatrix} 0,1 & 0,2 \\ 0,3 & 0,3 \end{bmatrix}$$



## *Leontief inverse*

$$(\mathbf{I} - \mathbf{A}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0,1 & 0,2 \\ 0,3 & 0,3 \end{bmatrix} = \begin{bmatrix} 0,9 & -0,2 \\ -0,3 & 0,7 \end{bmatrix}$$

$$(\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1,228 & 0,351 \\ 0,526 & 1,579 \end{bmatrix}$$

## Changes in final demand

- In year t, the final demand matrix is as follows:

$$\mathbf{Y_t} = \begin{bmatrix} C_1 + I_1 \\ C_2 + I_2 \end{bmatrix} = \begin{bmatrix} 500 \\ 1100 \end{bmatrix}$$

- In year t+1, the final demand becomes the following:

$$\mathbf{Y_{t+1}} = \begin{bmatrix} 700 \\ 1400 \end{bmatrix}$$

- Therefore, the output of sectors 1 and 2 in t+1 becomes

$$\begin{aligned} \mathbf{X_{t+1}} &= (\mathbf{I} - \mathbf{A})^{-1}\mathbf{Y_{t+1}} \\ &= \begin{bmatrix} 1{,}228 & 0{,}351 \\ 0{,}526 & 1{,}579 \end{bmatrix} \begin{bmatrix} 700 \\ 1400 \end{bmatrix} = \begin{bmatrix} 1350{,}877 \\ 2578{,}947 \end{bmatrix}. \end{aligned}$$



## In incremental form

$$\begin{aligned} \Delta\mathbf{X_{t+1}} &= (\mathbf{I} - \mathbf{A})^{-1}\Delta\mathbf{Y_{t+1}} \\ &= \begin{bmatrix} 1{,}228 & 0{,}351 \\ 0{,}526 & 1{,}579 \end{bmatrix} \begin{bmatrix} 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 350{,}877 \\ 578{,}947 \end{bmatrix}. \end{aligned}$$

# Additional concepts



## Direct and Indirect Effects

- If there is additional final demand, it must be produced – and automatically is a part of additional output.

  In the above example, there is additional final demand (i.e., 200) for sector 1. Automatically, output of sector 1 must increase by 200. This is the DIRECT EFFECT

- But, that is not all!

  Producing that additional output requires inputs from sector 2. For sector 2, this is additional demand for output. In its production process, sector 2 also requires inputs from sector 1 → thus, output of sector 1 must increase again. Chains of reactions like this occur because there are intersectoral linkages. This is the INDIRECT EFFECT

## Round-by-round effect – the first 6 rounds

| Round # | Additional output needed to produce additional output in the previous round | Additional output created in respective round |
|---|---|---|
| 1 | **The economic shock** | $\Delta Y_1 = \begin{bmatrix} 200 \\ 300 \end{bmatrix}$ |
| 2 | $\begin{bmatrix} 0{,}1 & 0{,}2 \\ 0{,}3 & 0{,}3 \end{bmatrix} \begin{bmatrix} 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 80 \\ 150 \end{bmatrix}$ | $\Delta Y_2 = \begin{bmatrix} 80 \\ 150 \end{bmatrix}$ |
| 3 | $\begin{bmatrix} 0{,}1 & 0{,}2 \\ 0{,}3 & 0{,}3 \end{bmatrix} \begin{bmatrix} 80 \\ 150 \end{bmatrix} = \begin{bmatrix} 38 \\ 72 \end{bmatrix}$ | $\Delta Y_3 = \begin{bmatrix} 38 \\ 72 \end{bmatrix}$ |
| 4 | $\begin{bmatrix} 0{,}1 & 0{,}2 \\ 0{,}3 & 0{,}3 \end{bmatrix} \begin{bmatrix} 38 \\ 72 \end{bmatrix} = \begin{bmatrix} 18{,}2 \\ 33 \end{bmatrix}$ | $\Delta Y_4 = \begin{bmatrix} 18{,}2 \\ 33 \end{bmatrix}$ |
| 5 | $\begin{bmatrix} 0{,}1 & 0{,}2 \\ 0{,}3 & 0{,}3 \end{bmatrix} \begin{bmatrix} 18{,}2 \\ 33 \end{bmatrix} = \begin{bmatrix} 8{,}42 \\ 15{,}36 \end{bmatrix}$ | $\Delta Y_5 = \begin{bmatrix} 8{,}42 \\ 15{,}36 \end{bmatrix}$ |
| 6 | $\begin{bmatrix} 0{,}1 & 0{,}2 \\ 0{,}3 & 0{,}3 \end{bmatrix} \begin{bmatrix} 8{,}42 \\ 15{,}36 \end{bmatrix} = \begin{bmatrix} 3{,}914 \\ 7{,}134 \end{bmatrix}$ | $\Delta Y_6 = \begin{bmatrix} 3{,}914 \\ 7{,}134 \end{bmatrix}$ |



## Keep doing that infinitely ……

- How to proof that if the rounds are carried out infinitely to the level where additional outputs required are zero, then the total additional output needed can be expressed as

$$X=(I-A)^{-1}Y$$

## Here is the proof:

The total output X needed to satisfy the final demand, as shown by the above round-by-round effect is

$$\Delta X = A^0\Delta Y + A^1\Delta Y + A^2\Delta Y + \ldots + A^n\Delta Y$$
$$= (I + A^1 + A^2 + \ldots + A^n)\Delta Y.$$

Multiply the right hand side with **(I-A)**. We obtain

$$(I + A^1 + A^2 + \ldots + A^n)(I - A)\Delta Y$$
$$= (I + A^1 + A^2 + \ldots + A^n - A^1 - A^2 - \ldots - A^n - A^{n+1})\Delta Y$$
$$= (I - A^{n+1})\Delta Y = \Delta Y$$

The last expression assumes that as $n \to \infty$, $A^{n+1}$ will approach zero.
Since $(I + A^1 + A^2 + \ldots + A^n)(I - A)\Delta Y = \Delta Y$

Then it must be true that $(I + A^1 + A^2 + \ldots + A^n) = (I - A)^{-1}$.

That means: We can approach the infinite round-by-round analysis with the Leontief inverse



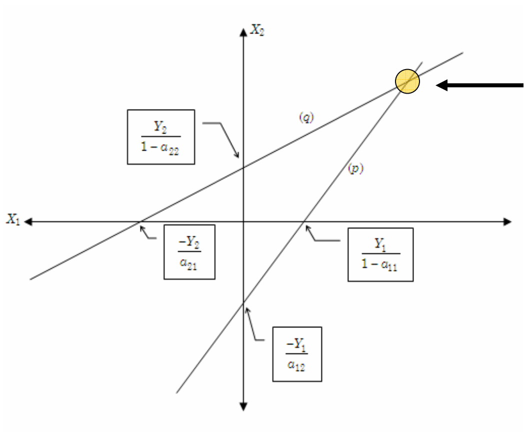
## Because of the direct effect

- Values of the main diagonal of Leontief inverse must be greater than 1

## Graphical representation of system solution

In 2-sector model, the system of equation is:

$$\begin{cases} (1-a_{11})X_1 - a_{12}X_2 = Y_1 \\ -a_{21}X_1 + (1-a_{22})X_2 = Y_2 \end{cases}$$




Graphically, we need the solution in Quadrant I. Solution for both inputs must be positive.

Both equations can be written as:

$X_2 = f(X_1)$

In order to have solution in Quadrant I, the gradient of each line must satisfy certain conditions

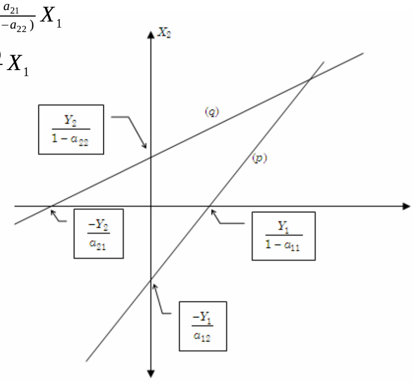


## The conditions for solution

Two line equations:

$$-a_{21}X_1 + (1-a_{22})X_2 = Y_2 \quad \rightarrow \quad X_2 = \frac{1}{(1-a_{22})}Y_2 + \frac{a_{21}}{(1-a_{22})}X_1$$

$$(1-a_{11})X_1 - a_{12}X_2 = Y_1 \quad \rightarrow \quad X_2 = \frac{1}{a_{21}}Y_1 + \frac{(1-a_{11})}{a_{12}}X_1$$




Must satisfy that:

$$\frac{1-a_{11}}{a_{12}} > \frac{a_{21}}{1-a_{22}}$$

$$(1-a_{11})(1-a_{22}) - a_{12}a_{21} > 0\,.$$

Two components Must be positive

This is determinant of matrix A, so that **| I – A | > 0**

**Hawkin-Simons Condition**