

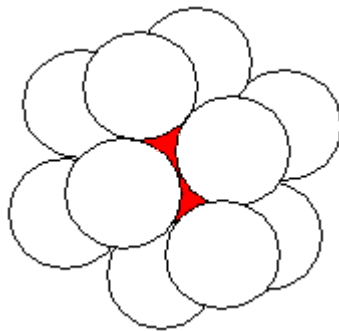
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Nature's Way of Measuring

(inspired by Buckminster Fuller)

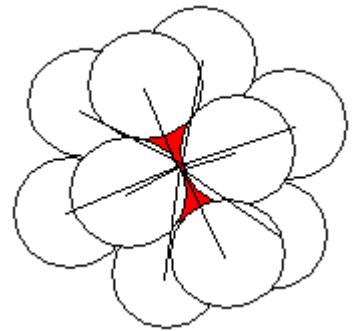
Chapter One The Isotropic Vector Matrix



The Isotropic Vector Matrix comes from the closest packing of unit radius spheres. Each sphere within the isotropic vector matrix has 12 surrounding spheres.

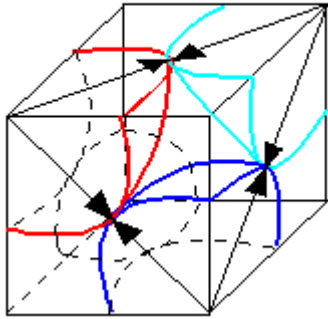
Connecting the centers of each of these 12 surrounding spheres to the center of the nuclear sphere are 12 double radii radiating from the nuclear sphere, each 2 adjacent radii within a hexagonal plane being separated by

60° . This angle of 60° is a property of the adjacency of the spheres.

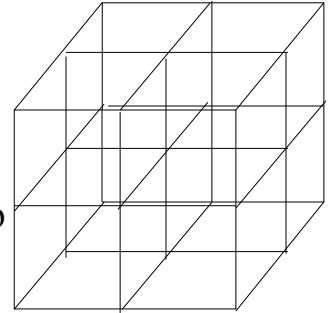


The radius of a sphere touching another sphere is defined as a vector pointing to that other sphere, and the two touching radii of the two touching spheres oppose each other and touch at the point of contact of the two spheres, and thus the name of *Vector Matrix*. (Isotropic because each sphere has vectors pointing in each of 12 directions due to the fact that each sphere is surrounded by 12 spheres.) The shape of the sphere determines the points of adjacency which is 12. Connecting these 12 points on the surface of the sphere with straight lines we come up with a cuboctahedron which is a cube with its corners sliced off.

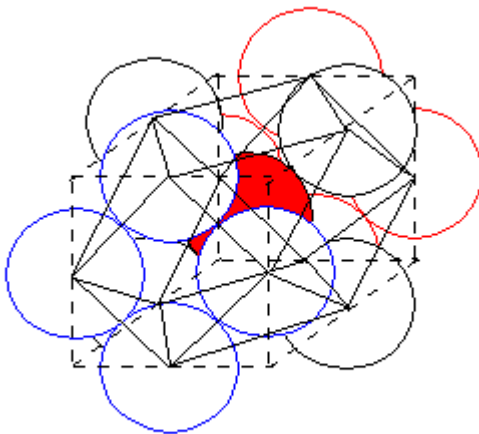
A very large cube that is divided into smaller and smaller cubes in multiples of eight is called a cubic vector matrix.



Four $\frac{1}{8}$ unit spheres within the isotropic vector matrix come together within the cubic vector matrix such that the cube's face diagonals coincide with the sphere's radii, making the diagonals two units each, made up of two opposing vectors.

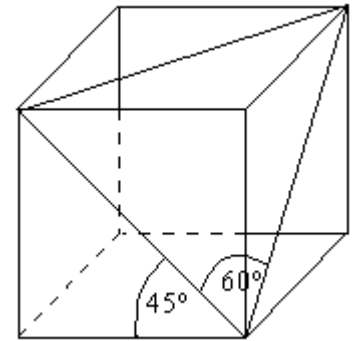


The triangle made by connecting any three corners of this cube using the radii of the spheres is an equilateral and equiangular 60° triangle. The 60° triangle is at a 45° angle to the edge of the cube, therefore, the isotropic vector matrix is separated from the cubic vector matrix by 45° . Only the diagonals of the cube's faces are set within the isotropic vector matrix.

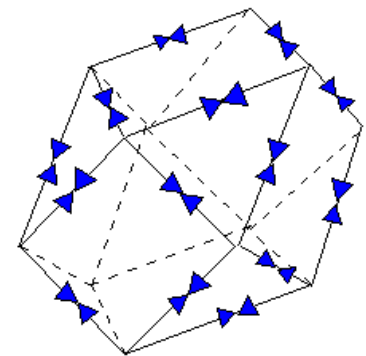


Stretched open to show the inside sphere.

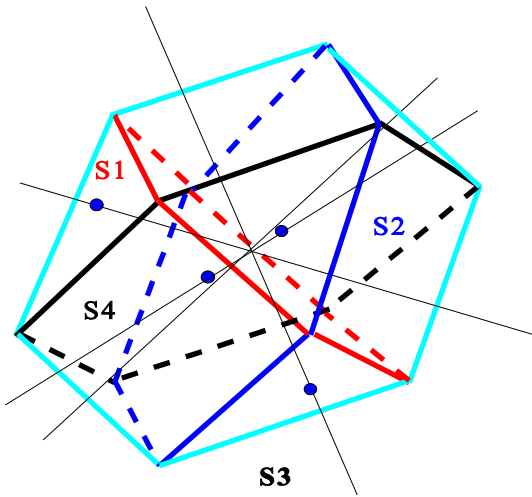
Each center of 12 spheres surrounding one sphere of the isotropic vector matrix, is located as the center point of each of the 12 edges of a cube. If these 12 spheric centers are connected along the faces of this cube, using the radii of these spheres, a cuboctahedron is formed. At each corner of the cube, a tetrahedron is removed to form the cuboctahedron.



Each edge of this cuboctahedron is formed from the double vector radii of two touching spheres, each vector opposing the other one. Adding up all the vectors results in a zero vector. Therefore, this cuboctahedron is named the **Vector Equilibrium**.



Using the cube that encases the Vector Equilibrium, a diagonal of the body of the



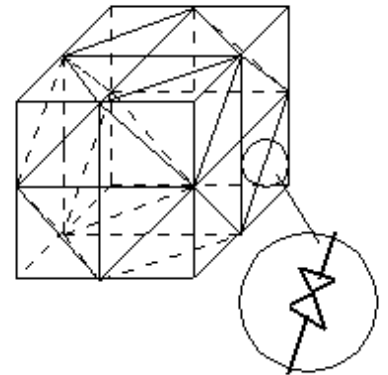
cube is from one corner to it's opposite corner. Bisecting each of these diagonals at 90° is a hexagon outlined by six edges of the Vector Equilibrium. There are four of these diagonals and four hexagons slicing through the Vector Equilibrium, each sharing the same center point with the Vector Equilibrium. There are thus $4 \times 6 = 24$ edges on the Vector Equilibrium (and thus 48 vectors).

smaller cubes, each of which have diagonals two units long.

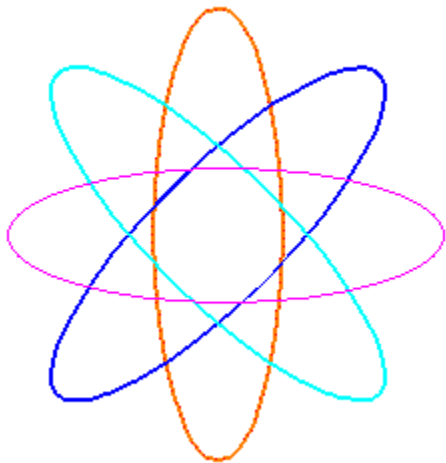
Each face of the cube encasing the Vector Equilibrium encases a 45° askew square bisecting each edge of the cube. Each one of the hexagons slicing through the Vector Equilibrium borrows one side of each askewed square from each of the 6 faces of the cube. Thus the 6 sides of the cube

times the 4 sides of one askewed square equals the 24 sides of the Vector

Equilibrium, the same as 4 hexagons times the 6 sides of one hexagon equals the 24 sides of the Vector Equilibrium.



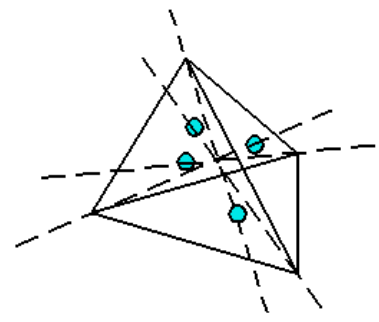
Each diagonal is actually two vectors pointing to each other.

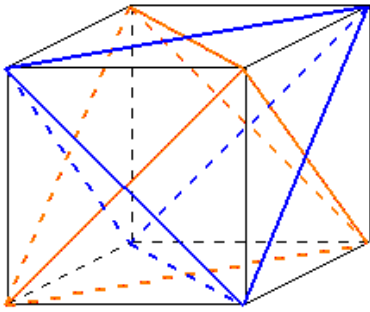


This relationship of the 12 points of contact around a sphere and the 12 centers of the 12 surrounding spheres and the 12 points of contact between the Vector Equilibrium and its encasing square are all brought together by rotating each of the hexagonal planes slicing through the Vector Equilibrium getting the four great circles of a sphere. This uses the four major axes of rotation of the Vector

Equilibrium (through the 8 corners of the cube). Each circle touches 3 other circles. Each circle has three points of contact. That is $3 \times 4 = 12$. That cements the relationships. (Also, the 3 dimensions of the sphere times the 4 great circles of the sphere equals 12.)

(Another geometric solid that has four major axes of rotation, but from face to apex, is the tetrahedron.)

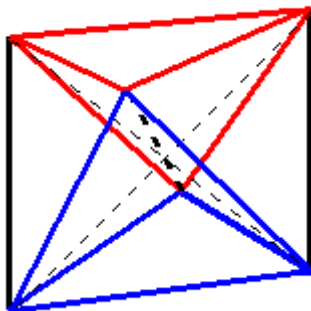
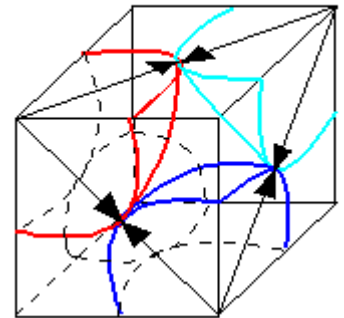




The diagonals of the six faces of a cube which only line up with the isotropic vector matrix are two interlacing tetrahedrons.

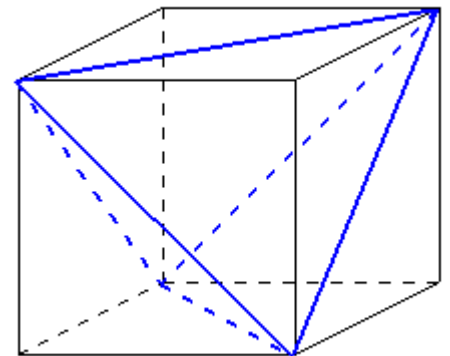
Taking one of these tetrahedrons and adding a tetrahedron to each one of the four faces, it takes five tetrahedrons to make one cube, therefore, Tetrahedron is more basic than Cube for measurement of volume. When the diagonal of the square face of the cube is unity, then the edges of the tetrahedrons are unity, giving a unit tetrahedron for using as the basic measure of volume.

Using the unit sphere with its unit radius making up half the diagonal of the cubic face, the length of the diagonal of the cube becomes 2. Using the Pythagorean Theorem, each edge of the cube then becomes $\sqrt{2}$. The volume of the cube is found by the cube of $\sqrt{2}$ or $(\sqrt{2})^3 = 2.828428$ or near 3..



The octahedron is made up of four tetrahedrons. Let each of these tetrahedrons be the unit tetrahedron. Therefore, the volume of the octahedron is 4 tetrahedrons.

Taking the 4 tetrahedrons that make up the octahedron, which has a volume of 4, and cutting each tetrahedron in half making eight $\frac{1}{2}$ - octahedrons, each having a volume of $\frac{1}{2}$, when four $\frac{1}{2}$ - octahedrons are added to each face of the unit tetrahedron, the smallest cube is created because $4(\frac{1}{2}) + 1 = 3$, an easier way of calculating the smallest cube than using the $\sqrt{2}$. (That is 5 tetrahedrons equals a volume of 3.)



Comparing this rational volume of 3 to the calculated volume using the $\sqrt{2}$, we get the

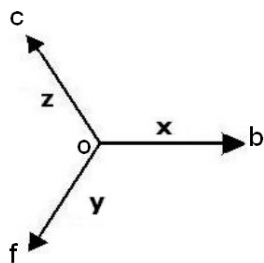
synergetics conversion factor of $\sqrt[3]{2.828428} = 1.06066$. Using this conversion factor on conventional areas and volumes, they are converted to rational areas and volumes. This conversion factor of $1.06066 = \sqrt{(9/8)}$.

For areas, 2 dimensions, $\sqrt{(9/8)}$ is triangled to become $\frac{9}{8}$.

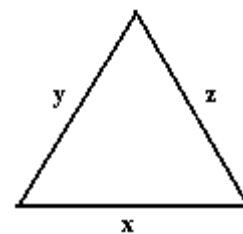
For volumes, 3 dimensions, $\sqrt{(9/8)}$ is tratedhedroned to become 1.193243.

Chapter Two

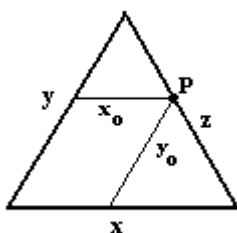
Linear Measurement



Any directional line segment or vector can be translated as long as its angle with respect to some horizontal line remains constant. Let there be 3 vectors $ob = x$, $of = y$, and $oc = z$ such that they are separated by equal angles. If the length ob is always equal to the lengths oc and of , then $x + y = z$. These three vectors can be translated to form an equilateral, equiangular triangle in which all

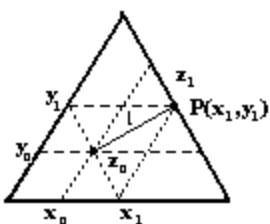
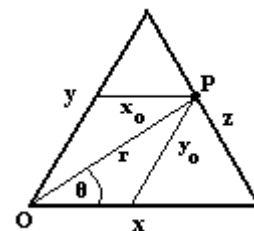


three internal angles are 60° angles.



If we take a line x_0 from y to z parallel to x and take a line y_0 from that intersection of x_0 and z to x parallel to y , z is divided into two segments at a point $P(x_0, y_0)$ on z , and there is an equilateral, equiangular triangle with all sides equal to x_0 above point P , and an equilateral, equiangular triangle with all sides equal to y_0 below point P such that z is divided by these two triangles into x_0 and y_0 . x_0 and y_0 are called the coordinates of P on z . The two portions x_0 and y_0 of z add up to form z , so $x_0 + y_0 = z$.

Now from the origin O (where x and y come together), draw a line r to the point P . Call x_0 and y_0 the components of r such that the distance of r from O to P is $r = x_0 \cos\theta + y_0 \sin\theta$, where $0^\circ \leq \theta \leq 60^\circ$. As θ approaches 0° , r approaches x_0 , and as θ approaches 60° , r approaches y_0 .



For an arbitrary line segment l in the middle of the triangle, we draw a z_0 at the lower end of l and z_1 at the other end of l . (In this case, the z of the triangle encasing l is z_1 .) l divides z_0 into $(x_1 - x_0)$ and $(y_1 - y_0)$ and divides z_1 into x_1 and y_1 . Extending y_1 to P , such that $y_1P = x_1$, and extending y_0 to z_1 , such that x_0P' intersects y_1P , and extending x_1 to P such that $x_1P = y_1$, then the intersecting lines create a parallelogram with sides $(x_1 - x_0)$ on the bottom and top and $(y_1 - y_0)$ on each side with l as the diagonal. This is like taking z_0 and bending it around l above and below to extend z_0 to z_1 at P . Unbending the sides of the parallelogram to make straight lines, the triangle with sides x_1 and y_1 is duplicated with the new origin on z_0 and the far side opposite lying on z_1 . It is the same triangle as $O x_1 y_1$ with l going from the origin O out to the side z_0 . The length of l then is the same as the length r in the above equation.

Taking the above equation for a line,

$$l = x_1 \cos\theta + y_1 \sin\theta, \text{ where } 0^\circ \leq \theta \leq 60^\circ.$$

l is rotated between x_1 and y_1 .

If l is treated as a vector, then

$$l = (x_1 - x_0) + (y_1 - y_0).$$

The part of l that is within the hexagon which extends from the middle of z_0 is

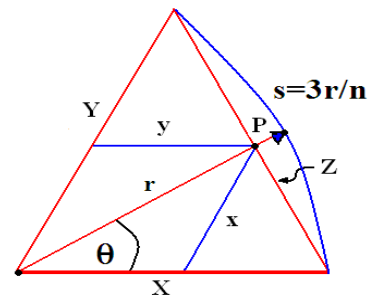
$$l = (x_1 - x_0) \cos\theta + (y_1 - y_0) \sin\theta .$$

This agrees with the definition of a vector, that is, $v = i(\cos\theta) + j(\sin\theta)$ where v is a unit vector. Multiplying both sides of the equation by a constant c , where $u = cv$, $u = ci(\cos\theta) + cj(\sin\theta)$. If $x = ci$ and $y = cj$, then $u = x \cos\theta + y \sin\theta$. (It must be remembered that the trigonometric functions used here are based upon the 60° coordinate system, so the numbers will be different.)

$x + y$ is the length of the sides of the equilateral, equiangular triangle that encases the line l . $x \cos\theta + y \sin\theta$ is the length of any other line within the triangle.

If $\theta > 60^\circ$, then l is being rotated into the next hexant, and the calculations of length are similar, and is similar for each of the six hexants.

Another line r that can be considered has the same length of any side, say, x . When it is rotated about the angle θ , then r intersects z at any point P as it rotates from 0° to 60° . The coordinates of P are the components x and y of r such that $x + y = z$, and $x + y = r$ since $r = x = y = z$.



To add r_1 and r_2 ,

$$r_1 = x_1 + y_1, \text{ and}$$

$$r_2 = x_2 + y_2, \text{ so}$$

$$r_1 + r_2 = x_1 + y_1 + x_2 + y_2 = (x_1 + x_2) + (y_1 + y_2) \text{ which adds up to a larger triangle.}$$

$$r_1 - r_2 = (x_1 + y_1) - (x_2 + y_2) = (x_1 - x_2) + (y_1 - y_2) \text{ which is a decrease in the triangle.}$$

Looking at the previous figure, you can see that $z_0 = x_0 + y_0$ and that the x_0 and y_0 are components of z_1 . Seeing that

$$z_1 = x_0 + (x_1 - x_0) + y_0 + (y_1 - y_0) \text{ and,}$$

$$r = x_1 = y_1 \text{ then,}$$

$$r = z_1 - z_0 = x_0 + (x_1 - x_0) - x_0 + y_0 + (y_1 - y_0) - y_0 \text{ and,}$$

$$r = z_1 - z_0 = (x_1 - x_0) + (y_1 - y_0) \text{ so,}$$

$$r = z_1 - z_0.$$

The Distance Formula

Instead of, as in a 90° coordinate system, the distance formula being

$$d = \sqrt{[(x_1 - x_0)^2 + (y_1 - y_0)^2]},$$

inside an equilateral, equiangular triangle,

$$d = (x_1 - x_0)\cos \theta + (y_1 - y_0)\sin \theta \text{ where } 0^\circ \leq \theta \leq 60^\circ, \text{ or}$$

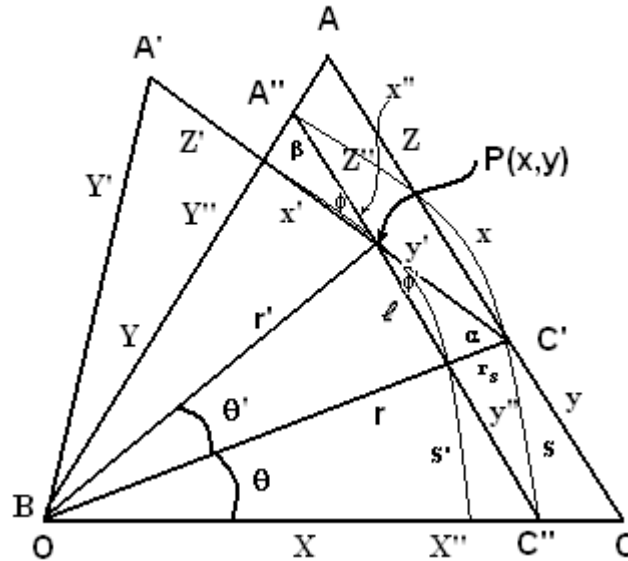
when r is rotated outside the triangle

$$d = r_1 + r_2 = (x_1 + x_2) + (y_1 + y_2) \text{ or,}$$

$$d = r_1 - r_2 = (x_1 - x_2) + (y_1 - y_2) \text{ or,}$$

$$d = r = z_1 - z_0 .$$

The Length of r



Within an equilateral triangle ABC with sides X, Y, and Z, draw a line r from O to the opposite side Z.

Line r is the base of another equilateral triangle A'BC' and is also a radius of arc s .

Swing a copy of r down to X so that r coincides with X. From the endpoint of r , draw a line l parallel to Z up to Y. Line l completes another equilateral triangle A''BC''. The triangle A'BC' therefor is only the triangle A''BC'' rotated at an angle of θ .

Therefore, $Z' = Z''$, $r = X''$, and $Y' = Y''$ by definition.

Draw a line r' from O to Z' where triangles A'BC' and A''BC'' intersect at point P(x,y).

Because Z' crosses Z'', produce angles ϕ and ϕ' , the angles ϕ and ϕ' are equal.

Radius r' divides Z' into x' and y' and Z'' into x'' and y'' .

All angles of equilateral triangles are equal, so α and β are equal by definition.

Since $Z' = Z''$, $y' / x' = x'' / y''$.

The distance between l and Z is r_s . The distance between the two arcs s and s' of which both extend from X'' to Y'' is also r_s .

Angle α is made up of sides y' and r_s , and angle β is made up of sides x'' and r_s .

Because $y' / x' = x'' / y''$ and $\alpha = \beta$ and $\phi = \phi'$, then the triangles A'A''P(x,y) and C'C''P(x,y) are congruent.

The triangles $A'A''P(x,y)$ and $C'C''P(x,y)$ are both shortened by sides r_s . We can call the side opposite ϕ , y_s .

Because $\alpha = \beta$ and $\phi = \phi'$ and $y_s = r_s$, then $y' = x''$.

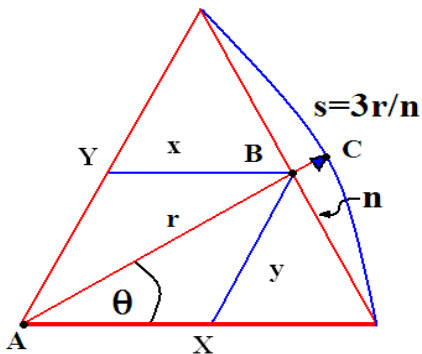
If $y' = x''$ and $Z' = Z''$, then $x' = y''$.

By definition, $Z'' = X'' = Z' = r$.

Therefore, $r = x' + y' = x'' + y''$.

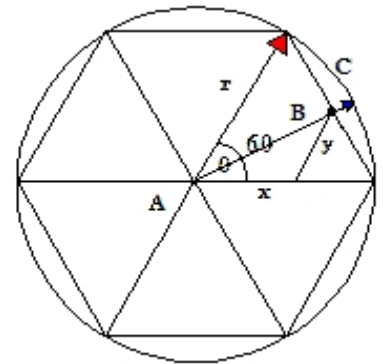
Angles

Starting with an equiangular, equilateral triangle, let there be a vector extending from the lower left corner through the opposite side such that the vector's length is the same as that of any side of the triangle.



Referring to the figure to the left, at the point C , the length of the vector $r = X = Y$. But at the point B , the full length of $r = x + y$. (In the figure, $n = y$.)

The figure to the right shows a hexagon. It represents the cycle of the circle based upon the chords of the hexagon instead of the ratio of the circumference to the diameter, π . Using a unit triangle where each side is one unit, and therefore, each chord is one unit, the circumference of the circle is divided into 6 parts. Let one rotation of r , beginning at the diameter of the circle or the x -axis, stand for one cycle. Each chord represents $1/6^{\text{th}}$ of a cycle. Then it is logical to keep that division of 6 and divide each chord into 6 equal parts or $1/36^{\text{th}}$ of the hexagon. Then there would be 36 divisions of the circle. If there are then 10 divisions between each $1/36^{\text{th}}$ mark, there will be 360 divisions around the hexagon. Extending those divisions to the enclosing circle, the circle then receives 360 divisions. The angle θ can be represented by these divisions being projected onto the circle, which would be 360° . Divisions on the hexagon would be 360 radians. Using radians, larger divisions would be in terms of $n/36$, and smaller divisions, $n/360$.



The trigonometric functions can be defined as follows:

$$\sin \theta = y/(x+y)$$

$$\cos \theta = x/(x+y)$$

$$\tan \theta = y/x$$

$\cot \theta = x/y$,
 and $\theta = \tan^{-1} y/x$ where y will always be in divisions of $n/36$ or $n/360$.

A trigonometric table within the unit triangle of the hexagon using radian measure s based upon $1/36^{\text{th}}$ of a cycle and $1/360^{\text{th}}$ of a cycle:

s	$\text{Sin } s$	$\text{Cos } s$	$\text{Tan } s$
0	0	1	0
$1/360, 1/36$	$1/360, 1/36$	$359/360, 35/36$	$1/359, 1/35$
$1/180, 1/18$	$1/180, 1/18$	$179/180, 17/18$	$1/179, 1/17$
$1/120, 1/12$	$1/120, 1/12$	$119/120, 11/12$	$1/119, 1/11$
$1/90, 1/9$	$1/90, 1/9$	$89/90, 8/9$	$1/89, 1/8$
$1/72, 5/36$	$1/72, 5/36$	$71/72, 31/36$	$1/71, 5/31$
$1/60, 1/6$	$1/60, 1/6$	$59/60, 5/6$	$1/59, 1/5$
$7/360, 7/36$	$7/360, 7/36$	$353/360, 29/36$	$7/353, 7/29$
$1/45, 2/9$	$1/45, 2/9$	$44/45, 7/9$	$1/44, 2/7$
$1/40, 1/4$	$1/40, 1/4$	$39/40, 3/4$	$1/39, 1/3$
$1/36, 5/18$	$1/36, 5/18$	$35/36, 13/18$	$1/35, 5/13$
$1/30, 1/3$	$1/30, 1/3$	$29/30, 2/3$	$1/29, 1/2$
$1/20, 1/2$	$1/20, 1/2$	$19/20, 1/2$	$1/19, 1$
$1/10, 1$	$1/10, 1$	$9/10, 0$	$1/9, +1$

Notice that in the Tangent column above all the denominators are one less than the denominators in the s column, and also in the Tangent column, the numerator added to the denominator equals the denominator in the s column.

$36 * 1/360$ segments = $1/10$ ($\sin 6^\circ$)

$18 * 1/360$ segments = $1/20$ ($\sin 3^\circ$)

$12 * 1/360$ segments = $1/30$ ($\sin 2^\circ$), $5 * 1/30 = 1/6$, and then $6 * 1/6 = 1$

$10 * 1/360$ segments = $1/36$, and then $6 * 1/36$ segments = $1/6$. Then $6 * 1/6$ segments = 1 unit

$09 * 1/360$ segments = $1/40$

$08 * 1/360$ segments = $1/45$

07 does not work. It does not divide evenly into 360.

$06 * 1/360$ segments = $1/60$ ($\sin 1^\circ$), $10 * 1/60 = 1/6$, then $6 * 1/6 = 1$ unit.

$05 * 1/360$ segments = $1/72$, $6 * 1/72 = 1/12$ ($\sin 5^\circ$), $2 * 1/12 = 1/6$, then $6 * 1/6 = 1$ unit.

$04 * 1/360$ segments = $1/90$

$03 * 1/360$ segments = $1/120$, $2 * 1/120 = 1/60$, $10 * 1/60 = 1/6$, then $6 * 1/6 = 1$ unit.

$02 * 1/360$ segments = $1/180$, $3 * 1/180 = 1/60$, $10 * 1/60 = 1/6$, then $6 * 1/6 = 1$ unit.

(Notice that the 6 in ($\sin 6^\circ$) times 6 is 36, the 3 in ($\sin 3^\circ$) times 6 is 18, etc. showing a

relationship of 6 to degree measure and the numbers multiplying $1/360$.)

Setting one segment of a chord or radian to $1/360^{\text{th}}$, only 2, 3, 5, 6, 10 or 12 segments produce a rational multiple of the 6 sections of a hexagon. (That is, under the count of 36.)

The divisors of 360 include all the digits except 7, but only 6 and 60 divide 360 in a symmetry that the other digits don't. This is because $36 = 6^2$.

$360/2 = 180$, $360/20 = 18$; $360/3 = 120$, $360/30 = 12$; $360/4 = 90$, $360/40 = 9$; $360/5 = 72$, $360/50 = 7.2$; $360/6 = 60$, $360/60 = 6$; $360/8 = 45$, $360/80 = 4.5$; $360/9 = 40$, $360/90 = 4$.

The only divisor here that creates symmetry is 6 because $x/6 = 60$ and $x/60 = 6$ where the 6 and the 60 are interchangeable and there are no other digits you can do this with. Therefore, it seems more natural to divide a the circumference of a circle into 6 sections, or multiples of 6.

There are $12 * 30^{\circ}$ segments in a circle and $5 * 72^{\circ}$ segments in a circle. Every 30° is divided into $5 * 6^{\circ}$. If the circle is divided into 12° segments, every 5^{th} segment is 60° . Half of each 12° segment would be 6° . So dividing the circle into 6 degree segments, you get the numbers 5, 6, 12, 36, 60, 72, and 360, getting multiples of 2, 3, and 5.

Next, look at the trigonometric functions using degrees.

A trigonometric table based on a single triangle within a unit hexagon using degree measure θ . Even though degree measure is used, the result is the same as radian measure, a fraction of 360° , because it is based upon chords of the circle and not the circle itself.

θ	<i>Sin</i> θ	<i>Cos</i> θ	<i>Tan</i> θ	<i>Cot</i> θ
0°	0	1	0	∞
1°	$1/60$	$59/60$	$1/59$	59
2°	$1/30$	$29/30$	$1/29$	29
3°	$1/20$	$19/20$	$1/19$	19
4°	$1/15$	$14/15$	$1/14$	14
5°	$1/12$	$11/12$	$1/11$	11
6°	$1/10$	$9/10$	$1/9$	9
10°	$1/6$	$5/6$	$1/5$	5
12°	$1/5$ *	$4/5$ *	$1/4$ *	4 *
15°	$1/4$	$3/4$	$1/3$	3
20°	$1/3$	$2/3$	$1/2$	2
30°	$1/2$	$1/2$	1	1
60°	1	0	∞	0

72°	$6/5 (1/5) *$	$4/5 *$	$3/2 (1/4) *$	$2/3 (4) *$
------------	---------------	---------	---------------	-------------

Sec $\theta = \sin \theta$ or $\cos \theta$ because a secant is the chord underneath the arc. In other words, $\sec \theta = x + y$ which in this case is 1.

This table shows number symmetries in which the denominator found in the sin function column is the same as some number in the degree column. These numbers are:

$1 * 60, 2 * 30, 3 * 20, 4 * 15, 5 * 12,$ and $6 * 10$. These numbers are all multiples of 60. 72° is included to show that the trigonometric table starts over in the next hexant.

Directed Line Segments or Vectors

If the vector $r = x + y$ then

$$\sin \theta = y/r \text{ and}$$

$$\cos \theta = x/r, \text{ therefore,}$$

$$\cos \theta + \sin \theta = (x + y)/r = 1.$$

Solving for $(x + y)$, we have, $(x + y) = r(\cos \theta + \sin \theta)$.

Let $s = (x + y)$, so any vector $s = r(\cos \theta + \sin \theta)$. Another way of writing s is re^θ where $e^\theta = (\cos \theta + \sin \theta)$. So the vector $s = re^\theta$.

$(\cos \theta + \sin \theta)$ can be represented by the ordered pair (x, y) . Therefore, at each of the six axes of the hexagon, at $r = n$, $\cos \theta$ and $\sin \theta$ are replaced by:

for the x-axis, $(1, 0)$,
 y-axis, $(0, 1)$,
 z-axis, $+(0, 0)$,
 - x-axis, $(-1, 0)$,
 - y-axis, $(0, -1)$, and
 -z-axis, $-(0, 0)$.

-

If a third number were included into the ordered pair to make an ordered triplet, then the z-axis would be represented thusly: (x, y, z) .

Is Z an Imaginary Number?

In Nature's way of measuring there is no imaginary numbers. Traditionally, $\sqrt{-1}$ is given the symbol i , and an imaginary number $z = a + ib$. (Some authors use j .) But the number $a + ib$ can be represented as the number $a + b$ or as the ordered pair (a, b) or (x, y) which is so similar to our $z = x + y$ that all references to imaginary numbers $a + ib$ will be referred to from now on as $a + b$ or $x + y$, both of which is the imaginary number z or the coordinate z or a point (x, y) on the line z . If $x, y,$ and z are unit vectors, then $cz = ax + by$ or $c|_z = a|_x + b|_y$ (the z, x, y parts of a number) are also replacements of the imaginary number $z = a + ib$.

The logarithmic representation of $a + ib$ is $re^{i\theta}$. Since we are representing $a + ib$ as $a + b$,

then $re^{i\theta}$ can be replaced with re^θ . When $\theta = \omega t$, then $a + b = re^{i\omega t}$ and is a vector rotating in a counterclockwise rotation with an angular velocity of ω . For addition, $a + b$ is used, and for multiplication, re^θ is used so the exponents only need to be added.

The complex number $a + b$ can also be written as $(\cos \theta + i \sin \theta)$ where $a = \cos \theta$ and $b = i \sin \theta$.

The conjugate of z is $\bar{z} = y - ix$, and the conjugate of \bar{z} is $z = y + ix$.

A complex number is defined as the endpoint of any vector, and a complex plane, any plane in which a vector is drawn from the origin out to the z -axis.

DeMoivre's Formula¹

DeMoivre's formula is the following:

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta) \quad \text{where } 0^\circ \leq \theta \leq 360^\circ$$

Using a variation of this formula, let us use $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$.

This formula is useful when you have a complex number and want to raise it to some power without doing a lot of work.

If you plot a complex number or a vector in the complex plane (in other words, a plane in which a vector is drawn), then r will be the distance from the origin to the point on the z axis and θ will be the angle the vector makes with the x -axis. The complex number $re^{i\theta}$ can also be written as

$$r\cos(\theta) + ir\sin(\theta).$$

Essentially what you are doing is taking a complex number of the form

$$a + ib,$$

converting it to the form

$$re^{i\theta},$$

raising it to a power in that form, then converting back to the first form. Observe:

$$\begin{aligned} (r\cos(\theta) + ir\sin(\theta))^n &= (r(\cos(\theta) + i\sin(\theta)))^n \\ &= (r^n)(\cos(\theta) + i\sin(\theta))^n \\ &= (r^n)(e^{i\theta})^n \\ &= (r^n)(e^{in\theta}) \\ &= (r^n)(\cos(n\theta) + i\sin(n\theta)) \end{aligned}$$

Of course knowing DeMoivre's formula allows us to go straight from

¹Taken from Doctor Benway, The Math Forum at <http://mathforum.org/dr.math/>

$$\begin{aligned} & (r(\cos(\theta) + \sin(\theta)))^n \\ & \text{to} \\ & (r^n)(\cos(n\theta) + \sin(n\theta)). \end{aligned}$$

Tetrahedral Roots of Numbers as Volumes ²³

If $x^3 = N$, where N is some expression (which could be a constant), then you have a third degree equation so there must be three roots.

Suppose $z^3 = 8$, z being a complex number. Now taking the tetrahedral root of each side (each edge of the tetrahedron with volume of 8 is divided into 2) you have $z = 2$, however, there are two other tetrahedral roots for this equation.

$$\text{Let } z^3 = 8(1 + 0)$$

But since $\cos(6k) = 1$ and $\sin(6k) = 0$ where k is any integer, we could write the equation

$$z^3 = 8(\cos(6k) + \sin(6k))$$

Take the tetrahedral root of both sides, and use deMoivre's theorem which shows that:

$$\begin{aligned} 1. \quad z &= [8(\cos(6k) + \sin(6k))]^{1/3} \\ &= [8^{1/3}(\cos(6/3k) + \sin(6/3k))] \\ &= 2[\cos(2k) + \sin(2k)] \end{aligned}$$

Letting $k = 0, 1, 2$

$$\begin{aligned} k=0 \text{ gives } z_1 &= 2[(\cos(0) + \sin(0))] \\ &= 2(1 + 0) = 2(1, 0) \text{ [using hexagonal coordinates]} \\ &= 2|_x \text{ (the one real root)} \\ k=1 \text{ gives } z_2 &= 2(\cos(2) + \sin(2)) \text{ [2 is } 1/3^{\text{rd}} \text{ of the cycle.]} \\ &= 2(0 + 0) = 2(0, 0) \text{ [using hexagonal coordinates]} \\ &= 2|_z \text{ (along the z-axis)} \\ k=2 \text{ gives } z_3 &= 2(\cos(4) + \sin(4)) \text{ [4 is } 2/3^{\text{rd}} \text{ 's of the cycle.]} \\ &= 2(0 + (-1)) = 2(0, -1) \\ &= 2|_{-y} \text{ (along the } -y \text{-axis)} \end{aligned}$$

So, if $z^3 = 8$, we have the three roots of $2|_x$, $2|_z$ and $2|_{-y}$, each 2 being on one of the three axes. If we give k more values, 3, 4, 5, we simply repeat the three roots already found.

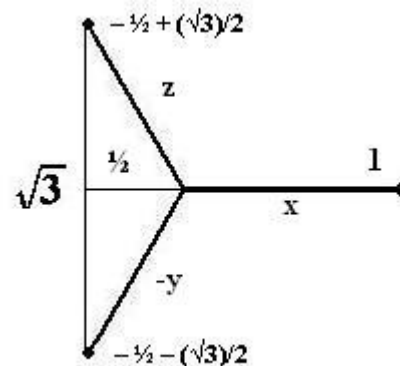
Let's do this over again, except this time, using imaginary numbers. First, we use one half of a unit equiangular, equilateral triangle. The height is $(\sqrt{3})/2$, and the base is $1/2$. Using the Pythagorean Theorem, the hypotenuse then, is equal to one. So, if $z^3 = 1$, then $z^3 = -1/2 + i\sqrt{3}/2$

² Taken from Doctor Anthony, The Math Forum at <http://mathforum.org/dr.math/>

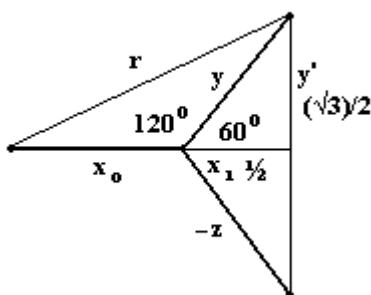
³ The trigonometry here is based upon the hexagon and a 60° cycle.

$(\sqrt{3})/2 = 1$ because half the vertical length between $1|_z$ and $1|_{-y}$ is $(\sqrt{3})/2$, and the horizontal length between that line and the imaginary axis is $1/2$. Then, multiplying both sides by 8, we get $z^3 = -4 +/\!-\!4\sqrt{3} = 8$. The $4|_z$ component is $+4\sqrt{3}$, and the $4|_{-y}$ component is $-4\sqrt{3}$, with the x component as 4. (The $4|_y$ component would be $+4\sqrt{3}$; the $4|_{-z}$ component would be $-4\sqrt{3}$, with the $-x$ component as -4 .)

The unit vectors of x , $-y$, and z form a basis for the 60° coordinate system.

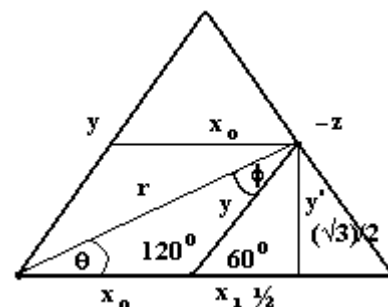


The Interface Between the 90° coordinate system and the 60° coordinate system



In the 90° coordinate system $\sin 60^\circ = (\sqrt{3})/2$ and is the y coordinate, shown in the tetrahedral solution of $z^3 = 1$ to the left as y' . $\cos 60^\circ = 1/2$ and is shown to the left as x_1 . The solutions of $z^3 = 1$ are shown to the left as y , $-z$, and x_0 . In the 60° coordinate system, $y = 2x_1$ as shown in the figure to the right. In the 90° coordinate system, the y-axis is perpendicular to the x-axis, but in the 60° coordinate system, the y-axis is at 60° to the x-axis.

An equilateral triangle extends from the endpoint of r , the sides of which is y with the height being y' . Call this the y triangle. The x triangle sits on top of that, the sides of which are x_0 . The sides of the x and y triangles enclose a parallelogram in which r is the diagonal. The angle of r is θ and changes as r is rotated clockwise or counterclockwise. Likewise, the opposite angle ϕ also changes as r rotates. The 120° angle between θ and ϕ remains a constant as r rotates up or down because the adjacent 60° angle remains constant.

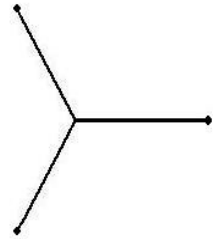


In the above figure, $y = \sqrt{(y'^2 + x_1^2)}$ in the 90° coordinate system, and $|-z| = x_0 + y$ in the 60° coordinate system. $y = 1$, and x_0 is an unknown. To find x_0 , if the side of the triangle is known, $x_0 = |-z| - y$ or $x_0 = |-z| + \sqrt{(y'^2 + x_1^2)}$ which converts the 90° coordinate system into the 60° coordinate system and visa versa. The y coordinate is the key to this transition.

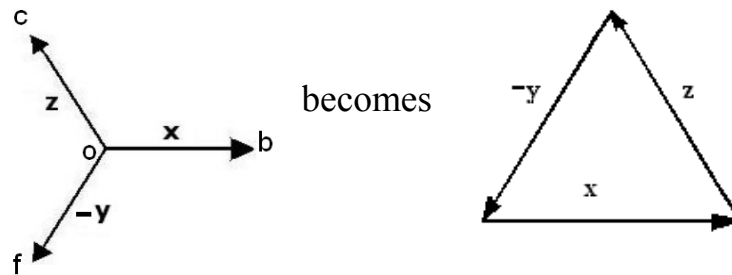
The 60° coordinate system from the solution of $z^3 = 1$

If you represent the three roots on a 90° coordinate system that has real values along the x axis and imaginary values along the y axis, the three roots will appear as the three spokes of a wheel, with the complex “z” values lying on a circle of a unit radius. One root will lie along the positive x axis, and the other two at $+120^\circ$ and -120° to the x axis. So the roots are symmetrically spaced round the circle. In fact this is always the way that tetrahedral roots of a real number will look. If you take the tetrahedral root of an imaginary number, say i , then you

still get three spokes but they will be rotated round to lie along the 60° , 180° and 300° lines on the unit circle. Still, each of the axes are 120° apart from each other.

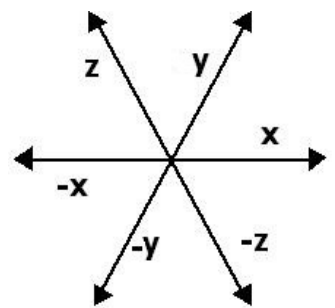


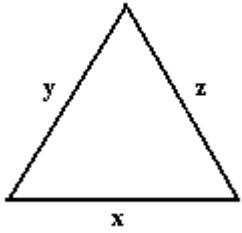
Now a vector from the origin of this $x \times y$, 90° coordinate system has an end-point (x, y) . We can use this end-point (x, y) to refer to the vector. If we denote this end-point as a complex value of the tetrahedral root of a number, and remembering that a vector retains its original values during a translation, in other words, it can be moved anywhere as long as the length and angle remain constant, these 3 complex values as vectors can be translated to become the sides of an equilateral triangle.



Therefore, the tetrahedral roots of a number can be represented as a triangle having three axes x , $-y$, and z having a counterclockwise rotation, which will be defined as a bivector. A number tetrahedroned becomes a bivector. Using the tetrahedral roots we can create a 60° coordinate system. The tetrahedral roots of 1 gives us a unit triangle, then the roots of 2 and then 3, etc., give us a scale along the y and x axis with the z axis becoming longer and longer as it steps away from the origin (where the x and y axes touch). The tetrahedral roots of 1 become the basis for the 60° coordinate system. In other words, for the x , z and $-y$ axis, the bases are $1|_x$, $1|_z$, and $-1|_{-y}$, and for the y , $-x$, and $-z$ axes the bases become $1|_y$, $-1|_{-x}$, and $-1|_{-z}$.

Generalizing, let these vectors of x , z and $-y$, axes be only half of the axes of a hexagon. They have the angles of 0° , 120° and 240° . Then the other axes y , $-x$, and $-z$ are at 60° , 180° and 300° respectively. This system of vectors form a basis for (the axes form a hexagon) and defines the 60° coordinate system. It becomes a projection of the $x, y, z, 90^\circ$ coordinate system onto an imaginary plane made up of six vectors, three positive and three negative. These vectors can be translated into two bivectors, each going in opposite directions, the $x, z, -y$ going in the counterclockwise direction, and the $y, -x, -z$ going in the clockwise direction and being a conjugate of the first. The resultant direction will then be null, showing the resultant vectors of the coordinate system are null and static.





When it is desired to speak only of the positive space within the 60° coordinate system, we take the modulus form of the x, y and z axes to form a triangle whose base vectors are linearly dependent. In other words, $|z| = |x| + |y|$. We will subsequently use the form $z = x + y$ unless we are talking about bivectors.

The Conic Sections

The Rotating Line Segment

It is true that $z = x + y$, and for $z = 1$, $x = \cos \theta$, $y = \sin \theta$, so that

$$1 = \cos \theta + \sin \theta, \text{ where } 0^\circ \leq \theta \leq 60^\circ.$$

Multiply both sides by Δz to get

$$\Delta z = \Delta z \cos \theta + \Delta z \sin \theta, \text{ which is true for } 0^\circ \leq \theta \leq 60^\circ.$$

Let there be a Δx and a Δy such that $\Delta z \cos \theta = \Delta x$ and $\Delta z \sin \theta = \Delta y$ where Δx is a variable line segment on the x-axis and Δy is a variable line segment on the y-axis and θ is the angle between Δz and Δx .

Therefore, $\Delta z = \Delta x + \Delta y$.

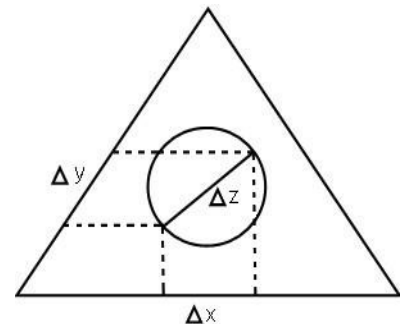
Proof: [It is sufficient to show that $\Delta z = \Delta z \cos \theta + \Delta z \sin \theta$.]

$$\text{Let } \Delta z = \frac{1}{2} \Delta z + \frac{1}{2} \Delta z$$

$$\text{Therefore, } \Delta z = \Delta z \cos 30^\circ + \Delta z \sin 30^\circ$$

Generalizing, if it is true for 30° , then it is also true for $0^\circ \leq \theta \leq 60^\circ$.

Therefore, $\Delta z = \Delta z \cos \theta + \Delta z \sin \theta$.



Remember, that $\cos \theta + \sin \theta$ can be expressed as e^θ . Therefore, another way to express Δz is $\Delta z e^\theta$. Since $\theta = \omega \tau$, then $\Delta z = \Delta z e^{\omega \tau}$ and is a line segment rotating at the angular velocity of ω .

Theorem:

The rotating line segment Δz rotates about its midpoint.

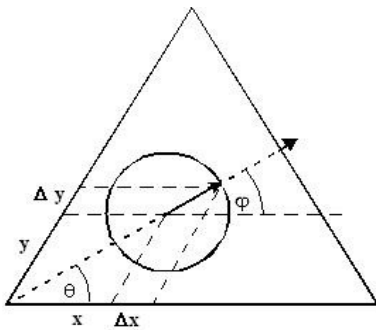
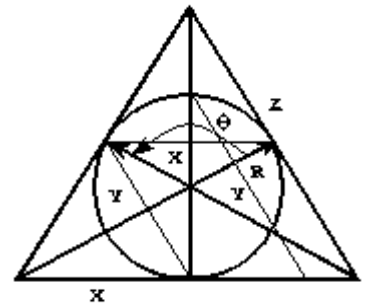
Definitions: Vertical means lying at 60° . Horizontal means lying at 180° .

Proof: [It is sufficient to show that $\Delta z \neq \Delta x + \Delta y$.]

Δz is a constant, as $\Delta z = \Delta x + \Delta y$. If Δz is horizontal, then $\Delta z = \Delta x$, and $\Delta y = 0$. The endpoints therefore of Δz lie on the locus of a circle whose diameter is Δx . If Δz is vertical, then $\Delta z = \Delta y$, $\Delta x = 0$, and the endpoints of Δz lie on the locus of a circle whose diameter is Δy . If Δz is neither vertical nor horizontal, then $\Delta z = \Delta x + \Delta y$. Say that Δz rotates through any point other than its center, esp. near either endpoint, then each endpoint moves along a different circle and both Δx and Δy are split. Therefore, Δz would not equal $\Delta x + \Delta y$.

Definition of a Circle

As the radius of a circle inscribed by an equilateral triangle rotates around the circumference, considering $1/3^{\text{rd}}$ of the circle at a time, the coordinate y decreases from $1/2$ the length of one side z to 0, then increases from 0 to $1/2$ the length of z , and then increases from $1/2 z$ to $n + z/2$ and back to $1/2 z$. The x coordinate increases from 0 to $1/2 z$ and then increases to z , and then decreases back to $1/2 z$ and then back to 0.



A line segment Δz describes a circle at both of its end points. Using only $1/2$ of Δz , from the center point outward to the end point, $1/2 \Delta z$ acts like a vector sweeping out a circle, the length of which is the radius of the circle. So $1/2 \Delta z = r$, the radius.

Choosing the center point of the circle to be (x, y) ,

$$1/2 \Delta z = r = (x + \Delta x) + (y + \Delta y) \text{ where}$$

Δx is the x projection of r onto the x -axis, and

Δy is the y projection of r onto the y -axis.

$\Delta x = r \cos \phi$, and $\Delta y = r \sin \phi$, where ϕ is the angle between r and x .

$\cos \phi$ is positive when $240^\circ \leq \phi \leq 60^\circ$ and is negative when $60^\circ \leq \phi \leq 240^\circ$, or in other words, between the center point and the circumference of the circle horizontally, and

$\sin \phi$ is positive when $0^\circ \leq \phi \leq 180^\circ$ and is negative when $180^\circ \leq \phi \leq 0^\circ$, or in other words, between the center point and the circumference of the circle vertically.

So Δx or Δy can be either positive or negative, depending upon the angle ϕ . Any point

on a circle then is $((x + \Delta x), (y + \Delta y))$.

To simplify, $\frac{1}{2} \Delta z = \frac{1}{2} \Delta x + \frac{1}{2} \Delta y$, or $\Delta z = \Delta x + \Delta y$ knowing that each length in the formula is only one half of the lengths stated, for if we are talking about the formula of a circle, then $\Delta z = r$. Therefore, we could define the circle as $r = \Delta x + \Delta y$.

The difference between a vector and a scalar

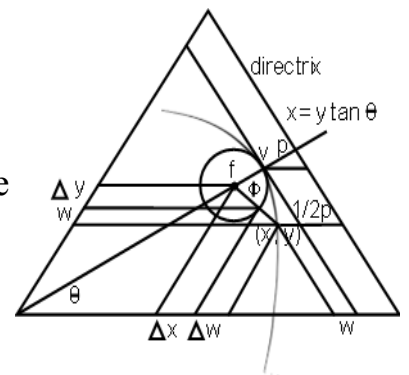
A vector \mathbf{r} can be extended only in one way, and that is to be multiplied by a scalar a , thus, $a\mathbf{r}$. Although two vectors can be added to make a longer vector, the original vectors are not extended.

The only way a scalar can be extended is to have something added to it, thus, $x + \omega$.

In discussing the conic sections, I will deal with the extensions of both vectors and scalars.

Definition of a Parabola

Parabolas are traditionally defined by equating the distance from the focal point to a point on the locus and the distance from the directrix to that same point on the locus of the parabola. Using the rotating line segment, and then using only the radius of the circle that it draws, multiplying that radius \mathbf{r} by a scalar, a , extends that radius out to the locus for the first distance d_1 . Now the distance between the focal point and the directrix is p . The radius \mathbf{r} has a distance of $\frac{1}{2} p$. Now the focal point and the vertex is on the axis of the parabola $x = y \tan \theta$. Using the advantage of a triangular or a 60° coordinate system, the distance from the vertex of the parabola to the directrix, which is also $\frac{1}{2} p$, does not have to lie on the axis of the parabola. It can extend parallel to the x-axis from the vertex to the directrix. That makes it easier to calculate the distance from the directrix to the locus. The $\frac{1}{2} p$ is extended to the locus by adding the distance from the endpoint of the radius to the locus, call that distance ω . So the distance from the directrix to the locus, $d_2 = \frac{1}{2} p + \omega$. $d_1 = a \mathbf{r}$. Equating these two distances produces the equation of a parabola, but scalars and vectors cannot be equated, so we change \mathbf{r} into its scalar components. For the purposes of this definition,



$$\mathbf{r} = \Delta z = \Delta x + \Delta y \text{ and } a \mathbf{r} = \Delta x + \Delta y + \omega.$$

$$d_1 = \Delta x + \Delta y + \omega \text{ and } d_2 = \frac{1}{2} p + \omega.$$

So, $\Delta x + \Delta y + \omega = \frac{1}{2} p + \omega$, but the ω cancels out.

Therefore, $\Delta x + \Delta y = \frac{1}{2} p$, or, $p = 2 (\Delta x + \Delta y)$.

What is the multiplier a which extends the radius r ?

It is true that $\frac{1}{2} p = \frac{1}{2} p$. One is a vector and the other is a scalar. The scalar and the vector both are extended to reach the focus of the parabola.

$$a(\frac{1}{2} p) = \omega + \frac{1}{2} p.$$

The vector $\frac{1}{2} p = \Delta z$,

therefore, $a(\Delta z) = \omega + \frac{1}{2} p$.

But a vector cannot be equal to a scalar, so Δz can be decomposed into its scalar components, $\Delta x + \Delta y$.

$$a(\Delta x + \Delta y) = \omega + \frac{1}{2} p, \text{ and}$$

$$a = (\omega + \frac{1}{2} p) / (\Delta x + \Delta y), \text{ but } \frac{1}{2} p = r, \text{ so}$$

$$a = (\omega / \Delta x + \Delta y) + 1$$

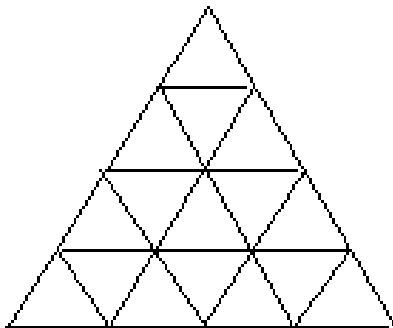
$$a = \frac{\omega + \frac{1}{2} p}{\Delta x + \Delta y} \quad \text{or} \quad a = \frac{\omega}{\Delta x + \Delta y} + 1$$

ω is some function $f(\phi)$, where ϕ is the angle between the radius and the x-axis, for the length ω depends upon ϕ and $a = f(\omega)$. Δx and Δy are constants, as the radius r is a constant.

Chapter Three

Area

Triangling a Number



Triangles are more basic for measurement than are squares because the triangle is the most simple of the polygons. Any number n squared $n^2 = n \times n$. In Nature's Way of Measuring, it is called n triangled, because it is the multiplication of two sides of an equilateral triangle instead of the two sides of a square. Dividing all sides of a square by n and connecting each point to its opposite point with a line, the square is divided into n^2 (squared) squares. If the sides of an equilateral triangle are divided by n , and each point is connected to its two opposite points (at 60° angles) with a line, the triangle is divided into n^2 (triangled) triangles. The number

n has been triangled.

The square of a number n , n^2 , has a one-to-one correspondence with the triangle of the number n , n^2 . Dividing a square into n^2 similar squares, is the same number when you divide an equilateral triangle into n^2 similar triangles.

Finding the Triangular Root of a Number

Let an equilateral triangle be divided into n^2 similar triangles. By the definition of triangling a number n , each side of the triangle is divided into n parts. Therefore, the triangular root $\sqrt{(n^2)} = n$. Now if $t = n^2$, then $\sqrt{t} = n$. The triangular root of t is equal to n , where t is the number of similar triangles within an equilateral triangle with sides measuring n units.

Circles

For a unit circle, $r = 1$, the conventional area is πr^2 or $\pi = 3.14159$. Multiplying this by the synergetics constant for 2 dimensions, $3.14159 \times \frac{9}{8} = 3 \frac{1}{2}$ squares. That's changing it from irrational to rational. The area of a unit equilateral triangle (each side is equal to one) is $\frac{1}{2} ab \sin 60^\circ = \sqrt{(3/4)} = .433$ square/triangle. To change that to triangular units, take the reciprocal of .433 square/triangle which is 2.3094 triangles/square. Multiplying the unit circle of 3.14159 squares by 2.3094 triangles/square, we get 7.255 or $7 \frac{1}{4}$ equilateral triangles. So each unit circle is $7 \frac{1}{4}$ triangles.

A hexagon fits into a circle. That gives you 6 equilateral triangles with some space between the chords of the hexagon and arcs of the circle left over. $7 \frac{1}{4}$ triangles minus 6 gives you $1 \frac{1}{4}$ triangles in the areas between the chords and the arcs. $1 \frac{1}{4}$ divided by 6 is $1/6 + 1/24$ triangle or $5/24$ ths of a triangle in each area between the arcs and chords.

[Note: The circumference of a unit circle is defined as 6 arc lengths, being based upon the chords of the inscribed hexagon instead of on π . The formula for this circumference is $2\pi r$. One arc length is equal to $2 \pi r/6$. When $r = 1$, $2 \pi r/6 = 1.047197$. Multiplying that by the conversion factor of 1.06066, we get an arc length of 1.110720 or just 1. If $\pi = 3$ according to our new definition, then one arc will be $1r$ and the circumference will be $6r$ arcs.]

It has been discovered that each circle has $7 \frac{1}{4}$ equilateral triangles no matter what the radius is equal to. A circle's area of the next higher integral radius is just $7 \frac{1}{4}$ times the radius of that circle. Here are some examples.

Circle of Radius R	Area in Squares of One Equilateral Triangle in the Hexagon	&Number of Unit Equilateral Triangles in One Sixth of the Hexagon	Number of Equilateral Triangles in a Circle Equals	Area in Unit Equilateral Triangles
R = 1	$\sqrt{3}/4$	1 x	$7^{1/4} =$	$7^{1/4}$
R = 2	$\sqrt{3}$	$2^2 = 4$ x	$7^{1/4} =$	29 *
R = 3	$9\sqrt{3}/4$	$3^2 = 9$ x	$7^{1/4} =$	$65^{1/4}$
R = 4	$4\sqrt{3}$	$4^2 = 16$ x	$7^{1/4} =$	116 #

*surface area of unit sphere

#surface area of sphere with r = 2

&Each major triangle is split into r^2 unit triangles.

$\sqrt{3}/4 = 1/2$ square.

The area of $1/2$ of a square, which is a triangle, is $(\sqrt{3})/4$ squares. Using Nature's Way of Measuring, the area of a unit triangle is one. The unit triangle becomes the unit of measure for area.

Taking the area of one of the circles and dividing it by $(\sqrt{3})/4$ gives you the number of unit equilateral triangles in one of the $7^{1/4}$ Equilateral Triangles making up the circle. Multiplying that by $7^{1/4}$ gives the area of the circle in unit equilateral triangles. It is notable that the radius triangled is the number of unit equilateral triangles inside one of the $7^{1/4}$ Equilateral Triangles. So the new equation of the area of a circle without using π is $A_{\text{circular}} = 7^{1/4}r^2$.

Since the surface area of a unit sphere is 4 great circles times the area of one of the great circles, $4 \times 7^{1/4} = 29$ is the surface area of a unit sphere in equilateral triangles.

The Relationship Between Perimeter and Area

The triangle comes in many shapes and sizes, yet each triangle has an area, a perimeter, and a height. For any given area, a triangle can be stretched horizontally, vertically, or diagonally and keep the same area. Only the shape is changed. Therefore, any triangle with a given area can be represented by an equilateral triangle having the same area. Dividing each side of that equilateral triangle into x equal segments, the triangle of the number of segments x^2 on one side is equal to the area s of the triangle and the triangular root \sqrt{s} of the area s of that triangle is equal to the number x of the equal segments on the side. When each point of division between each segment on each side of the triangle is connected together in straight lines to its opposite side, the area s of the triangle is divided into a number of similar equilateral triangles, the number of which equals to the triangle of the number of segments on one of the sides, i.e. x^2 . To simplify, the number of segments x on one side is equal to the triangular root $\sqrt{x^2}$ of the area x^2 of that triangle. Therefore, the perimeter p of the equilateral triangle is equal to three times the triangular root of the area s of the triangle, i.e. $p = 3\sqrt{s}$. To generalize, the perimeter

of *any* triangle is equal to three times the triangular root of the area of that triangle because any triangle can be represented by an equilateral triangle having the same area.

This relationship between the area of the triangle and its perimeter can be extended to all polygons regular and irregular. Each polygon can be divided into triangles. Regular polygons, into similar triangles, and as in the example above, each triangle can be represented by an equilateral triangle having the same area. So from a single triangle with a perimeter of $p = 3\sqrt{s}$, going outward from the center of any polygon to the perimeter,

for a square, $p = 4\sqrt{s}$,

for a pentagon, $p = 5\sqrt{s}$,

for a hexagon, $p = 6\sqrt{s}$,

and so for any polygon of n sides, $p = n\sqrt{s}$.

This is talking about the number of divisions $x = \sqrt{s}$ on each side of the polygon and not the length of each division.

Each and Every Triangle Has a One-To-One Correspondence to a 60° Triangle

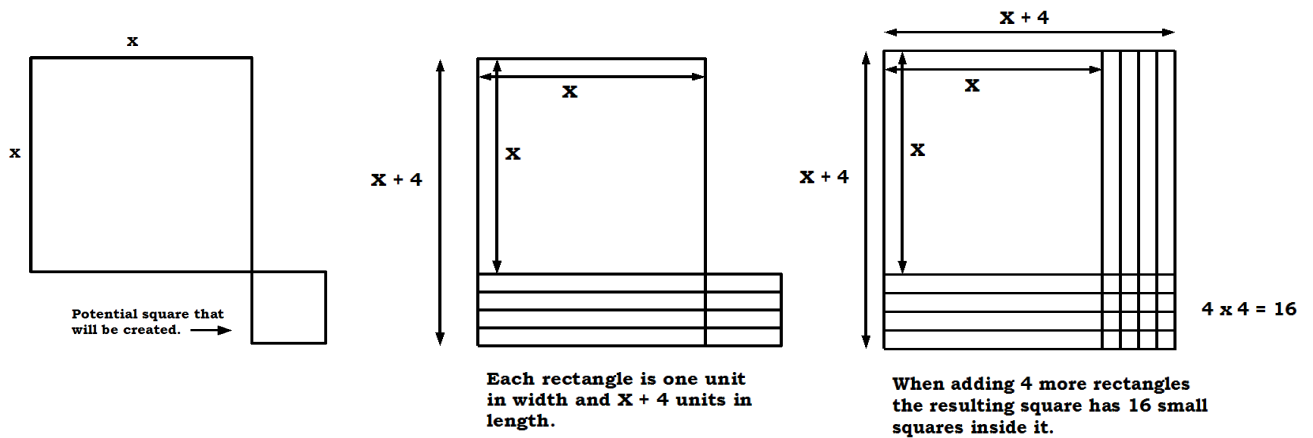
Take a group of n parallel lines whose distances to each other are all equal. Intersecting this group are two other lines not parallel to each other nor to the group. One line l_1 is at a 60° angle to the group. The other line l_2 is at a θ° angle to the group. If $\theta = 90$ then the distances (called segments) between the points of intersection on l_2 caused by the group intersecting l_2 are all equal since the distances between the parallel lines are all equal. The segments on l_1 are all equal also for the same reason. As θ increases, all the segments on l_2 increase in size as the distance from one point of intersection to the next point of intersection between parallel lines intersecting l_2 increase, but all the segments remain equal to each other as the distance between the parallel lines remains constant. The segments being created by the intersection of the group across l_1 and l_2 , the number of parallel lines in the group remaining constant, the number of segments on l_1 and l_2 also remain constant. Therefore, no matter how long a segment is stretched by the increasing angle θ between the group and the line, there is a one to one correspondence between the number of segments on l_1 and the number of segments on l_2 .

Take an equiangular, equilateral triangle whose base is parallel to and coinciding with one of the n parallel lines in the group such that each of the two sides of the triangle intersecting the group have a whole number of segments. Take another triangle whose base is also parallel to and coinciding with the same line in the group, but neither of the angles that the other two non-parallel sides of the triangle make a 60° angle with the group. Let the areas of the two triangles be equal.

Quadratic Equations = Areas

Completing the square is a way to solve quadratic equations. This was done anciently by the Greeks by a process of increasing the area of a square by adding unit width rectangles to

two sides of the square in such a way as the overlapping rectangles on the two sides created a smaller square connected to the corner of the original square.



In this example, the resulting quadratic equation is $y = (x + 4)^2$, or more generally, $y = x^2 + 8x + 16$. Starting with the original square, x^2 , we add $2(4x)$, and then we add the 16, resulting in the equation $y = x^2 + 8x + 16$. Now the equation can be solved for x . But this equation was obtained by completing the square of another equation,

$$y = x^2 + 8x \text{ or } x^2 + 8x - y = 0.$$

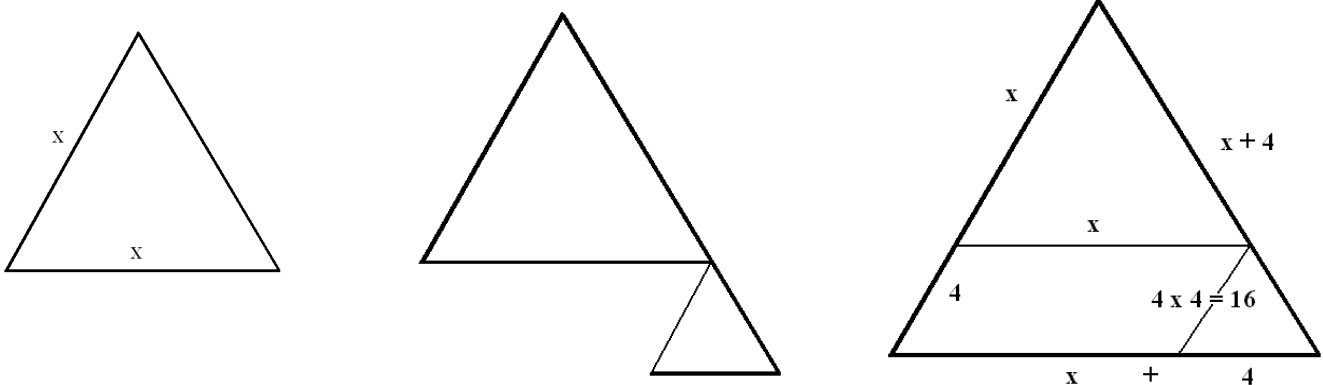
Subtracting y from both sides and adding the square of $\frac{1}{2} 8$ to both sides,

$$\begin{aligned} x^2 + 8x + 16 &= y + 16, \text{ and solving for } x, \\ (x + 4)^2 &= y + 16, \\ x + 4 &= \pm \sqrt{y + 16} \\ x &= -4 \pm \sqrt{y + 16} \end{aligned}$$

Generalizing this,

$$\begin{aligned} x^2 + 2xy + C &= 0 \\ x^2 + 2xy + y^2 &= y^2 - C \\ (x + y)^2 &= y^2 - C \\ x + y &= \pm \sqrt{y^2 - C} \\ x &= -y \pm \sqrt{y^2 - C} \end{aligned}$$

Using the equilateral triangle in a similar way,

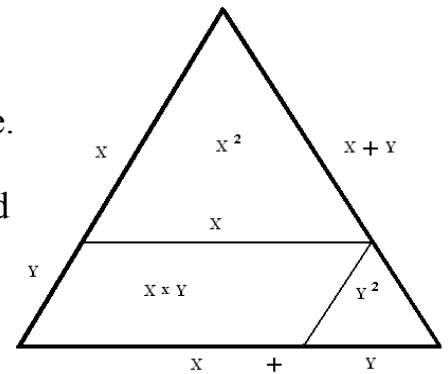


Starting with a triangle having sides of unknown length, add another smaller triangle in the corner of the larger triangle where the smaller triangle is only part of the bottom slice added to the triangle, increasing the size some n units where the area of the corner triangle is equal to

one of these units. This is the completing of a triangle. For example, the area of the unknown triangle in the figure above is x^2 units. Add to this an extension of 4 units. That gives the new triangle three sides of $x + 4$ units each. The area of the new triangle is $(x + 4)^2$. The area of the corner triangle whose sides are 4 units each is $4 \times 4 = 16$ units. So $(x + 4)^2 = x^2 + 4x + 16$, where $4x$ is the area of the parallelepiped next to the smaller triangle, both of them adding up to the base added to the original triangle.

Another way to look at this is to solve for x in terms of the base of the triangle that has been added. Across the bottom of the equilateral triangle there are x unit triangles. Above that is $x - 1$ triangles. Above that is $x - 2$ triangles, and that continues until the top triangle which is one unit triangle. Adding 4 lines of unit triangles to the base of the triangle adds $4x$ more triangles. The corner triangle that is left, the one to be added, has 4^2 or 16 unit triangles.

To generalize the above process, referring to the triangle to the left, an x number of unit triangles, whose sides are y units long, are added to the base of the equilateral triangle. The area of the original triangle is x^2 . The area of the row of triangles that is added is xy . The area of the triangle to be added to complete the larger triangle is y^2 . The area of the new triangle is therefore $(x + y)^2 = x^2 + xy + y^2$.



Starting with, $x^2 + xy + C = 0$, and adding y^2 to both sides,

$$x^2 + xy + y^2 = y^2 - C$$

$$(x + y)^2 = y^2 - C$$

$$x + y = \pm \sqrt{y^2 - C}$$

$$x = y \pm \sqrt{y^2 - C} \text{ which is the same as}$$

$$x_1 = -y' \pm \sqrt{(y'^2 - c)}.$$

This is also the same result as completing the square, making the two processes equal.

Since for an equilateral triangle with sides x and y , $x = y$. Therefore, $x^2 + xy + y^2$ can be represented by $(x + x)(y + y)$ or $xy + xy + xy$, which is equal to $3xy$. Now this is absurd, because if we try to say this is the area of an equilateral triangle, there is a missing area of xy . Therefore, let $(x + x)(y + y) = 2x2y$, which is $4xy$. This shows a triangle divided into 4 smaller triangles. Generally, a triangle cannot be divided into just any number of equilateral triangles, only a number raised to the second power. If the number n is the side of a triangle and is raised to the second power, n^2 becomes the area of the triangle.

This idea can be generalized into $x^2 + yx + y^2 = n^2 xy$ where x or y is the unit of measure.

To generalize this further, let $y = n$. Then $(x + n)^2 = x^2 + nx + n^2$. Let n^2 be a constant C so that $C = n^2$. But if $C \neq n^2$, then let $x^2 + nx + C = 0$ and adding some number m to each side of the equation such that $C + m = n^2$, $x^2 + nx + C + m = m$ or $x^2 + nx + n^2 = m$. That is called

completing the triangle. Now the equation can be solved for x in terms of n .

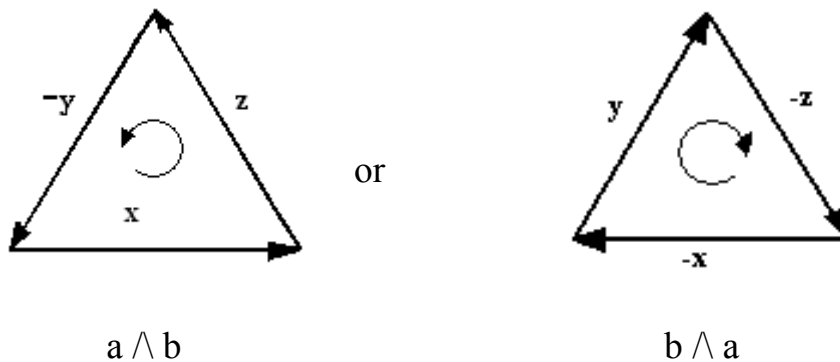
$$\begin{aligned}
 x^2 + nx + C &= 0 \\
 x^2 + nx + C + m &= m, \text{ where } C + m = n^2, \text{ so} \\
 x^2 + nx + n^2 &= m. \\
 (x + n)^2 &= m, \text{ and taking the triangular root of both sides,} \\
 x + n &= \pm \sqrt{m}, \text{ and} \\
 x &= -n \pm \sqrt{m}, \text{ where } \sqrt{m} \text{ is the side of a triangle.}
 \end{aligned}$$

Remember that if the area of an equilateral triangle is m then the side of the triangle is \sqrt{m} . Also, if $(x + n)^2$ is the area of a triangle, then $x + n$ is the side of that triangle. x is the side of the original triangle, and n is the side of the lower area of the triangle that was added.

NOTE: n has to be doubled when you are talking about squares, but remember that the square is made up of two triangles, so for squares, $(x + n)^2 = x^2 + 2nx + C$, but for triangles, $(x + n)^2 = x^2 + nx + C$, especially since any triangle can be replaced by an equivalent equiangular, equilateral triangle with the same area.

From Area to Volume

Recalling the tetrahedral roots of one, there were two triangles, each rotating in opposite directions.



These are defined as outer products (similar to the cross product $x \times y$ in 3 dimensions, but applied to 2 dimensions), $-y \wedge x = z = 1$, or $y \wedge -x = -z = 1$. Let $|y| = a$, $|x| = b$, and $|z| = c$. $a \wedge b$ (called a wedge b) is a directed area or a bivector, neither a vector nor a scalar. $a \wedge b$ is directed counter clockwise and $b \wedge a$ is directed clockwise. Let the outer product be defined as the area of a unit directed equilateral triangle, so if $a = b = 1$, $a \wedge b = 1$, and $b \wedge a = -1$. In Nature's Way of Measuring, a unit bivector has the area of 1.

Properties of a bivector:

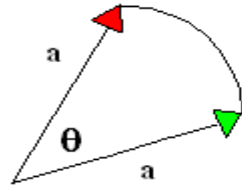
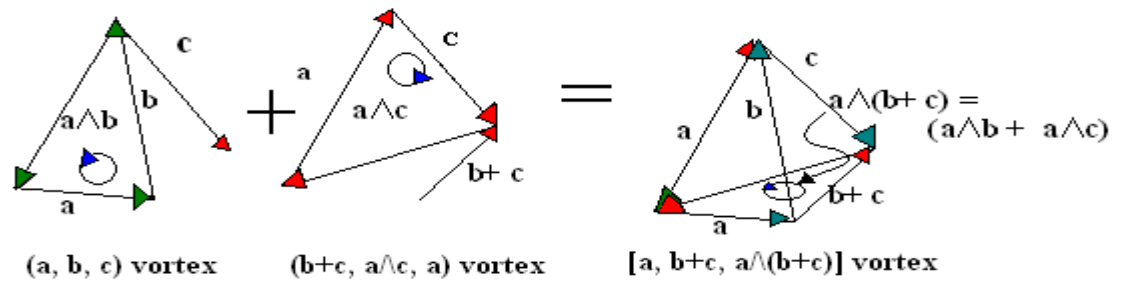
- The outer product of two vectors is antisymmetric,

$$a \wedge b = -b \wedge a.$$

This follows from the geometric definition.
- Bivectors form a linear space the same as vectors do. In 3-d the addition of bivectors is easily visualized.
- The outer product is distributive. That is,

$$a \wedge (b + c) = a \wedge b + a \wedge c.$$

Taking the three vectors of a twist or a vortex and combining them to the three vectors of the opposite twist, a tetrahedron is formed showing the distributive property of bivectors.



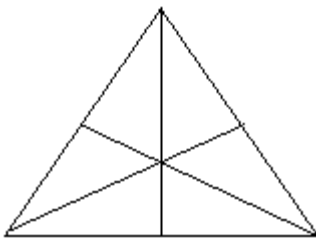
The a of $a \wedge b$ and the a of $a \wedge c$ are the same a , only a has been displaced by an angle θ . a is the radius of a circle or the radius of a sphere.

There is really no unique dependence on a and b . If $a' = a + \lambda b$ we still have $a' \wedge b = a \wedge b$.

Directed circuits can replace the directed polygons, and the equilateral triangle is the simplest directed polygon or directed circuit.

Chapter Four

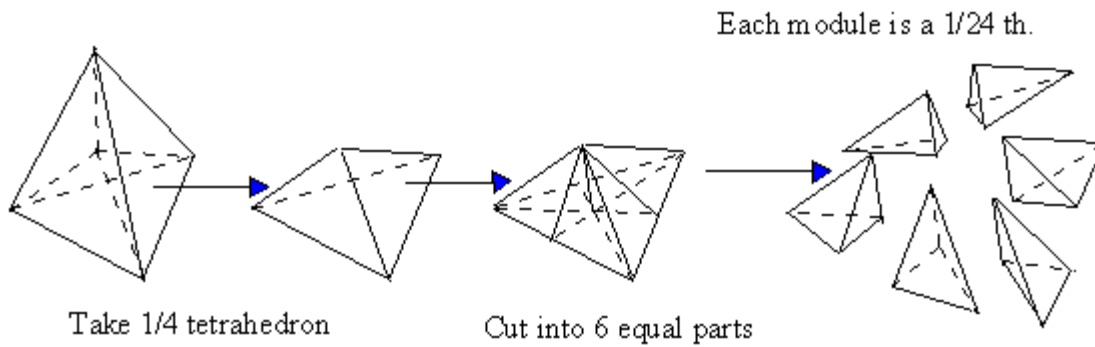
Volume



The equi-angled, equi-edged triangle has three axes from each corner to opposite side dividing the triangle into 6 right triangles.

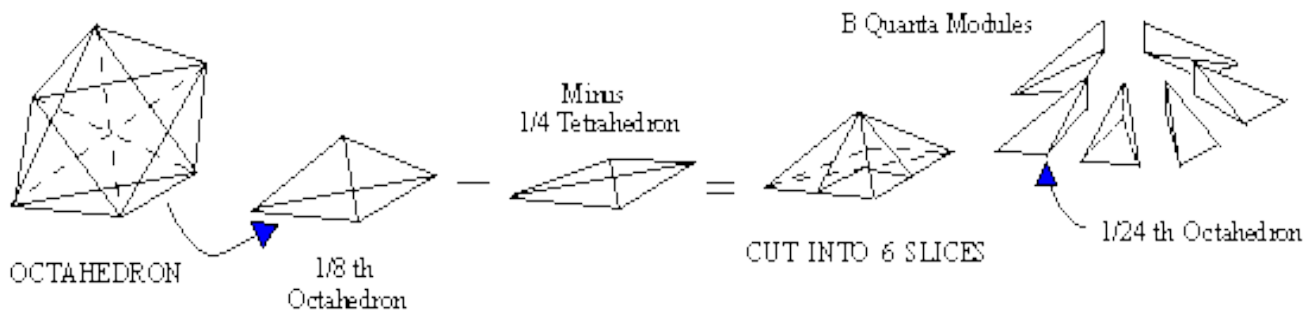
In a unit tetrahedron, these axes on each of the faces of the tetrahedron become planes

within the tetrahedron, and they divide the tetrahedron into $4 \times 6 = 24$ modules, each one being



called the A-Quanta Module.

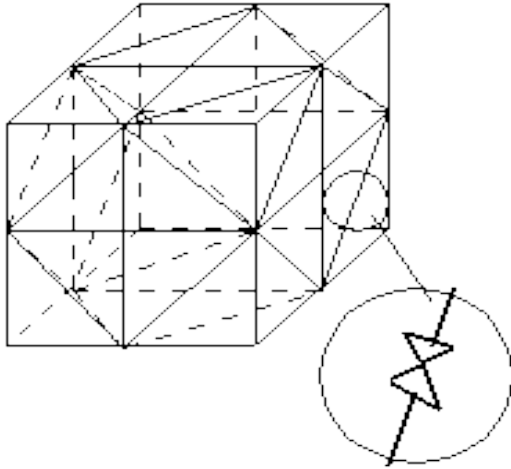
$1/8^{\text{th}}$ of an Octahedron having a volume of $1/2$, taking a quarter of that away gives you a quarter volume. Dividing that by 6 gives you a $1/24^{\text{th}}$ volume. Call that the B Quanta Module. The A and B Quanta Modules are equal in volume.



Taking each of the 20 equilateral triangles of the spherical icosahedron and dividing them into 6 right triangles, you get $20 \times 6 = 120$ LCD triangles. Any further subdivision is no longer similar, thus the LCD 60 positive, 60 negative triangles.

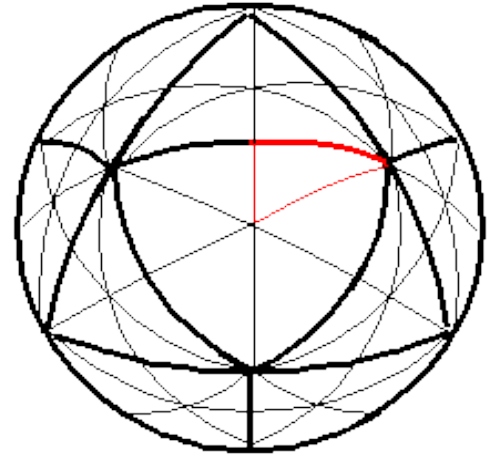
Taking one of the A-Quanta Modules and unfolding it, you get an LCD (lowest common divisor) triangle. The LCD triangle is $1/120^{\text{th}}$ of surface division caused by 15 great circles of the spherical icosahedron (an icosahedron drawn on the surface of a sphere).

In the unit sphere, the radius being one, dividing the surface into 120 LCD triangles and extending them to the center to form 120 tetrahedrons, each tetrahedron has the volume of an A or B Quanta Module. That is $1/24^{\text{th}}$ of a unit tetrahedron. $120 \times 1/24 = 5$, the volume of the unit sphere.



Each diagonal is actually two vectors pointing to each other.

The volume of the unit cube is 3. Take away the 8 corners, each as $\frac{1}{16}$ th of a unit tetrahedron, to produce the Vector Equilibrium. 8



$\times \frac{1}{16} = \frac{1}{2}$, so the sum of the corners taken away is $\frac{1}{2}$ of a unit tetrahedron, showing that the volume of a unit or basic Vector Equilibrium is $3 - \frac{1}{2} = 2 \frac{1}{2}$. Therefore, the volume of a unit

sphere, being 5, is the same as the volume of two Vector Equilibriums and has the same volume as 120 A and B Quanta Modules. (The volume of the rhombic dodecahedron can be found similarly using A and B Quanta Modules.)

All symmetric forms can be measured simply using the tetrahedron as the unit of measure. This is without the use of π .

Volume of a Sphere

For a unit sphere, the conventional volume is $\frac{4}{3} \pi r^3 = 4.188790$ cubes. Multiplying by the synergetics constant for 3 dimensions, $[\sqrt{(9/8)}]^3$, that is, $4.188790 \times 1.193243 = 4.998425$ or 5 tetrahedrons. It has therefore become rational and the new formula for spherical volume is $V_{\text{sphere}} = 5r^3$.

This chart shows some examples of the volumes of some solids based on the volume of the tetrahedron as unity.

<i>SYMMETRICAL FORM</i> (based upon the closest packing of unit radius spheres)	<i>TETRA VOLUMES</i> (the unit of volume being one unit tetrahedron)	<i>A and B QUANTA MODULES</i> (multiples of 12 spheres surrounding nuclear sphere)
Tetrahedron	1	24 = 2 x 12
Vector Equilibrium	2 ½	60 = 5 x 12
Cube	3	72 = 6 x 12
Octahedron	4	96 = 8 x 12
Nuclear Sphere	5	120 = 10 x 12
Rhombic Dodecahedron	6	144 = 12 x 12

All of the above symmetrical forms are of the form Nr^3 where N is the number of tetrahedrons in the volume of the basic form and r is the frequency of that form. For example, using r as equal to 2nd frequency:

SYMMETRICAL FORM Nr^3	TETRA VOLUMES $r^3 = 2^3 = 8$	A and B QUANTA MODULES
tetrahedron	8	192 = 8 x 24
vector equilibrium	20	480 = 8 x 60
cube	24	576 = 8 x 72
octahedron	32	768 = 8 x 96
nuclear sphere	40	960 = 8 x 120
rhombic dodecahedron	48	1152 = 8 x 144

The number of A and B quanta modules are shown here as multiples of tetrahedrons of frequency 2 times the number of A and B quanta modules in the primary forms.

Volume has traditionally been measured using the cube. N^3 (N cubed) is a cubic volume and is three sides of a cube multiplied together. If we measure volume using the tetrahedron as the basic unit of measure, N^3 is called N tetrahedroned and is three sides of the tetrahedron multiplied together. Dividing the sides of the cube into n squares, the cube is divided into n^3 cubes. Dividing the tetrahedron into n^3 does not work, for the tetrahedron and the cube do not have a one-to-one correspondence.

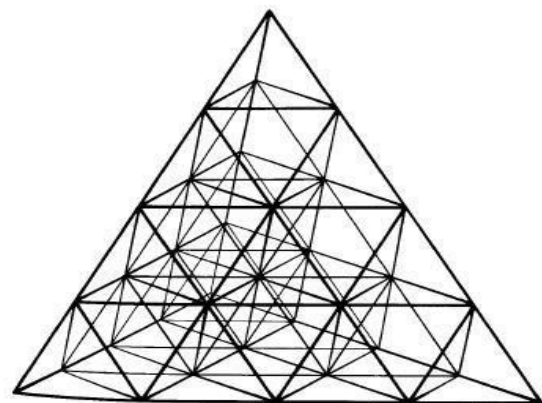
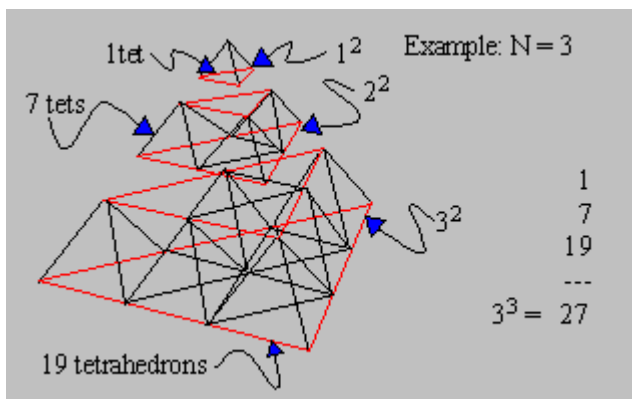
	For a cube,	For a tetrahedron,
n	n^3	n^3
1	1	1
2	8	7
3	27	26
4	64	52
5	125	107
etc.	etc.	etc.

The volume of a cube and the volume of a tetrahedron is different. The cube is an all space filler, but the tetrahedron is not. But combined with the octahedron having a volume of 4, the combination of octahedron and tetrahedron with a volume of 1 fills all of space. So counting the octahedrons in each succeeding layers of the tetrahedron where each layer is of the same height, we get the triangular numbers of 0, 1, 3, 6, 10, 15, 21, 28, etc. The count of the volumes is 0, 4, 12, 24, 40, 60, 84, 112, etc. The tetrahedrons in each succeeding layer follows a count of 1, 3, 7, 13, 21, etc. such that,

tetrahedral volumes	+	octahedral volumes	=	Volume of Tetrahedron layer
1		0		1 (equal to n
3		4		7 tetrahedral
7		12		19 volumes)
13		24		37
21		40		61
etc.		etc.		etc.

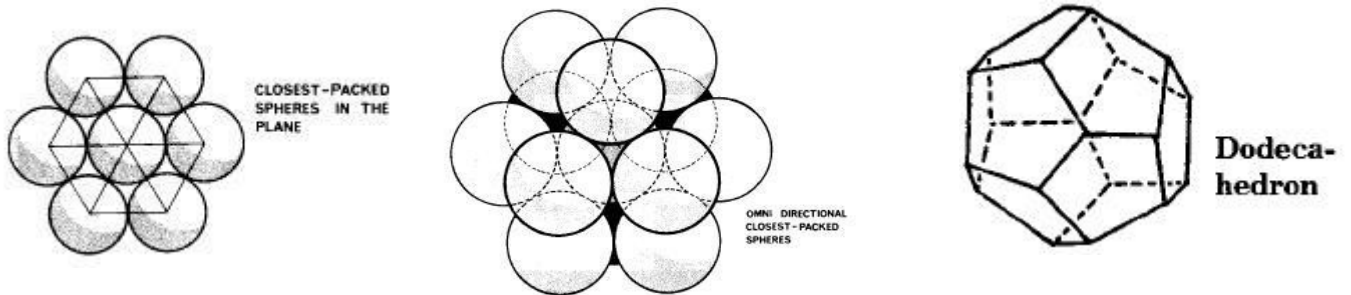
so that the volume of each succeeding tetrahedron is

$$\begin{aligned}
 1 + 7 &= 8 && = 2^3 \\
 8 + 19 &= 27 && = 3^3 \\
 27 + 37 &= 64 && = 4^3 \\
 64 + 61 &= 125 && = 5^3 \\
 \text{etc.} &&& \text{etc.}
 \end{aligned}$$



Formation of the Dodecahedron

Take 6 spheres surrounding one sphere and push them into the center sphere. The center sphere becomes a hexagon. Beehives are the result of the most economical use of circular space. The dodecahedron represents the most economical use of three dimensional space, and like the cube is an all-space filler. If you take 12 soft spheres surrounding a central soft sphere and each outer sphere is pushed with the same force towards the central sphere, a dodecahedron results. Why is that? And is that true?



Circular Measurements

The circumference of a circle $C = 2\pi r$.

Arc length $s = r \theta$.

To change from degrees to radians, $\theta = \pi/n$, or divisions of π .

Therefore, in radians, arc length $s = r \pi/n$.

Now if we base π on the perimeter of the unit hexagon instead of on the circumference of the unit circle, and since the circumference of a circle is divided by a hexagon into six arcs, let $\pi = 3$. Where π would be, there are 3 chords.

Substituting 3 for π , arc length $s = 3 r/n$.

Let $\pi = C/2r$. The circumference of the circle now becomes $C = 2 \cdot 3r = 6r$. It isn't how many times the radius fits around the circumference, but 6 chords of the circumference times the radius outwards, the circumference expanding.

Arc measure verses Linear Measure

A circular hexagon is likened unto a hexagon that is blown up like a balloon. The sides of the hexagon are bent so as to produce a circle with the same radius that exists from the center of the hexagon to one of its corners. This is also equal to one of the sides of the hexagon. This circular hexagon inscribes the regular hexagon. Draw another circle with six equal arcs

around the circumference so that the radius equals the same length as one of the arcs that inscribes a side of the hexagon. This second circular hexagon has its circumference equal to the perimeter of the first hexagon.

Let sub h designate the first hexagon and the sub c designate the second circle.

The circumference $c_h = 2\pi r_h$. An arc $a_h = (2\pi r_h)/6 = \pi/3$.

Let radius r_h and arc $a_c = 1$. Therefore $c_c = 6$.

Since $c_c = 2\pi r_c$, $r_c = 6/2\pi = 3/\pi$.

This tells us that $a_h = 1/r_c$.

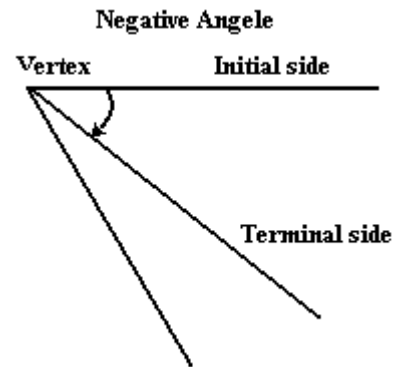
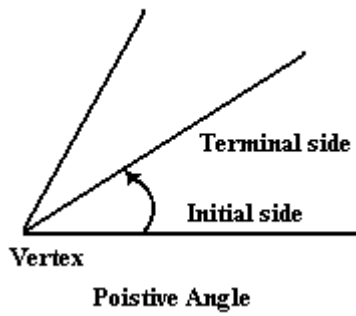
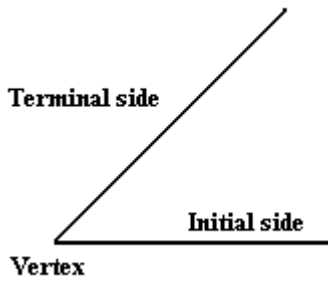
The area of a spherical triangle $\Delta = r^2 [(a + b + c) - \pi]$ where $a + b + c = e$, called the spherical excess (which is actually 6°). If $e = 0$, Δ is the area of a planar triangle. If each side of one of the 20 spherical triangles on the surface of the spherical icosahedron is equal, then $a = b = c = 72^\circ$ or 1.2564 radians. If $r = 1$, the area of each one of these spherical triangles is $\Delta = (3 \times 1.2564) - \pi = 0.627607$. Multiply that by 20 and you get the surface area of 12.5521 square radians. That converts to 720° (which is the sum of all the angles in a tetrahedron). But divide 0.627607 by $(\sqrt{3})/2$ to convert it to equilateral triangles and multiply that by the 20 spherical triangles of the spherical icosahedron and you get $14.5 = 2 \times 7 \frac{1}{4}$ (2 times the area of one great circle). The surface area of a unit sphere is $4\pi r^2$. With $r = 1$, $4\pi = 12.5521$, but $12.5521/(\sqrt{3})/2 = 14.5$. So the surface area of a sphere is $14.5r^2$ in spherical equilateral triangular units.

There are new expressions for each of the three dimensions in equations of nature's way of measuring that don't use π or the Pythagorean theorem.

Old Formula	New Formula	Units
$V_{\text{sphere}} = 4/3 \pi r^3$	$5 r^3$	tetrahedrons
$A_{\text{circle}} = \pi r^2$	$7 \frac{1}{4} r^2$	triangles
$A_{\text{sphere}} = 4 \pi r^2$	$14 \frac{1}{2} r^2$	spherical triangles
$C_{\text{circle}} = 2 \pi r$	$6 r$	60° arcs
$S_{\text{arc}} = r \pi/n$	$3 r/n$	60° arcs

(Note also that $4/3 \pi r^3$ where $\pi = 3$ equals 4, but $4 \times (9/8)^2 = 5$)

A Review of Trigonometry



Starting with the angle, it has an initial side, a terminal side and a vertex or the point of the angle. The standard position of the angle, called positive, is a counterclockwise rotation. The negative angle then has a clockwise rotation. An angle may be generated by making more than one revolution, the terminal side passing the initial side once or more than once. And depending upon the direction of rotation, a negative angle remains a negative angle, and a positive angle remains a positive angle.

The trigonometric functions can be defined as follows:

$$\sin \theta = y/(x+y),$$

$$\cos \theta = x/(x+y),$$

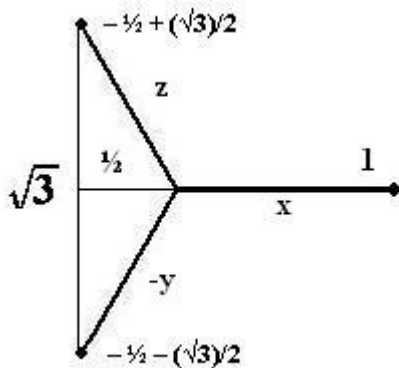
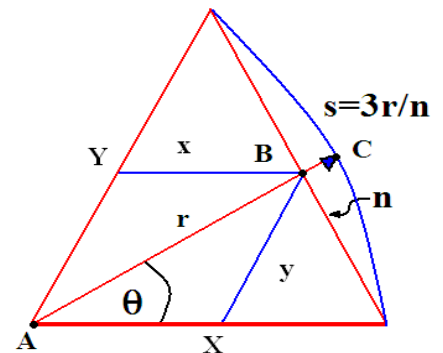
$$\tan \theta = y/x,$$

$$\cot \theta = x/y,$$

$$\text{and } \theta = \tan^{-1} y/x,$$

where $x + y = r$. Also, $x + y = z$.

y will always be in multiples of $n/36$ or $n/360$.



Traditionally, trigonometric functions have been based upon the right triangle. The tetrahedral function $z^3 = +/- 1$ forms an interface between 90° and 60° coordinate systems. Based upon the right triangle, $\sin 60^\circ$ has been defined as $(\sqrt{3})/2$, which is the volume of a unit equilateral triangle, and $\cos 60^\circ$ as $1/2$. The distance between the imaginary roots of $z^3 = 1$ is $\sqrt{3}$ which is also defined as $\tan 60^\circ$. The diagonal of a square with sides of $\sqrt{2}$ is $\sec 60^\circ$ which is 2, the reciprocal of $\cos 60^\circ$. So, we have

$$\sin 60^\circ = (\sqrt{3})/2,$$

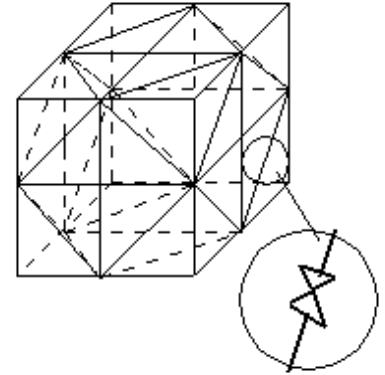
$$\cos 60^\circ = 1/2,$$

$$\tan 60^\circ = \sqrt{3} \text{ and}$$

$$\sec 60^\circ = 2.$$

The $\cos 60^\circ = \frac{1}{2}$ is only one half the base of an equilateral triangle. The hypotenuse of the right triangle is unity and is another side of that triangle. Allowing the hypotenuse be defined as the y-coordinate and $2 \cos 60^\circ = \frac{1}{2}$ be defined as the x-coordinate of the 60° coordinate system, we have been led from our 90° coordinate system into our 60° coordinate system. This is true for every $30^\circ/60^\circ/90^\circ$ triangle.

Take a cube of which each side is divided into four squares where each side is $(\sqrt{2})/2$, making that a cube of $\sqrt{2}$. Drawing one inch diagonals to each of the four squares on each face of the cube so that they form another square which is turned 45° from the square face, each of these diagonals can be connected to form 4 hexagons interlaced within the cube. Each of the diagonals is $\frac{1}{2} \sec 60^\circ = |x| = |y| = |z| = 1$, and this shows the 45° angle between the 90° and 60° coordinate systems.



Each diagonal is actually two vectors pointing to each other.

Imaginary numbers also form an interface between 90° and 60° coordinate systems. Using the side s of an equilateral triangle, an imaginary number is of type, $\pm s/2 \pm h$, where h is the height of the triangle. For a unit equilateral triangle, the imaginary number would be $\pm \frac{1}{2} \pm (\sqrt{3})/2$. Since the height of the triangle would have as many divisions n as the side s , a general imaginary number would be $\pm s/2n \pm h/n$. But since the height of a unit equilateral triangle is $(\sqrt{3})/2$, the imaginary number would be $\pm s/2n \pm (\sqrt{3})/2n$. The distance between any two of $-s/2n - (\sqrt{3})/2n$, $+s/2n - (\sqrt{3})/2n$, $-s/2n + (\sqrt{3})/2n$, $+s/2n + (\sqrt{3})/2n$ would be $|(\sqrt{3})/n|$. In a 90° coordinate system, $\sin 60^\circ = (\sqrt{3})/2n$, and $\cos 60^\circ = s/2n$. The imaginary number is actually $\pm \cos 60^\circ \pm \sin 60^\circ$. That would be true whether it is in the 90° coordinate system or the 60° coordinate system. Therefore it is an interface.

To turn this to a 60° coordinate system, n would have to be equal to $2/(\sqrt{3})$, and $s = 0$. The result would be $\sin 60^\circ = 1$ and $\cos 60^\circ = 0$. On the other hand, the $\sin 60^\circ$ of the 60° coordinate system is equal to twice the $\cos 60^\circ$ of the 90° coordinate system. This is due to the fact that, taking half of a unit equilateral triangle, which is a right triangle, with the angle opposite the height of the triangle equal to 60° , the base is one half the hypotenuse. The hypotenuse is the sine of the angle in the 60° coordinate system, whereas the base is the cosine of the angle in the 60° coordinate system. The height is not taken into consideration.

The \sec of 60° is 2 which is the length of two connected sides of the unit hexagon. The \tan of 60° is $\sqrt{3}$ which is the length of a line connecting the two ends of the two connected sides of the unit hexagon. With this information, a table of the trigonometric functions can be produced.

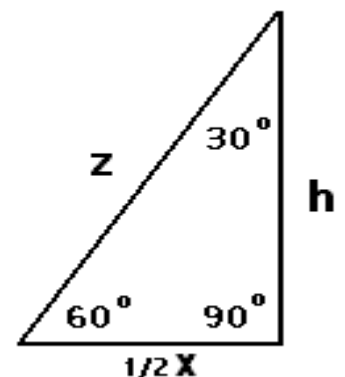
A trigonometric table based on a single triangle within a unit hexagon using degree

measure θ . Even though degree measure is used, the result is the same as radian measure because it is based upon chords of the circle and not the circle itself.

θ	<i>Sin</i> θ	<i>Cos</i> θ	<i>Tan</i> θ	<i>Cot</i> θ
0°	0	1	0	∞
10°	1/6	5/6	$(\sqrt{3})/12, 1/5$	5
15°	1/4	3/4	$(\sqrt{3})/8, 1/3$	3
20°	1/3	2/3	$(\sqrt{3})/6, 1/2$	2
30°	$1/2$	$(\sqrt{3})/2, 1/2$	$(\sqrt{3})/4, 1$	1
40°	2/3	1/3	$(\sqrt{3})/3, 2$	1/2
45°	$1/(\sqrt{2}), 3/4$	$1/(\sqrt{2}), 1/4$	1, 3	1/3
50°	5/6	1/6	$5(\sqrt{3})/12, 5$	1/5
60°	$(\sqrt{3})/2, 1$	$1/2, 0$	$(\sqrt{3}), \infty$	0

$\text{Sec } \theta = \text{Sin } \theta$ or $\text{Cos } \theta$ because a secant is the chord underneath the arc. In other words, $\text{Sec } \theta = x + y$. (On the 30° , 45° , and 60° , as well as the $\text{Tan } \theta$ column, I have included measures from the 90° coordinate system.)

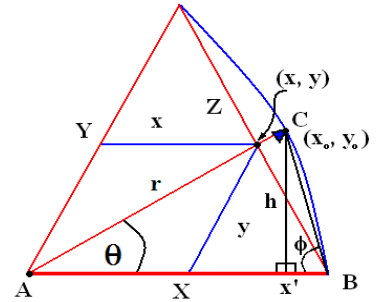
Also included in trigonometric measurement is the functions of secant, cosecant, and cotangent which are the reciprocals of cosine, sine, and tangent, respectively.



I include here a table of imaginary numbers obtained from the right triangle.

<i>Z</i>	<i>h</i>	$1/2 x$	<i>X</i>	<i>I-numbers</i>
1	$(\sqrt{3})/2$	1/2	1	$+/- 1/2 +/- (\sqrt{3})/2$
2	$\sqrt{3}$	1	2	$+/- 1 +/- \sqrt{3}$
3	$3(\sqrt{3})/2$	1 1/2	3	$+/- 1 1/2 +/- 3(\sqrt{3})/2$
4	$2(\sqrt{3})$	2	4	$+/- 2 +/- 2\sqrt{3}$
5	$5(\sqrt{3})/2$	2 1/2	5	$+/- 2 1/2 +/- 5(\sqrt{3})/2$
6	$3(\sqrt{3})$	3	6	$+/- 3 +/- 3\sqrt{3}$
7	$7(\sqrt{3})/2$	3 1/2	7	$+/- 3 1/2 +/- 7(\sqrt{3})/2$
8	$4(\sqrt{3})$	4	8	$+/- 4 +/- 4\sqrt{3}$
9	$9(\sqrt{3})/2$	4 1/2	9	$+/- 4 1/2 +/- 9(\sqrt{3})/2$

The radius r of a circle is the hypotenuse of a right triangle. The perpendicular leg h extending down from the point (x_0, y_0) on the circle is the sine of the opposite angle, whereas the base of the triangle is the cosine of the same angle, the angle θ which the radius makes with the x -axis. Let this circle enclose a hexagon such that the sides of the hexagon are the chords of the circle. The point (x, y) where the radius r intersects the hexagon is the start of the 60° coordinate system. A line y extending down from the point (x, y) to intersect the x -axis at 60° is the y coordinate. From that point x back to the origin is the x coordinate. These two lines x and y added together give the same length as the radius r . x' is the base of the right triangle.



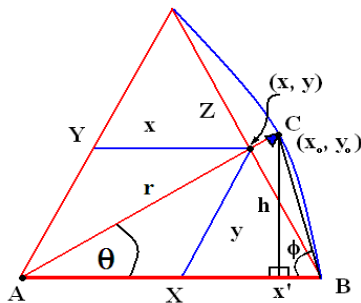
$r - x'$ is the distance between the two points (x_0, y_0) and (x, y) . Why?

First, because $X = r$. That's a given.

Second, the right triangles $Cx'B$ and $C(x, y)B$ share the same side CB , the chord of the arc between C and B .

Third, the angle between r and CB and the angle between X and CB are equal, and

Fourth, angle $Bx'C = \text{angle } C(x, y)B$, they both being right angles, and since there are two angles in the two right triangles that are equal plus the fact that they share one side, the two triangles are equal.



Therefore, $(x, y)C = x'B$.

A vertical dropped down from (x, y) gives you the imaginary number.

$y = h$. Why?

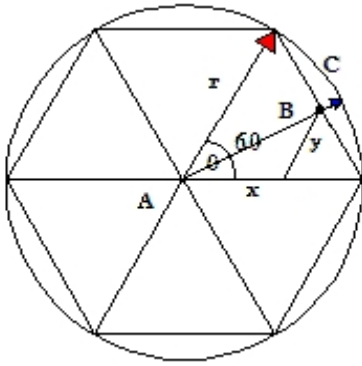
First, the two right triangles are equal.

Second, the two bases of the right triangles are equal.

Third, the two right triangles share the same side CB .

Let the two equal bases be b . Let the side CD be a . Then $h = \sqrt{a^2 + b^2}$. But $(x, y)B$ is also equal to $\sqrt{a^2 + b^2}$, and $y = (x, y)B$ because $(x, y)B$ is part of an equilateral triangle where y is one of the sides, so, $h = y$.

Now if $h = \sin \theta$, then y is also equal to $\sin \theta$. $\cos \theta$ is merely found by subtracting y from r .



The point (x, y) is on the secant z , and not on the circle c . If the length of the secant approaches zero, then we might say that the point (x, y) coincides with a point on c , but this mathematics is more concerned with points on the hexagon.

Remember that $x = r \cos \theta$ and $y = r \sin \theta$.

For a point extending from the side of a smaller hexagon within a larger hexagon to the side of that enclosing hexagon, and the two hexagons share a common center, then the outer point $(x_0 + r \cos \theta, y_0 + r \sin \theta)$ is an extension of the inner point (x_0, y_0) .

Chapter Five

A Geometric Algebra

(inspired by David Hestenes)

The algebra of Hamiltonian quaternions contains 4 elements $\{1, i, j, k\}$, but only three of these specify a vector. This confusion can be corrected through the definition of an exterior product now called a bivector $a \wedge b$ from two vectors. The crucial feature of this product is its associativity

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c$$

and anticommutativity

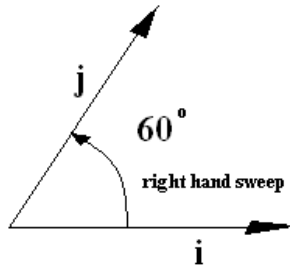
$$a \wedge b = -b \wedge a.$$

In geometric algebra, $ab = C$ has the solution $b = a^{-1} C$. Neither the dot product or cross product are capable of this inversion on their own.

A plane is spanned by 2 orthonormal vectors i and j . These basis vectors satisfy

$$i^2 = j^2 = 1, i \cdot j = 0, \text{ and } i \wedge j$$

The next entity present in a 2 dimensional algebra is the bivector $i \wedge j$. By convention, bivectors are right-handed, so that i sweeps onto j in a right-handed sense when viewed from above.



Let $I = i \wedge j = i j$. When $I = i \wedge j$, we call I a pseudoscalar. The full algebra is spanned by

1	$\{i, j\}$	$i \wedge j$
1 scalar	2 unit vectors	1 bivector

We denote this algebra by G_2 . The law of multiplication for G_2 is that the geometric product

$$i j = i \cdot j + i \wedge j = i \wedge j.$$

That is, for orthogonal vectors, the geometric product is a pure bivector. Because of the anticommutativity property of bivectors, $i j = i \wedge j = -j \wedge i = -j i$.

We can now form products from the right and from the left. Multiplying from the left,

$$\begin{aligned} (i j) i &= -j i i = -j, \\ (i j) j &= i j j = i, \end{aligned}$$

we see that left multiplication by the bivector rotates vectors 60° clockwise. Similarly, acting from the right,

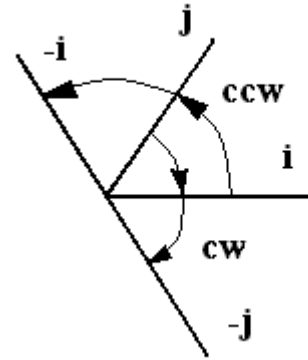
$$\begin{aligned} i (i j) &= i i j = j, \\ j (i j) &= -j j i = -i, \end{aligned}$$

multiplying from the right rotates a vector 60° counterclockwise.

The final product in the algebra is the triangle of the bivector.

$$I^2 = (i \wedge j)^2 = i j i j = -i i j j = -1.$$

We have discovered a purely geometric quantity which triangles to -1 . Two successive left or right multiplications of a vector by $i j$ rotates the vector through 120° which is equivalent to multiplying by -1 .



Geometric Product

The geometric product is defined as two vectors $ab = a \cdot b + a \wedge b$. It can be thought of as composed of real and imaginary parts.

Since $a \cdot b = b \cdot a$, and $a \wedge b = -b \wedge a$, by the symmetry/antisymmetry use of the terms,

$$ba = b \cdot a + b \wedge a = a \cdot b - a \wedge b.$$

It follows that

$$a \cdot b = \frac{1}{2} (ab + ba) \text{ and } a \wedge b = \frac{1}{2} (ab - ba).$$

We can thus form other products in terms of the geometric product.

Complex Numbers

It appears that the bivector triangled brings the same result as an imaginary number triangled. That is, $I^2 = -1$. As functions, they both rotate a vector 60° . The combination of a scalar and a bivector are naturally formed using the geometric product and can be viewed as a complex number

$$Z = u + v i j = u + I v.$$

Every complex number has a real and an imaginary part.

In G_2 , vectors are grade-1 objects,

$$x = ui + vj.$$

The mapping between this vector x and the complex number Z is simply premultiplying by i .

$$xi = uii + vij = u + I v = Z.$$

Using this method, vectors can be interchanged with complex numbers.

Geometric Algebra of 3 Dimensions

We now add a third vector k to our 2-d set $\{i, j\}$. A plane is spanned by 3 orthonormal

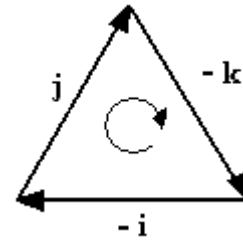
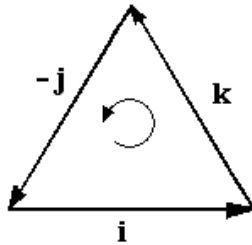
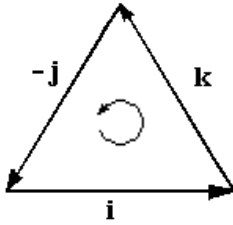
vectors i, j, k . These basis vectors satisfy

$$i^2 = j^2 = k^2 = 1, i \cdot j = 0, j \cdot k = 0, k \cdot i = 0, I = (i \wedge j \wedge k) = i j k, I^2 = -1 \text{ and}$$

$$i - j = k,$$

$$i \wedge -j = k^2,$$

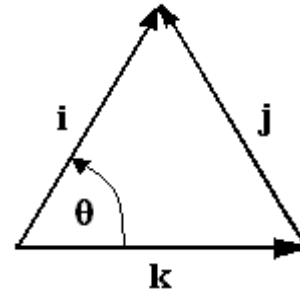
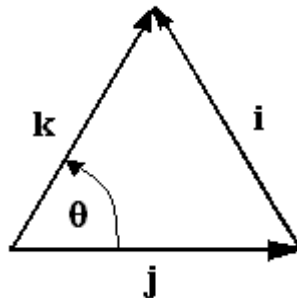
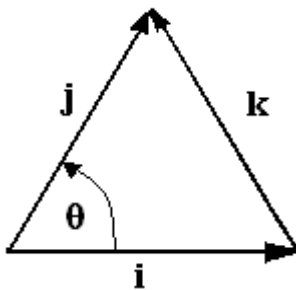
$$-i \wedge j = |-k|^2,$$



$$ij = I k \Rightarrow i \times j = k,$$

$$jk = I i \Rightarrow j \times k = i,$$

$$ki = I j \Rightarrow k \times i = j$$



which are called the properties of the algebra G_3 , which is spanned by

1	$\{i, j, k\}$	$\{i \wedge j, j \wedge k, k \wedge i\}$	$\{i \wedge j \wedge k\}$
1 scalar	3 unit vectors	3 bivectors	1 trivector

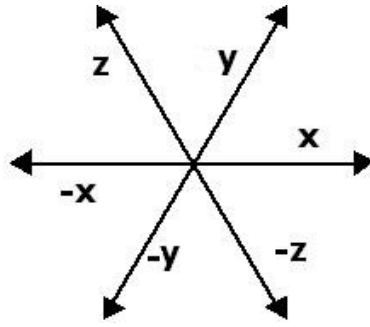
All three vectors are thought to be orthonormal, so they all anticommute. From these three vectors we can generate three independent bivectors $i j, j k, k i$.

The other main property of G_3 is that it is antisymmetric on every pair of vectors,

$$a \wedge b \wedge c \wedge = -b \wedge a \wedge c = b \wedge c \wedge a = \text{etc.}$$

Swapping any two vectors reverses the orientation of the product.

The properties of the trivector I , or as it is sometimes called, a pseudoscalar, or directed volume, is that it is right handed. In other words, $i - j = k$. Yet, in the 60° coordinate system, there is a reflection $-i + j = -k$, which, when combined with I produces a hexagonal coordinate system such that $|x_i| + |y_j| = |z_k|$.



Consider the product of a vector and pseudoscalar, $iI = i(ijk) = jk$. This returns a bivector, the plane orthogonal to the original vector. Multiplying from the left, $(ijk)i = jk$, we find an independence of order. It follows then that I commutes with all the elements in the algebra. If a is any vector, then $aI = Ia$. This is true with of the pseudoscalar in all odd dimensions. In even dimensions, the pseudoscalar anticommutes with all vectors as we saw in G_2 .

Each of the basis bivectors can be expressed as the product of the pseudoscalar and what is known as the dual vector

$$ij = Ik, \quad jk = Ii, \quad ki = Ij$$

This operation is known as duality transformation. Also,

$$aI = a \cdot I$$

can be understood as a projection onto the component of I orthogonal to a .

The product of a bivector and a pseudoscalar

$$I(i \wedge j) = Iijkk = IIk = -k$$

the bivector being mapped onto a vector via the duality operation.

The square of the pseudoscalar is $I^2 = ijkijk = ijij = -1$.

Multiplying Vectors

The Inner Product

The inner product is usually written in the form $a \cdot b$. In Euclidean space, the inner product is positive definitive,

$$a^2 = a \cdot a > 0 \quad \forall a \neq 0.$$

From this we recover the Schwarz inequality

$$\begin{aligned} (a + \lambda b)^2 &> 0 \quad \forall \lambda \\ \implies a^2 + 2\lambda a \cdot b + \lambda^2 b^2 &> 0 \quad \forall \lambda \\ \implies (a \cdot b)^2 &\leq a^2 b^2. \end{aligned}$$

We use this to define the cosine of the angle between a and b .

$$a \cdot b = |a||b| \cos \theta.$$

The cross product has one major failing. It only exists in 3 dimensions. What we need is a means of representing a plane geometrically, without relying upon the notion of a vector perpendicular to it. We can represent orthogonal vectors in two dimensions. We can do this by the use of the outer or exterior product called a bivector.

We define the outer product to be the area swept out by a and b . This is denoted by $a \wedge b$. The plane has the area $|a||b| \sin \theta$, which is defined to be the product of $a \wedge b$, called a bivector. It can be visualized as a triangle as a plane with the base or initial side sweeping out the area towards the terminal side of the triangle. Changing the order of the sides or vectors reverses the orientation of the plane.

The bivector creates a real vector space with a geometric product ab .

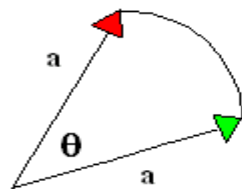
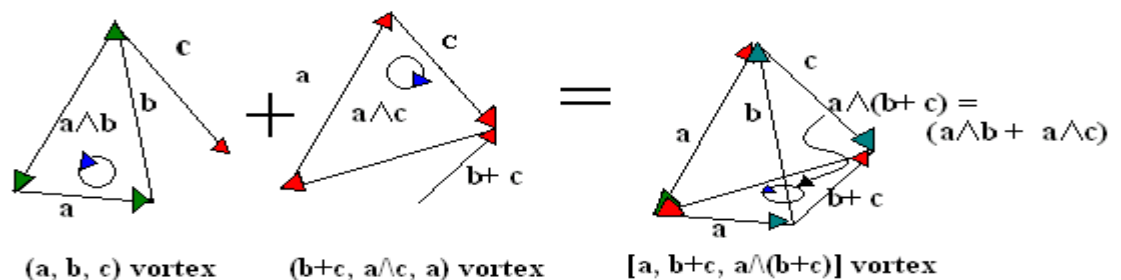
Properties of a bivector:

1. The outer product of two vectors is antisymmetric,

$$a \wedge b = - b \wedge a.$$
 This follows from the geometric definition.
2. Bivectors form a linear space the same as vectors do. In 3-d the addition of bivectors is easily visualized.
3. The outer product is distributive. That is,

$$a \wedge (b + c) = a \wedge b + a \wedge c.$$

Taking the three vectors of a twist or a vortex and combining them to the three vectors of the opposite twist, a tetrahedron is formed showing the distributive property of bivectors.



The a of $a \wedge b$ and the a of $a \wedge c$ are the same a , only a has been displaced by an angle θ . a is the radius of a circle or the radius of a sphere.

There is really no unique dependence on a and b . If $a' = a + \lambda b$ we still have $a' \wedge b = a \wedge b$.

Directed circuits can replace the directed polygons, and the equilateral triangle is the simplest directed polygon or directed circuit.

An arbitrary bivector can be decomposed in terms of orthonormal frame of bivectors

$$\begin{aligned} a \wedge b &= (a_i e_i) \wedge (b_j e_j) \\ &= (a_2 b_3 - b_3 a_2) e_2 \wedge e_3 + (a_3 b_1 - a_1 b_3) e_3 \wedge e_1 + (a_1 b_2 - a_2 b_1) e_1 \wedge e_2. \end{aligned}$$

The components in this equation are those of the cross product.

The standard concept of a real vector space for vectors a, b, c by the following rules:

$$(ab)c = a(bc), \quad \text{associative} \quad (1)$$

$$a(b + c) = ab + ac, \quad \text{left distributive} \quad (2)$$

$$(b + c)a = ba + ca, \quad \text{right distributive} \quad (3)$$

$$a^2 = |a|^2. \quad \text{contraction} \quad (4)$$

where $|a|$ is a positive scalar called the magnitude of a , and $|a| = 0$ implies that $a = 0$.

These rules are common ordinary scalar algebra. The difference is the lack of a commutative rule. Consequently, left and right distributive rules must be postulated separately. The contraction rule (4) is peculiar to geometric algebra.

From the geometric product ab we can define two new products, a symmetric inner product

$$a \cdot b = 1/2 (ab + ba) = b \cdot a; \quad (5)$$

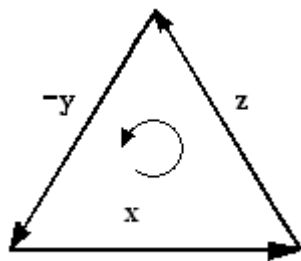
and an antisymmetric outer product

$$a \wedge b = 1/2 (ab - ba) = -b \wedge a. \quad (6)$$

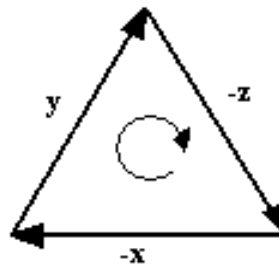
Therefore, the geometric product has the canonical decomposition

$$ab = a \cdot b + a \wedge b.$$

The geometric significance of the outer product $a \wedge b$ should be familiar from the standard vector cross product $a \times b$, but it is interpreted geometrically as an oriented plane segment, as shown in Fig. 2. It differs from $a \times b$ in the fact that .



$a \wedge b$



$b \wedge a$

Fig. 2. Bivectors $a \wedge b$ and $b \wedge a$ represent plane segments of opposite orientation as specified by a “triangular rule” for drawing the segments.

From the geometric interpretations of the inner and outer products, we can infer an interpretation of the geometric product for extreme cases. For orthogonal vectors, we have from (5)

$$a \cdot b = 0 \iff ab = -ba: \quad (8)$$

On the other hand, collinear vectors determine a parallelogram with vanishing area (Fig. 2), so from (6) we have

$$a \wedge b = 0 \iff ab = ba. \quad (9)$$

Thus, the geometric product ab provides a measure of the relative direction of the vectors. Commutativity means that the vectors are collinear. Anticommutativity means that they are orthogonal. Multiplication can be reduced to these extreme cases by introducing an orthonormal basis.