Loss of Structurally Stable Regulation Implies Loss of Stability

D. H. S. Maithripala and Jordan M. Berg
Department of Mechanical Engineering
Texas Tech University
{sanjeeva.maithripala,jordan.berg}@ttu.edu

W. P. Dayawansa
Department of Mathematics and Statistics
Texas Tech University
daya@math.ttu.edu

Abstract

Conditions for the existence of a structurally stable regulator at a particular plant are well known. We refer to systems for which these conditions fail as *critical*. At a critical point at least one of the following must occur for any compensator: 1) it does not asymptotically reject the exogenous system, 2) it does not stabilize the closed-loop system, or 3) it does one or both of the preceding, but without structural stability. However, exactly which of these occurs has not before been explicitly addressed. We show here that a structurally stable regulator is necessarily *destabilizing* at a critical point. Further, we characterize the unstable poles that appear and also show that destabilization is inherently linked with structural stability.

I. Introduction

For a plant to be *regulated* means that certain designated outputs asymptotically approach zero. Typically it is also desirable that all internal states of the plant and the controller be stable. This must be accomplished despite the effects of an exogenous system which influences the plant in a specified way. The problem of regulation of linear systems subject to perturbation has a long history in the control literature. The foundations of the subject were laid by Wonham [15] and Francis [7], who introduced the notion of *structurally stable* regulation to account for the possibility of unspecified, but small, uncertainties in the plant. As described in [2], [7], [15], the requirement that a regulating feedback compensator (a *regulator*) be structurally stable turns out to be quite severe for MIMO systems, leading to the conditions that the number of inputs be greater than or equal to the number of outputs of the open-loop plant and that the controller contain multiple copies of the exogenous system—exactly as many as the number of output channels.

It is clear why structural stability is desirable. We design using a model that is at best a close approximation of the real system. With structural stability we are assured that if a model is “close enough," the corresponding compensator will regulate. However without it even an arbitrarily small error in the system model may make regulation impossible. Having achieved this level of robustness, it is natural to ask for more; namely, to design a compensator that can maintain the regulation property in the face of large deviations from a design point. In the case of arbitrarily small perturbation, stability is not a consideration. This is because the closed-loop eigenvalues are continuous functions of the system parameters. Hence a nominally stabilizing compensator is stabilizing in some neighborhood of the nominal system. When perturbations are allowed to be large, however, the possibility arises that the closed-loop system will not be stable. Thus the analysis of regulating compensators under large perturbations must explicitly consider stability as well as regulation.

In Theorem 8.5 of [15] Wonham gives, in terms of the open loop plant, conditions for existence of a structurally stable regulator. One well-known necessary and sufficient condition is that the open-loop transmission zeros and the exogenous system poles have no common elements. We refer to this as the *zero condition*, and call systems that violate it *critical*. The zero condition may be used to bound the largest region in which a single compensator can simultaneously regulate. From the work of [15] it is straightforward to see that violation of this condition implies failure of the regulator, however the mechanism of this failure is not clear. Is it the asymptotic tracking property which is lost, or is it stability, or is it just structural stability? We show in this paper that it is necessarily stability that is lost.

Cevik and Schumacher [6] show that at any critical system the closed-loop stability margin is zero for any structurally stable regulating compensator. This follows from Theorem 3.6 and Proposition 3.8 of [6]. Our result could be obtained by extending the analysis of [6], however their results are proved over the course of four papers [3]–[6] by means of formulating the structurally stable regulator problem in the setting of subspace valued functions. In contrast, the proof presented here is much more direct. Furthermore, we develop in the process two results which are themselves of interest. The first of these is a reformulation of some classic results of [15] when the regulator is not assumed to be stabilizing. The second shows that when a transmission zero of the plant coincides with a pole of the exosystem, the common element must also be a pole of the closed-loop system. A limited version of these results first appeared in [11]. We mention here a remark by Wonham ([15], Remark 3 on p. 206) to the effect that once a structurally stable regulating compensator has been constructed, the output regulation property is not influenced by varying the plant. Hence any failure must be due to loss of stability. However no formal statement or proof is offered there.

Our proofs of the above results rely on the structural stability of the regulator. Hence it is natural to ask whether the result also applies when structural stability is relaxed. We refer to an example from [15] that shows the answer to be no. That is, a compensator which is regulating but not structurally stable is not necessarily destabilizing at a critical point.

Section II states definitions and provides needed theorems from the literature. Section III contains the statement and proofs of our main results.

II. Problem Formulation

We consider the system given by

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1 u, \tag{1}$$

$$\dot{x}_2 = A_{22}\, x_2, \tag{2}$$

$$y = E_1 x_1 + E_2 x_2, \tag{3}$$

where $x_1 \in \mathcal{X}_1$, $x_2 \in \mathcal{X}_2$, $u \in \mathcal{U}$ and $y \in \mathcal{Y}$ with $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{U}$ and $\mathcal{Y}$ being n_1, n_2, m and q dimensional vector spaces respectively. The exogenous system (2), or exosystem for short, models disturbances or reference inputs that act on the plant. The regulated output of the system is given by y and is assumed to be the same as the measured output. The following technical assumptions are fairly standard [2], [7], [15]: (A1) The pair (A_{11}, B_1) is stabilizable. (A2) The pair (E_1, A_{11}) is observable. (A3) The matrices B_1 and E_1 are of, respectively, full column and full row rank. (A4) All the eigenvalues of the exogenous system A_{22} are in the closed right half-plane. (A5) The exosystem A_{22} is not subject to perturbation.

An error feedback dynamic compensator of the form $\dot{z} = Nz + My; \quad u = Fz$, with $z \in \mathcal{Z}$, where $\mathcal{Z}$ is an n_c dimensional vector space, is employed to achieve regulation conditions with closed-loop stability for the given linear system. In contrast to the more traditional problem of finding a controller for a fixed plant, we seek to understand the behavior of a fixed controller as the plant it is applied to varies. We introduce nomenclature to reflect this viewpoint. The fixed controller is specified by the triple (F, N, M). Here we restrict our attention to the case that (F, N, M) is minimal, and N contains an exact q-fold copy of the maximal cyclic component of the exosystem A_{22}. We refer to such a triple as an *isolator*. This terminology is suggested by the key property of any such compensator (proved in Lemma 1 below), namely that when applied to a regular plant the exosystem is rendered unobservable from the closed-loop output; hence the output is “isolated” from the exosystem. The property of being an isolator relates only to the exosystem, and is independent of the plant. It is essential to note that the isolator is not necessarily stabilizing.

The closed-loop dynamics, omitting the exogenous system, can be expressed as

$$A_L = \begin{bmatrix} A_{11} & B_1F \\ ME_1 & N \end{bmatrix}, \quad B_L = \begin{bmatrix} A_{12} \\ ME_2 \end{bmatrix}, \quad E_L = \begin{bmatrix} E_1 & 0 \end{bmatrix}.$$

If all the eigenvalues of A_L have negative real parts, the closed-loop system is said to be *internally stable*. A complete compensator that provides both internal stability and regulation is called a *synthesis* ([15], page 200). If a compensator is a synthesis for every system in some neighborhood of a nominal point $p = (A_{11}, A_{12}, B_1, E_1, E_2)$, then it is referred to as a *structurally stable* synthesis ([15], page 200).

We now briefly review some classical results. The statements are made more compact using the notation for the linear matrix functions $L_1 := \left(A_{11} \otimes I_2^T - I_1 \otimes A_{22}^T\right)$ and $L_c := (B_1F \otimes I_2^T)$, and the subspaces $\mathcal{K}_1 := Ker(E_1 \otimes I_2^T)$ and $\mathcal{K}_c := Ker\left(N \otimes I_2^T - I_c \otimes A_{22}^T\right)$. Here $\otimes$ is the Kronecker product. These expressions result from rewriting matrix Sylvester equations in standard matrix-vector form.

The following theorem (Theorem 8.5, [15]) gives the necessary conditions for the existence of a structurally stable synthesis:

Theorem W 1: A structurally stable synthesis exists at p if and only if

$$L_1\mathcal{K}_1 + Im(B_1 \otimes I_2^T) = \mathcal{X}_1 \otimes \mathcal{X}_2'. \tag{4}$$

Equation (4) is exactly the zero condition, which is more commonly written in algebraic notation as

$$rank \begin{bmatrix} A_{11} - \lambda I & B_1 \\ E_1 & 0 \end{bmatrix} = n_1 + q \quad \forall \;\; \lambda \;\in\; \Lambda(A_{22}), \tag{5}$$

where $\Lambda(A_{22})$ is the spectrum of A_{22}. Thus, in keeping with our definition of critical systems, we refer to systems for which (4) fails as *critical*. Note that (4) also implies that the number of inputs, m, of the system (E_1, A_{11}, B_1) must be greater than or equal to the number of outputs, q.

Theorem W1 is concerned only with characterizing plants for which a structurally stable synthesis exists. Wonham [15] also states necessary and sufficient conditions under which a given stabilizing compensator is a structurally stable synthesis. However, we cannot use these results as stated in [15] because they intermingle the regulation and stability properties of a synthesis. In order to remove the stability assumption, we need to separate these cleanly. To this end we take a viewpoint similar to [2] and observe that the regulation requirement may be split into a condition on the existence of an unobservable closed-loop invariant subspace containing the exosystem, and another condition concerning the external stability of this subspace (which will be satisfied automatically if A_L is Hurwitz). The following re-formulation of the conditions from [15] for structural stability of a synthesis are the first step in that direction. It is obtained without difficulty from Theorem 8.6, Corollary 8.3 and Theorem 8.7 of [15].

Theorem W 2: Given a *stabilizing* compensator for the regular system p, denoted by the minimal triple (F, N, M), the following are equivalent:

W2.1 (F, N, M) is a structurally stable synthesis at p.

W2.2 $L_1\mathcal{K}_1 \oplus L_c\mathcal{K}_c = \mathcal{X}_1 \otimes \mathcal{X}_2'$ at p.

W2.3 N contains an exact q-fold copy of the maximal cyclic component of the exosystem, that is $dim(\mathcal{K}_c) = qn_2$. Hence in our terminology (F, N, M) is an isolator.

It is exactly the assumption of stability in Theorem W2 which we need to remove, in order to examine the effect of a fixed compensator on a family of plants. To that effect consider the following interpretation of *W2.2*: As shown in [15] (page 200), a dynamic compensator (F, N, M) makes the exosystem unobservable from the output if and only if there exist matrices X and Z such that

$$A_{11}X - XA_{22} + B_1FZ = -A_{12} + D, \tag{6}$$

$$E_1X = 0, \tag{7}$$

$$NZ - ZA_{22} = 0, \tag{8}$$

where $D = A_{11}\check{E}_1 E_2 - \check{E}_1 E_2 A_{22}$ and $\check{E}_1$ is the right inverse of E_1. Equations (6) – (8) state the existence of a closed-loop invarient subspace that entirely contains the dynamics of the exosystem, while (7) states that these dynamics are unobservable from the regulated output. If this subspace is externally stable then the regulation condition is satisfied and if, in addition, the portion of the plant and the controller lying within the subspace is stable then the internal stability condition is satisfied as well. In fact both of these are guaranteed if A_L is Hurwitz. This property is structurally stable by the continuity of the eigenvalues of A_L. Thus it only remains to characterize the structural stability of the properties of the unobservable subspace. This characterization is provided by the following weaker version of Theorem W2:

Lemma 1: For any minimal (F, N, M), (W2.2) implies both (4) and (W2.3). For the special case of square systems, given any isolator (F, N, M), (W2.2) and (4) are equivalent.

A solution (X, Z) to(6) – (8) is guaranteed to exist in any neighborhood of p if and only if the mapping represented by the left-hand side of (6) – (8) is full rank. This rank condition is equivalent to *W2.2*. Hence Lemma 1 says two things: 1) for the exosystem to be unobservable with structural stability the zero condition must be satisfied, and 2) the compensator (not necessarily stabilizing) must contain an exact q-fold copy of the exosystem (i.e., must be an isolator).

In the other direction, in the interests of space and clarity, we restrict ourselves to the square case. Here we see that if the compensator (not necessarily stabilizing) is an isolator and the zero condition is satisfied, then *W2.2* is satisfied. We stress that regulation need not follow, since external stability is not guaranteed. Note, however, if in adition the isolator is stabilizing (i.e., A_L is Hurwitz) then the regulation property as well as the internal stability property are guaranteed. The converse in Lemma 1 can be shown for general systems, but the proof is somewhat more involved.

We now prove Lemma 1. The following transversality condition is necessary. For a stabilizing isolator (hence a structurally stable synthesis) it is trivially true from Theorem W2, (W2.2). However we must show that it holds even when the isolator is not stabilizing. To our knowledge it is the first time the result has been reported and hence a proof is furnished. As shown below, this follows from a dimensionality check of the subspace equation (W2.2), and two classical results of [15].

Proposition 1: For any isolator (F, N, M), condition (W2.2) holds if and only if

$$L_1\mathcal{K}_1 \cap L_c\mathcal{K}_c = \{0\}. \tag{9}$$

Proof of Proposition 1: The following two lemmas are needed. The first, Lemma 8.7 of [15], is a geometric statement of the PBH test for observability:

Lemma W 1: The pair (E, A) is observable if and only if for any endomorphism $S : \mathcal{X}_k \to \mathcal{X}_k$

$$Ker\left(A \otimes I_k^T - I \otimes S^T\right) \cap Ker(E \otimes I_k^T) = \{0\}. \tag{10}$$

The second may be easily obtained from Theorem 8.7 and Corollary 8.3 of [15], and is a geometric statement of what it means to contain an q-fold internal model:

Lemma W 2: For any isolator (F, N, M)

$$\dim\left(Ker(N \otimes I_2^T - I_c \otimes A_{22}^T)\right) = qn_2. \tag{11}$$

Let (F, N, M) be a fixed isolator and assume condition (W2.2) holds. Then it is trivially observed that (9) is true. Conversely assume that (9) is true. Since by (A3) E_1 has full row rank, $dim(\mathcal{K}_1) = (n_1 - q)n_2$, and then by (A2) and Lemma 1, $dim(L_1\mathcal{K}_1) = (n_1 - q)n_2$. Further since the pair (F, N) is observable, and since from (A3) B_1 has full column rank, the pair (B_1F, N) is observable. Hence from Lemma W1 and Lemma W2, $dim(L_c\mathcal{K}_c) = q\,n_2$. Therefore it follows that $dim(L_1\mathcal{K}_1) + dim(L_c\mathcal{K}_c) = n_1\,n_2$, and hence from (9), condition W2.2 holds. $\Box$

We may now prove Lemma 1.

Proof of Lemma 1: Since $L_c = (B_1F \otimes I_2^T) = (B_1 \otimes I_2^T)(F \otimes I_2^T)$ it is easily seen that for any minmal (F, N, M) (W2.2) implies (4). Conversely, for square systems ($m = q$), assume that (W2.2) fails for any isolator. Then from Proposition 1, (9) fails. Which, once again, from $(B_1F \otimes I_2^T) = (B_1 \otimes I_2^T)(F \otimes I_2^T)$ implies that $L_1\mathcal{K}_1 \cap Im(B_1 \otimes I_2^T) \neq \{0\}$. Hence since $dim(L_1\mathcal{K}_1) + dim\,Im(B_1 \otimes I_2^T) = n_1\,n_2$, condition (4) fails for square systems. Condition (W2.2) implies $dim(L_c\mathcal{K}_c) = qn_2$, which implies W2.3. $\Box$

III. Loss of Regulation Implies Loss of Stability

Consider what may occur if an isolator (F, N, M) designed to stabilize one choice of a regular system p is applied to a different system, $\tilde{p}$. We noted earlier that any isolator applied to a *regular* point will make the exosystem unobservable from the output. (This was proved above for square systems, but may also be proved for general systems.) Hence the only way an isolator can fail to be a synthesis at a *regular* point $\tilde{p}$ is by failing to *stabilize* at $\tilde{p}$. It only remains to consider what happens when $\tilde{p}$ is a critical point. Clearly (F, N, M) cannot be a structurally stable synthesis at $\tilde{p}$, since (4) is a *necessary* condition for the existence of such a compensator. Thus at least one of three things must occur at $\tilde{p}$. The triple (F, N, M) may fail to regulate, that is, the dynamics of the exosystem may not be fully unobservable; or it may fail to stabilize the closed loop system; or finally it may be a stabilizing regulator, but not structurally stable. In this section we prove that necessarily it is stability that is lost when any isolator is applied at a critical point. Furthermore, the common element from the plant transmission zeros and the poles of the exosystem must appear as a pole of the closed-loop system. We begin by formally stating and proving this last fact.

Proposition 2: Given exosystem A_{22}, let (F, N, M) be any structurally stable synthesis at plant p. That is, let (F, N, M) be an isolator for A_{22} that also internally stabilizes plant p. In addition, let $\tilde{p}$ be a critical system with common element between the transmission zeros of the triple (E_1, A_{11}, B_1) and the eigenvalues of the exogenous system A_{22} denoted by λ^*. Then λ^* will also be an eigenvalue of the closed-loop system A_L for the plant $\tilde{p}$ compensated by the isolator (F, N, M).

Proof of Proposition 2: By assumption, (4) fails at $\tilde{p}$. Therefore from Lemma 1, W2.2 must fail, and so by Proposition 1, (9) must fail. Thus there exist X and Z satisfying (7) and (8) such that

$$A_{11}X - XA_{22} = -B_1FZ = W \in L_1\mathcal{K}_1 \cap L_c\mathcal{K}_c, \tag{12}$$

Pre-multiplying (7) by M and letting $X_L = [X^T \ Z^T]^T$ it follows that there exists a solution X_L to the Sylvester equation $A_LX_L - X_LA_{22} = 0$. The singularity of the Sylvester map implies the existence of a common element between the eigenvalues of A_L and the eigenvalues of A_{22}. Replacing A_{22} with the restriction of A_{22} to each of its eigenspaces, a straightforward calculation reveals that the common element must be λ^*. $\square$

Since the eigenvalues of A_{22} have been assumed to be in the closed right half-plane, Proposition 2 implies that any isolator (F, M, N) of A_{22} applied to a system for which (4) fails—that is, applied to any critical system—will result in a closed-loop system with at least one eigenvalue with non-negative real part.

Given the central role played in this analysis by the q-fold internal model, it is natural to ask whether the results still apply if the requirement of structural stability is dropped. The answer is no, as can be seen from Example 2, Section 8.5 of [15]. This example shows that a SISO system may be stabilized and regulated (but without structural stability) at a critical point. If we now construct a second SISO system, satisfying the zero condition, and associate with it a structurally stable synthesis, we can consider these two, independent, SISO systems as forming a single two-input, two-output system. This system violates the zero condition, so any compensator containing a 2-fold internal model must destabilize this system. However, we see that the 2×2 compensator formed in the natural way, which contains only *one* copy of the internal model, both regulates and stabilizes. We conclude from this example that the destabilizing effect we describe in this paper is inherently linked to the presence in the compensator of a q-fold internal model, and hence to the structural stability of the synthesis. We also note a related fact; namely that if the compensator contains *more than* q copies of the exosystem, exopoles appear in the closed loop at all points.

Our result may be applied to find an upper bound on the maximum stability margin achievable by a *robust* structurally stable regulator. Consider the stabilizing isolator (F, N, M) at p that achieves the maximal distance from p to the nearest system, $\tilde{p}$, destabilized by (F, N, M). Since Proposition 2 shows that *every* isolator is destabilizing at a critical point, clearly the achievable stability margin is bounded above by the distance from p to the nearest critical point. The following Proposition is a formal restatement of this bound:

Proposition 3: Let $\mathcal{M}$ be the set of all critical plants and let $d(\cdot, \cdot)$ be a metric on the space of systems. The distance to the nearest critical point, given by $\gamma_r = \inf_{\hat{p} \in \mathcal{M}} d(p, \hat{p})$ is an upper bound on the smallest unstructured perturbations to the nominal plant p that will cause closed-loop instability for any structurally stable synthesis at p.

Numerous metrics have been proposed to measure uncertainties in system space. These include the Graph, Gap ($\mathcal{H}_2$-gap), Pointwise Gap, $\mathcal{L}_2$-gap and the ν-gap [8], [10], [14]. We demonstrate the use of our result using the most easily computed, the $\mathcal{L}_2$-gap. This is defined as the gap between the $\mathcal{L}_2$ graphs of the two systems [12], [14].

Corollary 1: Let $\mathcal{M}$ be the set of all critical plants that correspond to some marginally stable exosystem. and let $N(s)D(s)^{-1}$ be a normalized stable right coprime factor representation of the plant p. Then if $d_g(\cdot,\cdot)$ is the $\mathcal{L}_2$-gap metric, $\gamma_r = \inf_{\hat{p}\in\mathcal{M}} d_g(p,\hat{p}) = \min_{\lambda\in\Lambda(A_{22})} \sigma_q(N(\lambda))$ where $\sigma_q(N(s))$ denotes the q^{th} singular value of $N(s)$ and $\Lambda(A_{22})$ denotes the spectrum of A_{22}.

Proof of Corollary 1: Let $N_iD_i^{-1}$ and $\tilde{D}_i^{-1}\tilde{N}_i$ be normalized stable right and left coprime factorizations of a plant p_i, respectively. Further let the critical plants p_2 that have a zero at $\lambda_k = j\omega_k$ be $\mathcal{M}_k$. That is let $\mathcal{M}_k = \{\tilde{D}_2^{-1}(s)\tilde{N}_2(s) : \sigma_q(\tilde{N}_2(j\omega_k)) = 0\}$. Therefore $\mathcal{M} = \cup_{\lambda_k\in\Lambda(A_{22})}\mathcal{M}_k$. Thus $\inf_{p_2\in\mathcal{M}} d_g(p_1,p_2) = \min_{\lambda_k\in\Lambda(A_{22})} \inf_{p_2\in\mathcal{M}_k} d_g(p_1,p_2)$. From [12], [14] we have $d_g(p_1,p_2) = ||\tilde{D}_2N_1 - \tilde{N}_2D_1||_\infty$ where $||\cdot||_\infty$ denotes the infinity norm. Since there exists $\bar{x}$ with $||\bar{x}|| = 1$ such that $\bar{x}^*\tilde{N}_2(j\omega_k) = 0$ and since $||\tilde{D}_2(j\omega_k)|| = 1$ we have the following,

$$\begin{aligned}
\inf_{p_2\in\mathcal{M}_k} d_g(p_1,p_2) &\geq \inf_{p_2\in\mathcal{M}_k} \left|\left| \left[\bar{x}^*\tilde{D}_2N_1 - \bar{x}^*\tilde{N}_2D_1\right](j\omega_k)\right|\right| \\
&\geq \inf_{p_2\in\mathcal{M}_k} \left|\left| \left[\bar{x}^*\tilde{D}_2N_1\right](j\omega_k)\right|\right| \geq \sigma_q(N_1(j\omega_k)).
\end{aligned} \tag{13}$$

Next we show that in fact there exists $p_2 \in \mathcal{M}_k$ such that this lower bound is achieved.

Let K be a constant matrix of dimension $q \times m$ selected as follows. We know from the singular value decomposition of $N_1(j\omega_k)$, that there exists unitary matrices U, V such that $N_1(j\omega_k) = U\Sigma V^*$ where, $\Sigma = \begin{bmatrix} \Sigma_q & 0 \end{bmatrix}$ with $\Sigma_q = diag(\sigma_1(N_1(j\omega_k)), \ldots\ldots, \sigma_q(N_1(j\omega_k))$. Let $\bar{\Sigma} = \begin{bmatrix} \bar{\Sigma}_q & 0 \end{bmatrix}$ where $\bar{\Sigma}_q$ is a $q \times q$ matrix with all zero elements except for the q^{th} diagonal entry which is $\sigma_q(N_1(j\omega_k))$. Selecting $K = U\bar{\Sigma}V^*$ we have that $\sigma_q\left(N_1(j\omega_k) - K\right) = 0$, and $||K|| = \sigma_q(N_1(j\omega_k))$. From [1] Theorem 5.1 it follows that there always exists a stable proper real rational matrix $X(s)$ such that $||X(s)|| \leq ||K|| = \sigma_q(N_1(j\omega_k))$ and $X(j\omega_k) = K$. This is a standard matrix boundary Nevanlinna-Pick interpolation problem. Thus if we let $Q(s) = N_1(s) - X(s)$ we have that $\sigma_q(Q(j\omega_k)) = 0$ from which follows that $Q(s)D_1(s)^{-1} \in \mathcal{M}_k$. Now selecting $N_2D_2^{-1}$ to be a normalized stable right coprime factorization of QD_1^{-1} we have the following.

$$\begin{aligned}
\inf_{p_2\in\mathcal{M}_k} d_g(p_1,p_2) &= \inf_{p_2\in\mathcal{M}_k} ||\tilde{D}_2N_1 - \tilde{N}_2D_1||_\infty = \inf_{p_2\in\mathcal{M}_k} \left|\left|\tilde{D}_2\left(N_1 - N_2D_2^{-1}D_1\right)\right|\right|_\infty \\
&\leq ||\tilde{D}_2\left(N_1 - Q\right)||_\infty = ||\tilde{D}_2X||_\infty \\
&\leq ||\tilde{D}_2||_\infty ||X||_\infty = \sigma_q(N_1(j\omega_k)).
\end{aligned} \tag{14}$$

hence from (13) it follows that $\inf_{p_2 \in \mathcal{M}_k} d_g(p_1, p_2) = \sigma_q(N_1(j\omega_k))$ $\Box$

A related result has been obtained for normalized coprime factor representations by Cevik and Schumacher [6]. They obtain an upper bound on stability when structural stable regulation is imposed. Interestingly, their result is exactly what we derive above for the $\mathcal{L}_2$-gap. Cevik and Schumacher further show in [6] that if a certain desired stability margin γ_d is less than γ_r and the maximum stability margin γ_{max} achievable with *any* compensator [9], then there exists an isolator with stability margin γ_d. They thus conclude that the minimum of $\{\gamma_r, \gamma_{max}\}$ is an upper bound on the maximum achievable stability margin for regulation with internal stability.

IV. Conclusions

Loss of structurally stable regulation implies loss of stability. Our proof of this follows from well-known theorems of the classical literature. A system for which the transmission zeros have an element in common with the poles of the exosystem is destabilized by any structurally stable regulator. Furthermore, the instability is guaranteed by the appearance of exactly that common element as a closed loop eigenvalue. However, as seen from an example discussed herein, this does not necessarily occur when the requirement of structural stability is dropped. This observation has relevance to control designers who must trade off possible loss of regulation versus possible loss of stability.

We have used our result to bound in any metric the largest stability margin that may be achieved using structurally stable regulation. We write an explicit formula for this bound in the case of the $\mathcal{L}_2$-gap. Finally we note that the mechanism of loss of stability, in which the closed-loop spectrum contains an element from the eigenvalues of the exosystem, has additional consequences for robust stabiliztion. Consider the case that the exosystem has poles deep in the right half-plane. This situation arises, for example, when considering nonlinear regulation of chaotic exosystems [13]. In this case, by the continuity of the closed-loop spectrum as the entries of the system matrices vary, the closed-loop must become unstable well before the transmission zero actually reaches the exosystem pole. Hence the margins in this case may be significantly less than either our bound or the results of [6]. (Note that the analysis of [6] explicitly assumes that the exopoles are on the imaginary axis.)

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