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AN ALGORITHM FOR STRAPDOWN INERTIAL NAVIGATION SYSTEM

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Introduction

The inertial navigation algorithms use measurements by accelerometers and gyroscopic angular rate sensors (ARS) called inertial sensors [2]. For solving the navigation problem it is desirable to install the inertial sensors closest to the flying vehicle (FV) mass center. Thus, after the first integration of the accelerometer readings we obtain FV velocities of movement. A second integration of the velocities obtained will give a path traveled or the FV location. The platform inertial systems use the gyroscopic sensors for stabilizing the platform (with the accelerometers fixed on it) on a horizontal position [1].

Equations of accelerometers placed on a horizontal platform

Let three accelerometers, assumed to be ideal (with no instrumental errors) be placed on some platform, rotating in relation to the inertial space at an absolute angular rate of $\vec{\omega}^{a}_{xyz}$. Besides, we assume that their axes of sensitivity are mutually perpendicular. From here onward we shall assume that they are placed at one point in space or we neglect their dimensions.

Vector $\vec{a}(t)$ of their readings will not depend on the angular rate $\vec{\omega}^{a}_{xyz}(t)$ of the platform and will be equal to:

$$\vec{a}(t) = \vec{\varpi}(t) - \vec{g}(t), \tag{1}$$

where: $\vec{\varpi}(t)$ is the absolute acceleration of a point of the object at which the accelerometers are placed; $\vec{g}(t)$ is a vector of the gravity field intensity at this point.

The absolute acceleration $\vec{\varpi}(t)$ is the derivative of the vector of the absolute rate $\vec{V}(t)$, taken in relation to the absolute (inertial) space

$$\vec{\varpi}(t) = \frac{d}{dt}\vec{V}(t), \tag{2}$$

where $\frac{d}{dt}$ is a symbol for the absolute derivative.

The platform is considering that to be connected with the coordinate system *xyz* whose axes pass through the accelerometer sensitivity axes. According to the theorem of the connection of the absolute derivative with the derivative of that same vector in relation to a moving coordinate system we have:

$$\vec{\varpi}(t)=\frac{d}{dt}\vec{V}(t)=\frac{\tilde{d}\vec{V}}{dt}+\vec{\omega}^{a}_{xyz}\times\vec{V}, \quad (3)$$

where $\frac{\tilde{d}}{dt}$ is a symbol for a derivative of the vector in the coordinate system *xyz*; $\vec{\omega}^{a}_{xyz}(t)$ is the angular rate of the system *xyz* (angular rate of the platform).

The vector equation (3) has a scalar form:

$$\begin{aligned}\varpi_x &= \dot{V}_x+\omega^a_y V_z-\omega^a_z V_y,\\ \varpi_y &= \dot{V}_y-\omega^a_x V_z+\omega^a_z V_x,\\ \varpi_z &= \dot{V}_z+\omega^a_x V_y-\omega^a_y V_x,\end{aligned} \quad (4)$$

where: V_x, V_y, V_z are projections of the vector of the absolute rate $\vec{V}$ along the moving axes x, y, z; $\omega^a_x, \omega^a_y, \omega^a_z$ are projections of the vector of the absolute angular rate $\vec{\omega}^{a}_{xyz}$ of the platform along the moving axes x, y, z.

In this case, as it follows from (1) and (4) the accelerometer readings will be:

$$\begin{aligned}a_x &= \varpi_x-g_x=\dot{V}_x+\omega^a_y V_z-\omega^a_z V_y-g_x,\\ a_y &= \varpi_y-g_y=\dot{V}_y-\omega^a_x V_z+\omega^a_z V_x-g_y,\\ a_z &= \varpi_z-g_z=\dot{V}_z+\omega^a_x V_y-\omega^a_y V_x-g_z.\end{aligned} \quad (5)$$

Equations (5) are called equations of a triplet of accelerometers placed on a rotating platform. Sometimes it is convenient to use them in a matrix form:

$$\begin{pmatrix}a_x\\ a_y\\ a_z\end{pmatrix}=\begin{pmatrix}\dot{V}_x\\ \dot{V}_y\\ \dot{V}_z\end{pmatrix}+\begin{pmatrix}0 & -\omega^a_z & \omega^a_y\\ \omega^a_z & 0 & -\omega^a_x\\ -\omega^a_y & \omega^a_x & 0\end{pmatrix}\begin{pmatrix}V_x\\ V_y\\ V_z\end{pmatrix}-\begin{pmatrix}g_x\\ g_y\\ g_z\end{pmatrix}. \quad (6)$$

There is another, ***second form of the equation***, for presenting the readings of accelerometers placed on a platform. In this form apparently participate projections W_x, W_y, W_z of the vector $\vec{W}$ of the ground speed and not projections V_x, V_y, V_z of the vector $\vec{V}$ of the absolute rate.

Let us take the absolute derivatives from the left-hand and right-hand part of the expression:

$$\vec{V}=\vec{W}+\vec{\Omega}\times\vec{R}, \quad (7)$$

where: $\vec{R}$ is the radius-vector of point *M* (the mass centre which in this case is the beginning of the basis *xyz*); $\vec{\Omega}\times\vec{R}$ is the linear peripheral velocity obtained as a result of the 24-hour rotation of the Earth. The result of the differentiation of (7) is:

$$\frac{d\vec{V}}{dt} = \vec{\varpi} = \frac{d\vec{W}}{dt} + \vec{\Omega} \times \frac{d\vec{R}}{dt} = \frac{d\vec{W}}{dt} + \vec{\Omega} \times \vec{V}. \quad (8)$$

In the latter equation the absolute derivative $\frac{d\vec{W}}{dt}$ will be presented in the form:

$$\frac{d\vec{W}}{dt} = \frac{\tilde{d}\vec{W}}{dt} + \vec{\omega}_{xyz}^{a} \times \vec{W}, \quad (9)$$

where: $\frac{\tilde{d}\vec{W}}{dt}$ is a symbol of a derivative of the vector in the coordinate system xyz connected with the platform.

By replacing of (7) and (9) in (8) we obtain:

$$\vec{\varpi} = \frac{\tilde{d}\vec{W}}{dt} + \vec{\omega}_{xyz}^{a} \times \vec{W} + \vec{\Omega} \times \vec{W} + \vec{\Omega} \times (\vec{\Omega} \times \vec{R}). \quad (10)$$

If in the latter vector equation the absolute angular rate $\vec{\omega}_{xyz}^{a}$ of the platform is expressed as the sum of the relative $\vec{\omega}_{xyz}$ and transfer rate $\vec{\Omega}$:

$$\vec{\omega}_{xyz}^{a} = \vec{\omega}_{xyz} + \vec{\Omega}, \quad (11)$$

then we obtain:

$$\vec{\varpi}(t) = \frac{\tilde{d}\vec{W}}{dt} + \vec{\omega}_{xyz} \times \vec{W} + 2\vec{\Omega} \times \vec{W} + \vec{\Omega} \times (\vec{\Omega} \times \vec{R}). \quad (12)$$

In this case vector $\vec{a}(t)$ of the readings of a triplet of accelerometers can be written in the form:

$$\vec{a}(t) = \vec{\varpi}(t) - \vec{g}(t) = \frac{\tilde{d}\vec{W}}{dt} + \vec{\omega}_{xyz} \times \vec{W} + 2\vec{\Omega} \times \vec{W} - \vec{g}^{T}, \quad (13)$$

where: $\vec{g}^{T}(t) = \vec{g}(t) - \vec{\Omega} \times (\vec{\Omega} \times \vec{R})$ is a free fall acceleration.

By switching from a vector form (13) to a scalar and matrix form, we obtain:

$$\begin{aligned}
a_x &= \varpi_x - g_x = \dot{W}_x + (\omega_y + 2\Omega_y)W_z - (\omega_z + 2\Omega_z)W_y - g_x^T, \\
a_y &= \varpi_y - g_y = \dot{W}_y - (\omega_x + 2\Omega_x)W_z + (\omega_z + 2\Omega_z)W_x - g_y^T, \\
a_z &= \varpi_z - g_z = \dot{W}_z + (\omega_x + 2\Omega_x)W_y - (\omega_y + 2\Omega_y)W_x - g_z^T,
\end{aligned} \quad (14)$$

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} \dot{W}_x \\ \dot{W}_y \\ \dot{W}_z \end{pmatrix} + \left[\begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{pmatrix} \right] \begin{pmatrix} W_x \\ W_y \\ W_z \end{pmatrix} - \begin{pmatrix} g_x^T \\ g_y^T \\ g_z^T \end{pmatrix}. \quad (15)$$

Equations of accelerometers fixed fast to the fuselage (straptdown variant)

Let us consider the case when the platform is fixed fast to the fuselage. Suppose the sensitivity axes *xyz* of the triplet of accelerometers coincide with the construction lines x_1 y_1 z_1 of the plane and three gyroscopic angular rate sensors measure the projections $\omega^a_{x1}, \omega^a_{y1}, \omega^a_{z1}$ of the vector of the absolute angular rate $\vec{\omega}^a_{x1y1z1}$ of the plane. Again we neglect the dimensions of the gyroscopes and assume that they are placed in the plane mass center (point *M*), their axes of sensitivity being oriented along x_1, y_1 and z_1.

According to the assumptions made above, equations (5) of the triplet of accelerometers will be:

$$\begin{aligned} a_{x1} &= \dot{V}_{x1} + \omega^a_{y1} V_{z1} - \omega^a_{z1} V_{y1} - g_{x1}, \\ a_{y1} &= \dot{V}_{y1} - \omega^a_{x1} V_{z1} + \omega^a_{z1} V_{x1} - g_{y1}, \\ a_{z1} &= \dot{V}_{z1} + \omega^a_{x1} V_{y1} - \omega^a_{y1} V_{x1} - g_{z1}, \end{aligned} \tag{16}$$

from where, by using matrix Π_a composed of the gyroscope readings, we obtain a matrix form:

$$\begin{pmatrix} a_{x1} \\ a_{y1} \\ a_{z1} \end{pmatrix} = \begin{pmatrix} \dot{V}_{x1} \\ \dot{V}_{y1} \\ \dot{V}_{z1} \end{pmatrix} + \Pi_a \begin{pmatrix} V_{x1} \\ V_{y1} \\ V_{z1} \end{pmatrix} - \begin{pmatrix} g_{x1} \\ g_{y1} \\ g_{z1} \end{pmatrix}, \tag{17}$$

$$\text{for } \Pi_a = \begin{pmatrix} 0 & -\omega^a_{z1} & \omega^a_{y1} \\ \omega^a_{z1} & 0 & -\omega^a_{x1} \\ -\omega^a_{y1} & \omega^a_{x1} & 0 \end{pmatrix} \tag{18}$$

in which g_{x1}, g_{y1}, g_{z1} are projections along the plane construction axes of the vector $\vec{g}(t)$ of intensity of the gravity field at point *M*.

SINS algorithm

The **Strapdown Inertial Navigation Systems** (SINS) use inertial sensors fixed fast to the fuselage. The plane absolute angular rate $\omega^a_{x1}, \omega^a_{y1}, \omega^a_{z1}$ measured by gyroscopic sensors is entered in unit 2. The accelerometer measurements a_{x1}, a_{y1}, a_{z1} are entered in unit 3. We shall present an algorithm of the ideal SINS operation as a list of formulae grouped in units according to their purpose:

Unit №1: Calculating the angular rates ω_ξ, ω_η, ω_ζ of rotation of the navigation frame of reference $\xi\eta\zeta$ in relation to the Earth, as well as its absolute angular rates $\omega^a{}_\xi, \omega^a{}_\eta, \omega^a{}_\zeta$:

$$
\begin{aligned}
&\xi^{-2} = 1 - e^2 u_{33}^2, \quad G = a\xi + h, \quad Q = a(1-e^2)\xi^3 + h,\\
&\delta = ae^2\xi^3 Q^{-1} G^{-1}, \\
&\omega_\xi = -G^{-1}W_\eta - u_{23}(W_\xi u_{13} + W_\eta u_{23})\delta, \quad \omega_\xi^a = \omega_\xi + \Omega u_{13},\\
&\omega_\eta = G^{-1}W_\xi - u_{13}(W_\xi u_{13} + W_\eta u_{23})\delta, \quad \omega_\eta^a = \omega_\eta + \Omega u_{23},\\
&\omega_\zeta = 0, \qquad \omega_\zeta^a = \omega_\zeta + \Omega u_{33}.
\end{aligned} \tag{19}
$$

The following are input quantities: W_ξ, W_η и W_ζ , entered from unit №3, as well as u_{13},u_{23},u_{33}, entered from unit №4. This unit estimates the Earth's shape as a reference-ellipsoid with a semi-minor axis *b*, semi-major axis *a* and square of eccentricity $e^2=(a^2-b^2)/a^2$=0.0066934. The flight altitude *h* above the reference-ellipsoid is also estimated as well as the speed Ω of the 24-hour rotation of the Earth. From the calculated angular rates in this unit we construct skew-symmetric matrices:

$$
\Pi = \begin{pmatrix} 0 & -\omega_\zeta & \omega_\eta \\ \omega_\zeta & 0 & -\omega_\xi \\ -\omega_\eta & \omega_\xi & 0 \end{pmatrix}, \qquad \Pi^a = \begin{pmatrix} 0 & -\omega_\zeta^a & \omega_\eta^a \\ \omega_\zeta^a & 0 & -\omega_\xi^a \\ -\omega_\eta^a & \omega_\xi^a & 0 \end{pmatrix}. \tag{20}
$$

Unit №2: Calculating the spatial position of the flying vehicle (FV). This unit integrates the matrix equation for the matrix of the direction cosines *D* [4]:

$$
\dot{D} = D\Pi_a - \Pi^a D, \quad D(t_0) = D_0[\psi_{ж c0}, \gamma_0, \vartheta_0]. \tag{21}
$$

Thus for calculation of a matrix of direction cosines is used the **Poisson's expanded equation** [4]. Its elements are used to calculate the angles of roll, gyroscopic heading and pitch:

$$
\begin{aligned}
&\gamma = Arctg\frac{-d_{33}}{d_{32}}[-\pi,\pi], \psi_c = Arctg\frac{d_{11}}{d_{21}}[0,2\pi],\\
&\vartheta = arctg\frac{d_{31}}{\sqrt{d_{32}^2+d_{33}^2}}[-\frac{\pi}{2},\frac{\pi}{2}],
\end{aligned} \tag{22}
$$

where : $D=$ (23)

$$
=\begin{pmatrix} \cos\vartheta\sin\psi_{жc} & \sin\gamma\cos\psi_{жc} - \cos\gamma\sin\vartheta\sin\psi_{жc} & \cos\gamma\cos\psi_{жc} + \sin\vartheta\sin\gamma\sin\psi_{жc} \\ \cos\vartheta\cos\psi_{жc} & -\sin\gamma\sin\psi_{жc} - \cos\gamma\sin\vartheta\cos\psi_{жc} & -\cos\gamma\sin\psi_{жc} + \sin\vartheta\sin\gamma\cos\psi_{жc} \\ \sin\vartheta & \cos\gamma\cos\vartheta & -\sin\gamma\cos\vartheta \end{pmatrix}.
$$

This unit forms the skew-symmetric matrix (18) from data entered about the absolute angular rate $\omega_{x1}^a, \omega_{y1}^a, \omega_{z1}^a$ of the FV. Matrix Π^a has already been calculated in Unit №1.

Unit №3: Calculating the ground speeds of movement of the FV mass center Data are entered about measurements made by accelerometers a_{x1}, a_{y1}, a_{z1} along the three construction axes $\vec{x}_1 \vec{y}_1 \vec{z}_1$ of the plane. Using the direction cosine matrix

D already calculated in unit №2 their projections performed on the axes $\vec{\xi}\vec{\eta}\vec{\zeta}$ of the navigation frame of reference:

$$\begin{pmatrix} a_\xi \\ a_\eta \\ a_\zeta \end{pmatrix} = D \begin{pmatrix} a_{x1} \\ a_{y1} \\ a_{z1} \end{pmatrix}. \qquad (24)$$

From a normal Earth's gravity field model we take projections g_ξ^T, g_η^T, g_n^T of the relative weight force.

The ground speeds of movement of the FV mass center are obtained as a result of integrating the equations:

$$\begin{aligned} \dot{W}_\xi &= a_\xi - (\omega_\eta + 2\Omega u_{23})W_\zeta + 2\Omega u_{33}W_\eta + g_\xi^T; \quad W_\xi(t_0) = W_\xi(0); \\ \dot{W}_\eta &= a_\eta + (\omega_\zeta + 2\Omega u_{13})W_\zeta - 2\Omega u_{33}W_\xi + g_\eta^T; \quad W_\eta(t_0) = W_\eta(0); \\ \dot{W}_\zeta &= a_\zeta - (\omega_\xi + 2\Omega u_{13})W_\eta + (\omega_\eta + 2\Omega u_{23})W_\xi + g_n^T; W_\zeta(t_0) = W_\zeta(0). \end{aligned} \qquad (25)$$

Unit №4: Calculating the location of the FV mass center From the matrix *П* (20) formed in Unit №1 we solve the matrix differential equation for a given initial location:

$$\dot{U} = -\Pi U, \quad U(t_0)=U[B(t_0), L(t_0), A(t_0)] \qquad A(t_0)=0; \qquad (26)$$

For calculation of a matrix of direction cosines *U* is used the **Poisson's normal equation** [1]. The elements of matrix *U* are used in Unit №1 also for calculating the geodetic latitude *B*, longitude *L* and azimuth *A* of the navigation frame of reference:

$$U = \begin{pmatrix} -\cos A \sin L + \sin A \sin B \cos L & \cos A \cos L + \sin A \sin B \sin L & -\sin A \cos B \\ -\sin A \sin L - \cos A \sin B \cos L & \sin A \cos L - \cos A \sin B \sin L & \cos A \cos B \\ \cos B \cos L & \cos B \sin L & \sin B \end{pmatrix},$$

$$B = arctg \frac{u_{33}}{\sqrt{u_{13}^2 + u_{32}^2}}, \quad L = Arctg \frac{u_{32}}{u_{13}}, \quad A = Arctg\left(-\frac{u_{13}}{u_{23}}\right) \qquad (27)$$

The true heading, estimated in relation to the geographical north, is calculated according to:

$$\psi = \psi_{жс} + A. \qquad (28)$$

The SINS operation algorithm described above is a closed system of differential equations. Its phase vector consists of the elements of matrices *D*, *U* and the ground speeds W_ξ, W_η, W_ζ. In Unit №2, by algebraic equations, we obtain the output values of the gyroscopic heading $\Psi_{жс}$, roll γ and pitch ϑ. In Unit №4 we similarly calculate the other output values: *B* - geodetic latitude; *L* -geodetic longitude and *A* -azimuth of a measured navigation frame of reference.

In the algorithm input data are: in Unit №2 data about the vector of the absolute angular rate from three mutually perpendicular angular rate sensors (ARS); in Unit №3 the projections of the apparent acceleration from three mutually perpendicular accelerometers along the same mutually perpendicular axes.

Conclusions

1. The gimbal inertial navigation system can use the same algorithm but instead of having computational unit 2, there will be a gyrostabilized platform. The angles of spatial position in this case will be taken as angles in the platform gimbal suspension. The platform itself is maintained horizontal and is forced to process (together with the inertial sensors fixed on it) with the values ($\omega^a{}_\xi, \omega^a{}_\eta, \omega^a{}_\zeta$) of the absolute angular rates calculated in Unit 1.

2. The gyrostabilized platform has a considerable volume because of the need for using gimbal suspension [2]. As a sophisticated electronic-mechanical system it has a high price comparable to the price of a light-flying vehicle. It is mainly for this reason that it is not suitable for gravimetric measurements.

3. Investigations on a strapdown variant of using inertial sensors are more difficult to perform, because it is necessary to take out from the accelerometer readings the inertial components of the plane movement around its mass center. But this is also necessary to be done when the inertial platform is positioned beside the mass center. The presence of a gyrostabilized platform, however, requires that instead of equation (21) we should use an analogous differential equation for estimating the dynamic errors resulting from the stabilization of the platform. The proposed SINS algorithm contains the basic equations of inertial navigation and possesses the dynamics of every inertial system.

4. The accuracy of maintaining the geodetic vertical axis in an inertial system depends on the accuracy of the weight force model [3] used in unit 3, equations (25). In flights at an altitude of several hundred meters, which are normal in aviation practice, the influence of the geoid is neglected and the geodetic vertical axis is assumed to be normal to a rotational ellipsoid

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АЛГОРИТЪМ ЗА БЕЗПЛАТФОРМЕНА ИНЕРЦИАЛНА НАВИГАЦИОННА СИСТЕМА

А. Н. МАДЖАРОВ

Статията представя математически зависимости в матрична диференциална форма за изчисляване на ъглите на крен, жироскопически курс и тангаж. Безплатформената инерциална система (БИНС) заменя универсалната жироплатформа с компютър, който моделира нейното присъствие виртуално. В БИНС жироскопите и акселерометрите са твърдо захванати за корпуса на транспортното средство, така че приемат всички негови движения. Обикновено, системите използват направляващи косинуси и преобразувание между инерциална координатна система и базис, твърдо свързан с корпуса на самолета. Тук се предлага изчислителна процедура за обработка на данните от датчици на ъглова скорост. По такъв начин, за изчисляване на матрицата на направляващи косинуси ще се използва разширеното, вместо обикновеното Уравнение на Пуасон.

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АЛГОРИТМ ДЛЯ БЕСПЛАТФОРМЕННОЙ ИНЕРЦИАЛЬНОЙ НАВИГАЦИОННОЙ СИСТЕМЫ

А. Н. МАДЖАРОВ

Статья представляет математические формулировки в матричной дифференциальной форме для вычисления угловых амплитуд бортовой качки, гироскопического курса и тангажа. Бесплатформенная система заменяет универсальную жироплатформу компьютером, который моделирует её присутствие виртуально. В бесплатформенной системе гироскопы и акселерометры твердо установлены к структуре транспортного средства так, чтобы они двигались с транспортным средством. Типичные системы используют направляющие косинусы и преобразование между инерциальной системы координат и трёхгранник, жестко связанный с самолетом. Здесь предлагается вычислительную процедуру обработки данных с измерителями угловой скорости. Таким образом, для вычисления матрицы направляющих косинусов будет использоваться расширенным, вместо нормального Уравнения Пуассона.

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The paper presents mathematical formulations in matrix differential form for calculating the angles of roll, gyroscopic heading and pitch. The strapdown system replaces gimbals with a computer that simulates their presence virtually. In the strapdown system the gyroscopes and accelerometers are rigidly mounted to the vehicle structure so that they move with the vehicle. Typical systems use direction cosines and the transformation between the inertial reference and body axes reference. Here is offered computing procedure of a data processing with spin-rate meters. Thus for calculation of a matrix of direction cosines will be used expanded, instead of a normal Poisson equation.