Almost-Global Tracking of Simple Mechanical Systems on a General Class of Lie Groups

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Abstract

We present a general intrinsic tracking controller design for fully-actuated simple mechanical systems, when the configuration space is one of a general class of Lie groups. We first express a state-feedback controller in terms of a function—the "error function"— satisfying certain regularity conditions. If an error function can be found, then a general smooth and bounded reference trajectory may be tracked asymptotically from almost every initial condition, with locally exponential convergence. Asymptotic convergence from almost every initial condition is referred to as "almost-global" asymptotic stability. Error functions may be shown to exist on any compact Lie group, or any Lie group diffeomorphic to the product of a compact Lie group and $\mathcal{R}^n$. This covers many cases of practical interest, such as $SO(n)$, $SE(n)$, their subgroups, and direct products. We show here that for compact Lie groups the dynamic configuration-feedback controller obtained by composing the full state-feedback law with an exponentially convergent velocity observer is also almost-globally asymptotically stable with respect to the tracking error. We emphasize that no invariance is needed for these results. However, for the special case where the kinetic energy is left-invariant, we show that the explicit expression of these controllers does not require coordinates on the Lie group. The controller constructions are demonstrated on $SO(3)$, and simulated for the axi-symmetric top. Results show excellent performance.

I. INTRODUCTION

Formally, a holonomic simple mechanical system consists of (i) a smooth manifold, corresponding to the configuration space of the system, (ii) a smooth Lagrangian corresponding to kinetic energy minus potential energy and (iii) a set of external forces or one-forms [1]. When some of these forces may be used for control, we refer to a simple mechanical control system [6], [13], [21], [30]. The study of mechanical systems from a modern geometric point of view can be found, for example, in the excellent texts of Abraham and Marsden [1], Arnold [5], Bloch [12], Bullo and Lewis [15], and Marsden and Ratiu [26]. In many systems of practical and theoretical interest, the configuration space of the system may be given the structure of a Lie group. Examples include underwater vehicles, satellites, surface vessels, airships, hovercraft, robots, and MEMS [4], [3], [11], [13], [14], [39], [40], [23], [25].

Ideally, control methodologies on a manifold should be intrinsic, that is, they should not depend on the choice of coordinates. To have globally well-defined behavior, the controller must be intrinsic. The problem of stabilizing a forced equilibrium of a simple mechanical system has been treated in the general setting, using purely intrinsic formulations, by many researchers [9], [11], [12], [10], [21], [30], [32], [33], [38]. The stabilization of a desired equilibrium of such a system by means of a suitable choice of a potential function is extensively treated by Koditschek [21]. Bullo and Murray use the Riemannian structure of the configuration manifold of a fully-actuated simple mechanical control system to derive a full state, feedback-plus-feedforward controller that locally exponentially tracks a general bounded reference trajectory [13].

Significant progress in geometric control has been made by specializing results from the general Riemannian framework to Lie groups. However this approach may fail to fully exploit the additional structure available in the latter case. In fact, the group structure may be used to transform the trajectory tracking problem into the better understood problem of stabilizing the identity element. This is not possible in a general Riemannian setting. The chief difficulty in the general tracking problem lies in defining the tracking error, and tracking error dynamics. When the configuration space is a Lie group we have a natural notion of error dynamics, that is globally well defined. Given two elements of the group, g and h, define the *configuration error* to be gh^{-1}. This is a generalization of the inverse of the left attitude error traditionally used in rigid body dynamics [13], [21]. The configuration error is also an element of the Lie group, and has no analog in the general Riemannian approach. A major contribution of the current work is our computation of the intrinsic globally defined error dynamics. In doing so we rely heavily on the Lie group structure. This allows us to improve upon tracking results obtained for general Riemannian manifold and specialized to Lie groups [13], as well as tracking results derived for specific Lie groups such as $SO(3)$ [21]. The derivatives of the configuration error and the velocity error define the tracking error dynamics on the tangent bundle of the Lie group. For fully-actuated simple mechanical systems on Lie groups we show that there exists a feedback control that transforms the tracking error dynamics to a simple mechanical control system, with kinetic energy, potential energy, and damping, arbitrarily assignable using additional state-feedback. Solution of the tracking problem is thus reduced to the task of stabilizing the identity element of this transformed system. Now the results of Koditschek [21] on set point control can be used to assign a suitable potential energy and damping and thereby stabilize the identity with "almost-global" asymptotic stability, that is, asymptotic stability for all initial conditions in an open and dense subset of the state space. The assigned potential energy should be a globally defined, smooth, proper Morse function, with a unique minimum at the identity. Koditschek [21] refers to such functions as navigation functions, but in this paper, to reflect the emphasis on tracking, we adopt the nomenclature [13] and refer to them as *error functions*. On a general Riemannian manifold the absence of the group operation precludes the definition of an intrinsic configuration error and associated tracking error dynamics on the state space. In this sense, the tracking problem on a Lie group is more closely related to tracking on R^n than it is to the general Riemannian case, for which the group operation is lacking.

Suitable error functions are constructed for Lie groups of practical interest by Koditschek [21], and by Dynnikov and Vaselov [18]. A result of Morse [29] shows that such functions exist on any compact connected manifold. By a straightforward extension of these results, such functions also exist on any Lie group diffeomorphic to a Lie group of the form $G \times \mathcal{R}^n$, where G is any compact connected Lie group. If the Lie group is not of the form given above, the existence of a globally defined error function is, to our knowledge, an open question. We show that almost-global stabilization of the identity element of the transformed system yields almost global tracking for the original system. Unless the Lie group is homeomorphic to $\mathcal{R}^n$, the presence of anti-stable equilibrium points and saddle points and their stable manifolds will limit the achievable asymptotic stability of the identity to be at best almost global [21]. This implies that our almost-globally asymptotically stabilizing tracking controller achieves the best possible convergence property.

Implementation of full state-feedback requires both configuration and velocity measurements. In application it is not unusual for only one of these to be available for measurement. In some cases it is the velocities [4], [28], [35], while in others it is the configuration [3], [37], [36]. This paper is concerned with the latter case, where dynamic estimation of the velocities is necessary. We show that a "separation principle" applies for the dynamic configuration feedback tracking control obtained by composing the full state-feedback compensator with a velocity observer. Specifically we show that if the configuration manifold is compact then the dynamic configuration-feedback tracking controller almost-globally converges for any initial observer error for which the velocity converges exponentially. A recent paper by Aghannan and Rouchon gives such an intrinsic observer that provides locally exponentially convergent estimates of the velocities of a simple mechanical system on a Riemannian manifold, given configuration variable measurements [2]. We use this observer to implement the full state-feedback controller without velocity measurements.

In summary, the paper provides two main contributions. First we derive an intrinsic, globally valid, full state-feedback tracking control for any fully-actuated simple mechanical system on a very wide class of Lie groups. This tracking control guarantees almost-globally asymptotic stability with locally exponential convergence to an arbitrary twice differentiable configuration reference signal. To the best of our knowledge it is the first time such a general result has been reported. We next show a separation principle for compact Lie groups, namely that the dynamic configuration-feedback controller obtained by composing the full state-feedback law with an exponentially convergent velocity observer is also almost-globally asymptotically stable with respect to the tracking error. To the best of our knowledge this too is the first time such a result has been reported. We emphasize that, for these two results to hold, no invariance properties are required of the kinetic energy, potential energy, or external forces. However, for the special case where the kinetic energy is left-invariant, we show that the explicit expression of these controllers do not require coordinates on the Lie group.

In section II we briefly present notation and review mathematical background for simple mechanical control systems on Lie groups and in section III-A we present the full state-feedback tracking control. In section III-B we then show that the separation principle holds on any compact Lie group for the combination of the full state-feedback controller with any exponentially convergent velocity observer. We show in section III-C that invariance properties of the kinetic energy and the external forces may be exploited to considerably simplify the explicit expression of the controller. In section IV we demonstrate this construction for any simple mechanical system with left-invariant kinetic energy on $SO(3)$. Finally we specify the inertia tensor and external forces corresponding to the axi-symmetric top and simulate tracking, with performance seen to be excellent.

II. Mathematical Background

This section briefly describes the notations and a few geometric notions that will be employed in the rest of the paper. For additional details the reader is referred to the texts of [1], [19], [26], [34]. Let G be a connected finite dimensional Lie group and let $\mathcal{G} \simeq T_eG$ be its Lie algebra. The left translation of $\zeta \in \mathcal{G}$ to T_gG will be denoted $g \cdot \zeta = DL_g\,\zeta$; the right translation of $\zeta \in \mathcal{G}$ to T_gG will be denoted $\zeta \cdot g = DR_g\,\zeta$. The adjoint representation $DL_g \cdot DR_{g^{-1}}$ will be denoted Ad_g. The Lie bracket on $\mathcal{G}$ for any two $\zeta, \eta \in \mathcal{G}$ will be denoted $[\zeta, \eta] = \mathrm{ad}_\zeta\,\eta$ and the dual of the ad operator will be denoted ad^*. Any smooth vector field $X(g)$ on G has the form $g \cdot \zeta(g)$ for some smooth $\zeta : G \mapsto \mathcal{G}$. Let $\{e_i\}$ be any basis for the Lie algebra $\mathcal{G}$ and let $\{E_i(g) = g \cdot e_i\}$ be the associated left invariant basis vector field on G. Now $[e_i, e_j] = C_{ij}^k e_k$, where C_{ij}^k are the structure constants of the Lie algebra $\mathcal{G}$ ($C_{ij}^k = -C_{ji}^k$), and $[E_i, E_j] = C_{ij}^k E_k$.

A. The Riemannian Structure

For each $g \in G$, $I(g) : \mathcal{G} \mapsto \mathcal{G}^*$ is an isomorphism such that the relation $\langle\langle \zeta\,,\,\eta \rangle\rangle_{\mathcal{G}} = \langle I(g)\zeta\,,\,\eta \rangle$ for $\zeta, \eta \in \mathcal{G}$ defines an inner product on $\mathcal{G}$. Here $\langle \cdot, \cdot \rangle$ denotes the usual pairing between a vector and a co-vector. Identifying $\mathcal{G}^*$ and $\mathcal{G}$ with $\mathcal{R}^n$, let $I_{ij}(g)$ and $I^{ij}(g)$ be the matrix representations of $I(g)$ and $I^{-1}(g)$ respectively. $I(g)$ is symmetric and positive definite. If $I(g)$ is globally smooth then such an $I(g)$ induces a unique metric on G by the relation $\langle\langle g \cdot \zeta\,,\, g \cdot \eta \rangle\rangle = \langle I(g)\zeta\,,\,\eta \rangle$. Further, it follows that every metric has such an associated family of isomorphisms. If the metric is left-invariant then I is a constant and any constant symmetric positive definite matrix induces a left-invariant metric on G.

In the remainder of the section we present expressions for the Levi-Civita connection and the Riemannian curvature corresponding to left-invariant metrics. These derivations are based on Cartan's structural equations as presented in sections 9.3b – 9.3e of [19]. Similar material may be found in [34]. We intensionally avoid the use of coordinate-frame fields to facilitate the coordinate-free expressions developed in section III-C. As a result we obtain connection coefficients and the Riemannian curvature two-forms in a general left-invariant frame field in place of the more familiar Christoffel symbols and corresponding curvature coefficients in a coordinate frame field.

Associated with any metric there exists a unique connection that is torsion free and metric called the Levi-Civita connection. For a vector field $X = X^k\, E_k$ and a vector $v = v^k E_k$ the Levi-Civita connection is given by

$$\nabla_v X = (dX^k(v) + \omega^k_{ij}(g) v^i X^j) E_k\,, \tag{1}$$

where $\omega^k_{ij}(g)$ are the connection coefficients in the frame $\{E_k\}$. For left-invariant metrics the connection coefficients turn out to be constants, given by

$$\omega^k_{ij} = \frac{1}{2}\left(C^k_{ij} - I^{ks}(I_{ir}C^r_{js} + I_{jr}C^r_{is})\right). \tag{2}$$

Since in general the E_k are not coordinate vector fields, ω^k_{ij}, are not the Christoffel symbols. The corresponding coefficients of the Riemannian curvature two-forms R^k_{jab} are also constant and can be shown to be [24],

$$R^k_{jab} = (-\omega^k_{rj}C^r_{ab} + 2\omega^k_{ar}\omega^r_{bj}). \tag{3}$$

The Riemannian curvature is then

$$R_c(\zeta,\eta)\xi = \{R^k_{jab}\xi^j(\zeta^a\eta^b - \zeta^b\eta^a) - \omega^k_{ij}C^i_{ab}\zeta^a\eta^b\xi^j\}e_k. \tag{4}$$

B. Simple Mechanical Control Systems on Lie Groups

A simple mechanical control system evolving on a Lie group G equipped with a metric $\langle\langle\cdot\,,\,\cdot\rangle\rangle$ is defined as a system with kinetic energy $E(\dot g) = \frac{1}{2}\langle\langle\dot g\,,\,\dot g\rangle\rangle$, conservative plus dissipative forces $f(g,\zeta) \in \mathcal{G}^*$ and a set of linearly independent forces $u_i f^i(g) \in \mathcal{G}^*$ for $i = 1,\cdots,m$, [1], [6], [12], [15], [30]. The scalar functions $u_i \in \mathcal{R}$ are the controls. If $m = n = dim(G)$ the system is said to be fully actuated. In what follows we only consider fully-actuated simple mechanical systems.

Let $I(g) : \mathcal{G} \mapsto \mathcal{G}^*$ be the isomorphism associated with the kinetic energy metric; $\langle\langle g\cdot\zeta\,,\,g\cdot\eta\rangle\rangle = \langle I(g)\zeta,\eta\rangle$ for $\zeta,\eta\in\mathcal{G}$. Then the Euler-Lagrange equations of motion of the system are

$$\nabla_{\dot g}\dot g \;=\; g\cdot I^{-1}(g)\left(f(g,\zeta) + \sum_i^m u_i f^i(g)\right), \tag{5}$$

where $\dot{g} = g \cdot \zeta$. This can also be expressed as a dynamical system on the left trivialization $G \times \mathcal{G}$ of TG as

$$\begin{aligned} \dot{g} &= g \cdot \zeta, & (6) \\ \dot{\zeta} &= \tilde{f}(g,\zeta) + I^{-1}(g)\left(f(g,\zeta) + \sum_{i}^{m} u_i f^i(g)\right), & (7) \end{aligned}$$

where $\tilde{f}(g,\zeta) = -\omega_{ij}^k(g)\zeta^i\zeta^j e_k$ are the inertial forces arising from the curvature effects. If the kinetic energy metric is left invariant then I is a constant and $\tilde{f}(g,\zeta) = I^{-1}\mathrm{ad}_\zeta^* \, I\zeta$.

III. Intrinsic Tracking for Simple Mechanical Systems

Let $g_r(t) \in G$ be an arbitrary twice-differentiable reference configuration trajectory to be tracked by the fully-actuated simple mechanical system (6)–(7). Let $\dot{g}_r(t) = g_r(t) \cdot \zeta_r(t)$. We introduce the *configuration error*,

$$e(t) = g_r(t)g^{-1}(t). \tag{8}$$

This is a generalization of the inverse of the left attitude error traditionally used in rigid body dynamics [13], [21]. It is intrinsic and globally defined. Most importantly, it is itself an element of the configuration space. In this regard it is a generalization of the usual notion of tracking error developed for the linear and nonlinear tracking problem on $\mathcal{R}^n$. Differentiating (8) and setting $\eta_e = \mathrm{Ad}_g(\zeta_r - \zeta)$, the error dynamics are computed to be

$$\begin{aligned} \dot{e} &= e \cdot \eta_e, & (9) \\ \dot{\eta}_e &= \mathrm{Ad}_g\left(\dot{\zeta}_r - \dot{\zeta} + [\zeta, \zeta_r]\right), & (10) \end{aligned}$$

where $\dot{\zeta}$ is given by (7). Observe that the error dynamics are defined on $TG \simeq G \times \mathcal{G}$ as well. As we now show, through a suitable choice of controls, the dynamics of the configuration error may be given the form of a fully-actuated simple mechanical system with arbitrarily assignable potential energy, kinetic energy, and damping.

A. Full State Feedback Tracking

Let $B = I^{-1}(g)[f^1(g) \;\; f^2(g) \;\; \cdots \;\; f^n(g)]$. Substituting

$$u = B^{-1}\Big(\dot{\zeta}_r - \tilde{f}(g,\zeta) - I^{-1}f(g,\zeta) + [\zeta,\zeta_r] - \mathrm{Ad}_{g^{-1}}\nu\Big) \tag{11}$$

in equation (10), we have the transformed error dynamics

$$\dot{e} = e \cdot \eta_e, \tag{12}$$

$$\dot{\eta}_e = \nu, \tag{13}$$

where $\nu \in \mathcal{G}$. These transformations reduce the problem of stably tracking the reference input to the problem of stabilizing $(id, 0)$ of (12)–(13). The error dynamics (12)–(13) are those of a fully-actuated simple mechanical control system on G. This is a key observation, since it has been shown that the stability of simple mechanical systems is completely determined by the nature of the potential energy. In particular, it is shown by Koditschek [21] that any given point of a compact configuration space with or without boundary may be made an almost-globally stable equilibrium by the appropriate choice of an potential energy function. Thus the task is now to assign a suitable potential energy, $F : G \mapsto \mathcal{R}$, such that the equilibrium $(id, 0)$ is almost-globally stable. The assigned potential energy function plays a role analogous to that of the norm in the case of tracking on R^n. The value of the potential energy for the configuration error is therefore a measure of the size of that error, and we refer to it as the *error function*. In the context of stabilization Koditschek uses the terminology navigation function [21]. The convergence properties are completely determined by the properties of the error function, and are independent of the specifics of the simple mechanical system. Thus for any given Lie group, solution of the tracking problem is reduced to the topological question of finding an appropriate error function for that space.

Remark 1: The system (12)–(13) is not in the standard form of a simple mechanical system since the inertial forces are missing. However it may be thought of as such a system in which a portion of the external forces exactly cancel the inertial terms arising from covariant differentiation, leaving only the controls.

Let $\tilde{I}$ be any symmetric positive definite matrix. This induces an inner product $\langle\langle \cdot\ ,\ \cdot \rangle\rangle_{\mathcal{G}}$ on $\mathcal{G}$ and a left invariant metric on G. This metric need not be related to the kinetic energy metric of the mechanical system under consideration. Let $\zeta_e = e^{-1} \cdot \operatorname{grad} F$, where $\langle dF, e \cdot \eta \rangle = \langle\langle e^{-1} \operatorname{grad} F, \eta \rangle\rangle_{\mathcal{G}}$ and

$$\nu = -\zeta_e - k\, \eta_e, \tag{14}$$

where k is a positive constant, yields the following error dynamics:

$$\dot{e} \;=\; e\cdot\eta_e, \tag{15}$$

$$\dot{\eta}_e \;=\; -\zeta_e - k\,\eta_e. \tag{16}$$

The complete state-feedback tracking control is

$$\begin{aligned} u \;=\;& B^{-1}\Big(\mathrm{Ad}_{g^{-1}}\zeta_e + k(\zeta_r-\zeta) + \dot{\zeta}_r - \tilde{f}(g,\zeta) \\ & - I^{-1}f(g,\zeta) + [\zeta,\zeta_r]\Big). \end{aligned} \tag{17}$$

Remark 2: Our approach implements a two-part composite control, in which the first component (11) is used to give the configuration error dynamics the structure of a simple mechanical control system and the second component (14) is used to assign a desired potential energy F and damping to the transformed system. The control (11) allows the full power of the results of [21] on set point control to be applied to tracking, via (14), in a very general setting.

Remark 3: The transformed system (12)–(13) is not a linear system unless the Lie group G is a vector space. Therefore the control (11) is different from feedback linearization. Unlike the approach presented above, linearization-based methods depend on the choice of coordinates and hence are, in general, not intrinsic nor globally well defined.

We now investigate the properties required by the assigned potential energy function F for almost-global tracking. A function with only non-degenerate critical points is called *Morse*.

Definition 1: An infinitely differentiable proper Morse function $F : G \mapsto \mathcal{R}$, bounded below by zero, and with a unique minimum at the identity is called an *error function*.

The following theorem shows that such an error function used with the state-feedback control (17) provides the strongest possible convergence properties.

Theorem 1: If $F : G \mapsto \mathcal{R}$ is an *error function* then the fully-actuated simple mechanical control system (6)–(7) with the control (17) almost-globally tracks any twice-differentiable configuration trajectory $g_r(t)$ with local exponential convergence.

Proof of Theorem 1: Consider $V(e,\eta_e) = F(e)+\frac{1}{2}\langle\langle\eta_e,\eta_e\rangle\rangle_{\mathcal{G}}$, where $F(e)$ is an error function. Taking the time derivative of V we have, $\dot{V} = -k\,\langle\langle\eta_e,\eta_e\rangle\rangle_{\mathcal{G}}$. Let $\mathcal{O}$ be the set of equilibria of (15)–(16). These are the points $(\bar{e},0)$ where $\bar{e}$ are the critical points of the error function F. By application of an extension of LaSalle's invariance theorem [30], and using the fact that all Lie groups are topologically complete, we find that all trajectories of the error dynamics

(15)–(16) converge to the set $\mathcal{O}$. The point $(id, 0)$ in $\mathcal{O}$ is the unique minimum of F. Since F is a Morse function, $\mathcal{O}_u = \mathcal{O} - (id, 0)$ consists of a finite number of points, all of which are anti-stable nodes or saddle points. Let the attracting set of the saddle points be denoted $\mathcal{S}$. Because F is a Morse function, no codimension-one set of saddle connections can separate the configuration space. Thus trajectories originating in the complement of $\mathcal{O}_u \cup \mathcal{S}$ converge to $(id, 0)$. $\mathcal{S}$ is contained within the closure of the stable manifolds of the finite number of saddle points. These stable manifolds are nowhere dense, hence $\mathcal{S}$ is also nowhere dense, and thus the complement of $\mathcal{O}_u \cup \mathcal{S}$ is open and dense. We conclude that the control (17) provides almost-global asymptotic tracking of the reference trajectory. Lemma 1, stated and proved in the appendix, shows that this convergence is locally exponential. $\square$

Remark 4: As pointed out in [21] no smooth vector field can have a global attractor unless the configuration manifold is homeomorphic to $\mathcal{R}^n$. Thus, in general, the global stabilization of the identity of the error dynamics is impossible, so almost-global asymptotic tracking is the best possible outcome. Previous work that specializes results on the general Riemannian tracking problem to Lie groups, notably [13], does not take full advantage of the Lie group structure, and hence guarantees only local convergence.

Remark 5: It is shown by Morse [29] that error functions always exist on any smooth compact connected manifold. By an extension of these results it also follows that such functions exist on any manifold diffeomorphic to a manifold of the form $G \times \mathcal{R}^n$ where G is compact and connected. These classes of Lie groups cover most of the situations of practical significance, including $SO(n)$, $SE(n)$, their subgroups and direct products. To apply theorem 1 on a general non-compact Lie group requires only a suitable error function. To the best of our knowledge it remains an open question as to whether such functions exist in the more general setting.

Remark 6: Local exponential convergence requires only that the assigned potential energy has a non-degenerate local minimum. Such functions can be constructed on any Lie group, and so the control (17) may be effective in a very general setting.

A *perfect* Morse function has exactly as many critical points as the homology of the underlying manifold requires. To minimize the number of unstable equilibrium points, whenever possible we use a perfect Morse function as the error function. Examples of perfect Morse functions on certain spaces, including $SO(n)$, $SE(n)$, $U(n)$, and $Sp(n)$, may be found in the literature [18]. Koditschek [21] gives an example of an error function that is a perfect Morse function on

$SO(3)$, which we use in Section IV.

B. Dynamic Output Feedback Tracking

The tracking control (17) involves both the configuration variable g and the velocity variable ζ. In this section we assume that only the configuration variables are available for measurement, and estimate the velocity. If ζ_o is the estimated value of ζ, the dynamic configuration-feedback tracking control obtained by composing a velocity observer with the state-feedback control (17) is

$$\begin{aligned} u &= B^{-1}(g)\left(\mathrm{Ad}_{g^{-1}}\zeta_e + k(\zeta_r - \zeta_o) + \dot{\zeta}_r \right. \\ &\qquad \left. -\tilde{f}(g,\zeta_o) - I^{-1}f(g,\zeta_o) + [\zeta_o,\zeta_r]\right), \end{aligned} \tag{18}$$

where g is measured. The following theorem provides a separation principle for this dynamic configuration-feedback control.

Theorem 2: Consider the fully actuated simple mechanical system (6)–(7) on a compact and connected Lie group G where the external forces are of the form $f(g,\zeta) = f^c(g) + f^d(g,\zeta)$ with $f^d(g,\zeta)$ linear in ζ. Then the dynamic configuration-feedback control (18) composed with any locally exponentially convergent velocity observer almost-globally tracks an arbitrary bounded twice-differentiable reference configuration trajectory $g_r(t)$ for sufficiently small initial observer errors.

We defer a proof of this theorem to the appendix.

Remark 7: The compactness assumption in theorem 2 can be relaxed if Ad_g, $f^d(g,\zeta)$ and $\omega_{ij}^k(g)$ are bounded for all $g \in G$.

One such locally convergent velocity observer for simple mechanical systems is presented in [2], for the case $f(g,\zeta) = f^c(g)$, that is, in the absence of damping. It is shown in [25] that velocity dependent external forces do not in fact affect convergence. Because this locally exponentially convergent observer is also intrinsic [2], its use in conjunction with (18) yields a globally well defined configuration-feedback tracking control. As guaranteed by theorem 2 convergence is almost-globally asymptotic for sufficiently small initial observer errors.

Remark 8: To this point no invariance properties have been assumed for the simple mechanical system. In particular neither the statements nor the proofs of theorems 1 and 2 have any such restrictions.

C. Coordinate-Free Representation

For many cases of interest the formulation of this dynamic configuration-feedback control can be considerably simplified and expressed explicitly in a coordinate-free manner. In particular if the kinetic energy of the system is left-invariant, the observer of [2] can be expressed explicitly without introducing coordinates on the Lie group [24], [25],

$$\begin{aligned} \dot{g}_o &= g_o \cdot (\zeta_o - 2\alpha\zeta_{oe}), && (19) \\ \dot{\zeta}_o &= I^{-1}\left(\mathrm{ad}^*_{\zeta_o} I\zeta_o - \alpha(\mathrm{ad}^*_{\zeta_{oe}} I\zeta_o + \mathrm{ad}^*_{\zeta_o} I\zeta_{oe})\right) + \alpha[\zeta_{oe}, \zeta_o] \\ &\quad +\Gamma(S(g, \zeta_0, u)) - R_c(\zeta_o, \zeta_{oe})\zeta_o - \beta\zeta_{oe}, && (20) \end{aligned}$$

where α, β are positive constants, and the configuration error $\zeta_{oe} \in \mathcal{G}$ is defined by $\exp(\zeta_{oe}) = g^{-1}g_o$ for g_o and g sufficiently close. Here $S(g, \zeta_o, u) = I^{-1}(f(g, \zeta_o) + \sum_i^m u_i(g, \zeta_o) f^i(g))$ and,

$$\Gamma(S) = (S^k - \omega^k_{ij} S^i \zeta^j_{oe})\, e^k. \tag{21}$$

The advantage of this formulation is that all the terms of the observer with the exception of the external forces S are independent of g. This leads to a compact and flexible representation that requires only changes to S and I to be adapted to different simple mechanical systems. Left-invariance of kinetic energy also allows the control (18) to be written as

$$\begin{aligned} u &= B^{-1}(g)\left(\mathrm{Ad}_{g^{-1}}\zeta_e + k(\zeta_r - \zeta_o) + \dot{\zeta}_r \right. \\ &\quad \left. -I^{-1}(\mathrm{ad}^*_{\zeta_o} I\zeta_o + f(g, \zeta_o)) + [\zeta_o, \zeta_r]\right), && (22) \end{aligned}$$

where now the inertial forces $\tilde{f}$ may be written in terms of ζ only. This is itself a significant simplification, and if in addition all external forces are also left-invariant then only the error feedback term $\mathrm{Ad}_{g^{-1}}\zeta_e$ is dependent on g. This last assumption is fairly common, see for example [13], [14], [39].

In the following section we explicitly compute the state- and dynamic-feedback tracking controller for any simple mechanical system on the Lie group $SO(3)$, with left-invariant kinetic energy. These expressions can be readily adapted to a particular application by specifying the inertia tensor I, and the external forces $f(R, \zeta)$. We make this specialization for the axi-symmetric top, and simulate the resulting performance.

IV. EXAMPLE: TRACKING ON $SO(3)$

The three-dimensional rotation group, $SO(3)$, is the Lie group of matrices $R \in GL(3, \mathcal{R})$ that satisfy $R\,R^T = R^T\,R = I$ and $det(R) = 1$. The Lie algebra $so(3)$ of $SO(3)$ is the set of traceless skew symmetric three-by-three matrices. Note that $so(3) \simeq \mathcal{R}^3$ where the isomorphism is defined by,

$$\xi \in \mathcal{R}^3 \mapsto \hat{\xi} = \begin{bmatrix} 0 & -\xi^3 & \xi^2 \\ \xi^3 & 0 & -\xi^1 \\ -\xi^2 & \xi^1 & 0 \end{bmatrix} \in so(3), \tag{23}$$

where $\xi = [\xi^1\ \xi^2\ \xi^3]^T$. We will use both ξ and $\hat{\xi}$ to mean the same element of $so(3)$, and identify $so(3)$ with $\mathcal{R}^3$ via the isomorphism (23). The adjoint representation $\mathrm{Ad}_R : so(3) \mapsto so(3)$ is explicitly given by $\mathrm{Ad}_R(\xi) = R\xi$, or $\mathrm{Ad}_R(\hat{\xi}) = R\hat{\xi}R^T$, respectively.

Define the isomorphism $I : so(3) \simeq \mathcal{R}^3 \mapsto so(3)^* \simeq \mathcal{R}^3$ by the positive definite matrix I. This induces a left invariant metric on $SO(3)$ by the relation, $\langle\langle R \cdot \xi\,,\, R \cdot \psi \rangle\rangle = \langle\langle \xi\,,\, \psi \rangle\rangle_{so(3)} = \langle I\xi, \psi \rangle$ for any two elements $R \cdot \xi, R \cdot \psi \ \in T_R SO(3)$. The Lie bracket on $so(3)$ is given by, $[\xi\,,\,\psi]_{so(3)} = \mathrm{ad}_\xi\,\psi = \xi \times \psi$. and the dual of the ad operator is given by, $\mathrm{ad}^*_\xi\,\Pi = \Pi \times \xi$, where $\Pi \in so(3)^* \simeq \mathcal{R}^3$.

From (6) – (7), a simple mechanical control system on $SO(3)$ with left-invariant kinetic energy takes the form,

$$\dot{R} = R\hat{\zeta}\,, \tag{24}$$

$$\dot{\zeta} = I^{-1}\left(I\zeta \times \zeta + f(R,\zeta) + \sum_i^m u_i f^i(R) \right). \tag{25}$$

A. Construction of the Controller

Let $R_r(t)$ be a twice-differentiable reference configuration trajectory to be tracked by (24)–(25) where $\dot{R}_r(t) = R(t)\hat{\zeta}_r(t)$. The intrinsic tracking error $e(t) \in G$ is given by $e(t) = R_r(t)R^T(t)$. Let $F(e)$ be an error function and let $\zeta_e = e^T grad\, F(e)$ with respect to the left-invariant kinetic energy metric induced by some three-by-three positive definite matrix $\tilde{I}$. The exact choice of $\tilde{I}$ is up to the designer and can be chosen based on transient performance considerations.

Consider the function $F(e) = \frac{1}{2} trace\{K(I - e)\}$, where K is a symmetric, positive definite three-by-three matrix. Using a key results of [17] it can be shown, as in [21], that F is a Morse function with four critical points and a unique minima at the identity. It can also be shown that $\zeta_e = \tilde{I}^{-1}\Omega_e$, where $\hat{\Omega}_e = (Ke - e^T K^T)$. This implies that the tracking control (17)

$$u = B^{-1}(R)\left(R^T\zeta_e + k(\zeta_r - \zeta) + \dot{\zeta}_r - I^{-1}(I\zeta \times \zeta + f(R,\zeta)) + \zeta \times \zeta_r\right), \tag{26}$$

achieves almost-global tracking with local exponential convergence. It is pointed out in [18], [21] that any Morse function on $SO(3)$ has at least four critical points. Thus this F is a perfect Morse function on $SO(3)$, and has the fewest possible unstable equilibria.

Remark 9: As an example of the application of the general results on set point control, Koditschek [21] derives an almost-globally asymptotically convergent full state-feedback law for the satellite tracking problem on $SO(3)$. This control is similar to (26) derived here by specializing (17) to $SO(3)$ with left-invariant kinetic energy. Three notable differences are that i) expression (26) is considerably simpler, ii) control (26) assigns arbitrary kinetic energy to the simple mechanical system of the error dynamics, giving an additional design degree of freedom, and iii) the approach described here admits systems with general conservative and damping forces.

The intrinsic observer (19) – (20) takes the form,

$$\dot{R}_o = R_o(\hat{\zeta}_o - 2\alpha\hat{\zeta}_{oe}), \tag{27}$$

$$\dot{\zeta}_o = I^{-1}\left(I\zeta_o \times \zeta_o - \alpha(I\zeta_o \times \zeta_{oe} + I\zeta_{oe} \times \zeta_o)\right) + \alpha\,\zeta_{oe} \times \zeta_o + \Gamma(S) - R_c(\zeta_o, \zeta_{oe})\zeta_o - \beta\zeta_{oe}, \tag{28}$$

where ζ_{oe} satisfies $exp(\zeta_{oe}) = R^T R_o$ and is given by,

$$\zeta_{oe} = \frac{\psi_o}{2\sin\psi_o}(R^T R_o - R_o^T R), \tag{29}$$

where, $\cos\psi_o = (tr(R^T R_o) - 1)/2$, for $|\psi_o| < \pi$ [26]. The parallel transport term $\Gamma(S)$ is calculated from (21) where $S(R,\zeta_o) = I^{-1}(f(R,\zeta_o) + \sum_i^m u_i f^i(R))$ and the curvature term $R_c(\zeta_o, \zeta_{oe})\zeta_o$ is calculated from (4).

With this observer the tracking feedback (26) can be implemented as,

$$u = B^{-1}(R)\left(R^T\zeta_e + k(\zeta_r - \zeta_o) + \dot{\zeta}_r - I^{-1}(I\zeta_o \times \zeta_o + f(R,\zeta_o)) + \zeta_o \times \zeta_r\right), \tag{30}$$

and achieves almost-global tracking with only the measurement of the configuration R.

B. Simulation Results

In this section we apply the feedback laws (17) and (18) to a simulation of a simple mechanical control system in $SO(3)$. We consider the classical problem of a axisymmetric top in a gravitational field. Let $P = \{P_1, P_2, P_3\}$ be an inertial frame fixed at the fixed point of the top and let $e = \{e_1, e_2, e_3\}$ be a body-fixed orthonormal frame with the origin coinciding with that of P. At $t = 0$, the two frames coincide. Let the coordinates of a point p in the inertial frame P be given by x, and in the body frame e be given by X. The coordinates are related by $x(t) = R(t)X$ where $R(t) \in SO(3)$. Let $-P_3$ be the direction of gravity and let I be the inertia matrix of the axisymmetric top about the fixed point. The kinetic energy of the top is $K = I\zeta^T\zeta/2$, where ζ is the body angular velocity and the potential energy is $U(R) = mgl\,(Re_3)^T P_3$. Here m is the mass of the top, g is the gravitational constant, l is the distance along the e_3 axis to the center of mass. For simplicity we assume the top to be symmetric about the e_3 axis, so $I = diag(I_1, I_1, I_3)$. The generalized potential forces $f(R)$ in the body frame are given by $\langle f(R), \zeta\rangle = -\langle dU, R \cdot \zeta\rangle = -mgl\,(R\hat{\zeta}e_3)^T P_3$ for any $\zeta \in so(3)$, which yields $f(R) = mgl\ (R^T P_3) \times e_3$. The system is a simple mechanical system on $SO(3)$, and the metric induced on $SO(3)$ by the kinetic energy is left-invariant. Thus the equations of motion on $SO(3) \times so(3)$ are, given by (24)–(25) where $f(R, \zeta) = mgl\ (R^T P_3) \times e_3$.

For convenience, let the desired reference configuration trajectory $R_r(t)$ be generated by the following simple mechanical system. It is not necessary for our results that the trajectory correspond to such a system.

$$\dot{R}_r = R_r\,\hat{\zeta}_r\,, \tag{31}$$

$$\dot{\zeta}_r = I^{-1}\,(I\zeta_r \times \zeta_r)\,. \tag{32}$$

We implement the controls (26) and (30). The simulation results shown in Figure-1 correspond to the full state-feedback (26). The rotation matrix R is parameterized by the unit quaternions (Euler parameters) [26]. The top parameters are $I_1 = I_2 = 1, I_3 = 2, mgl = 1$. The initial body angular velocity is $\zeta(0) = [1.3\ 1.2\ 1.1]$, and the initial configuration corresponds to a $\pi/2$ radian rotation about the $[1\ 1\ 1]^T$ axis. The reference trajectory $(R_r(t), \zeta_r(t))$ is generated by (31)–(32) for the initial conditions $\zeta_r(0) = [-.8\ \ -.3\ \ -.5]$ and $R_r(0) = id$.

The simulation results shown in Figure-2 correspond to the dynamic configuration-feedback (30) with $\alpha = \beta = 10$, and all other parameters as in the previous case. The initial observer velocity is zero and the initial observer configuration corresponding to a $0.9\pi/2$ radian rotation about the $[1\ 1\ 1]^T$ axis. Convergence properties are almost the same as for the full state-feedback control. The simulation results shown in Figure-3 correspond to the dynamic configuration-feedback stabilization of the unstable vertical equilibrium. The initial body angular velocity is $\zeta(0) = [3\ \ 2\ \ -1]$, and the initial configuration corresponds to a $\pi/3$ radian rotation about the $[1\ \ 1\ \ 1]^T$ axis. The initial observer velocity is $\zeta_o(0) = [3\ \ 2\ \ -1]$ and the initial observer configuration corresponds to a $0.9\pi/3$ radian rotation about the $[1\ \ 1\ \ 1]^T$ axis.

V. CONCLUSIONS

We have presented a globally-defined state-feedback controller for a fully-actuated, simple mechanical control system on a general class of Lie groups guaranteeing almost-global asymptotic stability, with locally exponential convergence . We have reduced this tracking problem to that of finding a suitable error function on the Lie group. Such functions exist on a wide class of Lie groups of interest, including compact Lie groups, such as $SO(n)$, Euclidian spaces, and their direct products, such as $SE(n)$. Extension of these results to infinite-dimensional and general non-compact Lie groups is a subject of future research. We have proved a separation principle for this controller with any locally exponentially convergent velocity observer. In particular using the intrinsic velocity observer of Aghannan and Rouchon [2] we obtain a globally well defined dynamic configuration-feedback controller. No invariance properties are required for the above results, but if available they allow a considerably simplified explicit formulation. The dynamic configuration-feedback controller was applied to the axi-symmetric heavy top, a simple mechanical system on the Lie group $SO(3)$. Simulation results show excellent performance.

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APPENDIX

For an error function $F(e)$ on G, it follows that the following two conditions are satisfied.

Condition 1: There exists a neighborhood $\mathcal{U}$ of the identity such that the identity is the only minimum of F in $\mathcal{U}$ and in addition the following holds for all $e \in \mathcal{U}$.

$$b_2^2\,||\text{grad}\,F||^2 \leq F(e) \leq b_1^2\,||\text{grad}\,F||^2, \tag{33}$$

for some constants b_1, b_2.

From Morse theory it follows that this, in fact, is true for any non-degenerate critical point of a function.

Condition 2: There exists a neighborhood $\mathcal{U} \times \mathcal{V}$ about $(id, 0)$ such that for all $(e, \eta) \in \mathcal{U} \times \mathcal{V}$

$$\frac{||H(e)\eta||}{\sqrt{F}} \leq M \quad \text{for some } M > 0, \tag{34}$$

away from the identity, where $H(e)$ is the Hessian of $F(e)$.

Since error functions are Morse, there always exists a neighborhood such that condition (33) holds, and thus there exists a neighborhood such that the Hessian is bounded (in fact there exists coordinates in which the Hessian is the identity in some neighborhood of the identity element). Thus for error functions there always exists a neighborhood $\mathcal{U} \times \mathcal{V}$ about $(id, 0)$ such that condition (34) holds away from $(id, 0)$. If the error function is such that $F(id) \neq 0$ then condition (34) holds every where in that neighborhood.

Lemma 1: Let $F(e)$ be an error function. Then the tracking control (17) achieves locally exponential tracking.

Proof of Lemma 1: Following the proof of local exponential stability in [13]. Let $\mathcal{U} \times \mathcal{V}$ be some neighborhood of $(id, 0)$ such that (33) – (34) holds. Since $F(e)$ is a Morse function such a neighborhood always exists. Since $(id, 0)$ is the only minimum of V in $\mathcal{U} \times \mathcal{V}$ and $\dot{V} \leq 0$ all trajectories of (15)–(16) that originate in $\mathcal{U} \times \mathcal{V}$ converge to $(id, 0)$. First we re-write $V(e, \eta) = F(e) + \frac{1}{2}||\eta_e||^2$ and $\dot{V}$ as follows,

$$V = \frac{1}{2}\begin{bmatrix} \sqrt{F} & ||\eta_e|| \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{F} \\ ||\eta_e|| \end{bmatrix} \tag{35}$$

$$\dot{V} = -\begin{bmatrix} \sqrt{F} & ||\eta_e|| \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} \sqrt{F} \\ ||\eta_e|| \end{bmatrix} \tag{36}$$

Let $W_c = \langle\langle \zeta_e, \eta_e \rangle\rangle_{\mathcal{G}}$ and $W = V + \epsilon W_c$ for some $\epsilon > 0$. From the Cauchy-Schwartz inequality we have that

$$\begin{aligned} W &\geq F - \epsilon||\zeta_e||||\eta_e|| + \frac{1}{2}||\eta_e||^2, \\ &\geq \frac{1}{2}\begin{bmatrix} \sqrt{F} & ||\eta_e|| \end{bmatrix} \begin{bmatrix} 2 & -\frac{\epsilon}{b_2} \\ -\frac{\epsilon}{b_2} & 1 \end{bmatrix} \begin{bmatrix} \sqrt{F} \\ ||\eta_e|| \end{bmatrix} \end{aligned} \tag{37}$$

and the right hand side of (37) is positive definite for sufficiently small ϵ and hence $V \leq \beta W$ for some $\beta > 0$. Consider $\dot{W}_c$.

$$\begin{aligned} \dot{W}_c &= \tilde{I}\dot{\zeta}_e \cdot \eta_e + \tilde{I}\zeta_e \cdot (-k\eta_e - \zeta_e), \\ &\leq ||\dot{\zeta}_e||||\eta_e|| + k||\zeta_e||||\eta_e|| - ||\zeta_e||^2, \\ &\leq ||\dot{\zeta}_e||||\eta_e|| + \frac{k}{b_2}\sqrt{F}||\eta_e|| - \frac{1}{b_1^2}F \end{aligned} \tag{38}$$

where the last inequality follows from (33). Letting $\alpha = -\frac{1}{2}(\frac{k}{b_2} - \frac{||\dot{\zeta}_e||}{\sqrt{F}})$ (38) becomes

$$\dot{W}_c \leq -\begin{bmatrix} \sqrt{F} & ||\eta_e|| \end{bmatrix} \begin{bmatrix} \frac{1}{b_1^2} & \alpha \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} \sqrt{F} \\ ||\eta_e|| \end{bmatrix}. \tag{39}$$

Since $W = V + \epsilon W_c$,

$$\dot{W} \leq -\begin{bmatrix} \sqrt{F} & ||\eta_e|| \end{bmatrix} \begin{bmatrix} \frac{\epsilon}{b_1^2} & \epsilon\alpha \\ \epsilon\alpha & k \end{bmatrix} \begin{bmatrix} \sqrt{F} \\ ||\eta_e|| \end{bmatrix}. \tag{40}$$

From condition (34) α is bounded and there exists a sufficiently small ϵ such that the right hand side of (40) is negative definite. Thus there exists $\gamma > 0$ such that $\dot{W} \leq -\gamma W$. Thus $W(t) \leq W(0)e^{-\gamma t}$. Since $V(t) \leq \beta W(t)$ and there exists a positive constant κ such that $W(0) \leq \kappa V(0)$,

$$V(t) \leq \kappa\beta V(0)e^{-\gamma t}.$$

Thus for any $(e(0), \eta_e(0)) \in \mathcal{U} \times \mathcal{V}$ the trajectories of (15)–(16) converge to $(id, 0)$ exponentially. □

Next we Consider the system

$$\dot{e} = e \cdot \eta, \tag{41}$$

$$\dot{\eta} = \tilde{I}^{-1}\left(\mathrm{ad}_\eta^* \tilde{I}\eta + f(e, \eta)\right) + \psi(e, \eta, q), \tag{42}$$

with $(e, \eta) \in G \times \mathcal{G}$ and $q \in \mathcal{R}^n$. The matrix $\tilde{I}$ is a positive definite symmetric 3×3 matrix and consider the Left invariant metric $\langle\langle \cdot, \cdot \rangle\rangle$ induced by $\tilde{I}$. Consider $V = F(e) + \frac{1}{2}||\eta||^2$ where $F(e)$ is an error function and also consider the following assumptions.

Assumption 1: The point $(id, 0)$ is an almost-globally stable equilibrium of (41)–(42) with $\psi \equiv 0$ and further more $\langle\langle e^{-1}\mathrm{grad}\, F, \eta \rangle\rangle + \langle\langle \tilde{I}^{-1} f, \eta \rangle\rangle \leq 0$.

The condition $\langle\langle e^{-1}\mathrm{grad}\, F, \eta \rangle\rangle + \langle\langle \tilde{I}^{-1} f, \eta \rangle\rangle \leq 0$ is satisfied by any simple mechanical system with Rayleigh type dissipation and with $F(e)$ as the potential energy function.

Assumption 2: The function $q(t) \in \mathcal{R}^n$ satisfies

$$||q(t)|| \leq c||q(0)||e^{-\lambda t}, \tag{43}$$

for some $c > 0, \lambda > 0$ and all $t > 0$.

Assumption 3: The interconnection term satisfies $\psi(e, \eta, 0) \equiv 0$ and the following linear growth condition in η,

$$||\psi|| \leq \gamma_1(||q||)||\eta|| + \gamma_2(||q||), \tag{44}$$

for two class $\mathcal{K}_\infty$ functions $\gamma_1(\cdot), \gamma_2(\cdot)$.

This assumption is automatically satisfied if the η dependence in $\psi(e, \eta, q)$ is linear and if either the Lie group is compact or $\psi(e, \eta, q)$ is independent of e.

Lemma 2: If assumptions 1 – 3 are satisfied, then the equilibrium $(id, 0)$ of the system (41) – (42) is almost globally stable. If the inequality in assumption 1 is strict the convergence is asymptotic.

Proof of Lemma 2: Under the assumptions 1 – 3 consider the following,

$$\begin{aligned} \dot{V} &= \langle\langle e^{-1}\mathrm{grad}\, F, \eta \rangle\rangle \\ &\quad + \langle\langle \tilde{I}^{-1} f, \eta \rangle\rangle + \langle\langle \eta, \psi \rangle\rangle \leq \langle\langle \eta, \psi \rangle\rangle, \\ &\leq ||\psi|| ||\eta||, \\ &\leq ||\eta|| \left(\gamma_1(||q||)||\eta|| + \gamma_2(||q||) \right), \end{aligned} \tag{45}$$

When $||\eta|| \geq 1$ (45) results in

$$\dot{V} \leq \left(\gamma_1(||q||) + \gamma_2(||q||) \right) ||\eta||^2, \tag{46}$$

The function $\gamma_1(||q||) + \gamma_2(||q||) = \gamma(||q||)$ is another class $\mathcal{K}_\infty$ function. Then (46) results in

$$\begin{aligned} \dot{V} &\leq \gamma(||q||)(||\eta||^2 + \sqrt{F}\, ||\eta||), \\ &\leq \begin{bmatrix} \sqrt{F} & ||\eta|| \end{bmatrix} \begin{bmatrix} \gamma & \frac{\gamma}{2} \\ \frac{\gamma}{2} & \gamma \end{bmatrix} \begin{bmatrix} \sqrt{F} \\ ||\eta|| \end{bmatrix}. \end{aligned} \tag{47}$$

The matrix defining the quadratic form of the right hand side of (47) is positive definite thus there exists a class $\mathcal{K}_\infty$ function $K_1(\cdot)$ such that the right hand side of (47) is less than

$K_1(||q||)V$. Thus

$$\begin{aligned} \dot{V} &\leq K_1(||q||)V, \\ &\leq K(||q(0)||)e^{-\theta t}V, \end{aligned} \tag{48}$$

for $K(\cdot)$ a class $\mathcal{K}_\infty$ function and $\theta > 0$. The last inequality follows from (43). Thus from (48)

$$V(t) \leq exp\left(K(||q(0)||)(1 - e^{-\theta t})\right)\ V(0), \tag{49}$$

therefore we have from the properness of V that $(e(t), \eta(t))$ remains bounded. Since $\lim_{t\to\infty} q(t) \to 0$ and (41) – (42) is almost-globally stable with $\psi \equiv 0$, the equilibrium $(id, 0)$ of (41) – (42) is almost-globally stable. Furthermore if the inequality in assumption 1 is strict then it follows that the equilibrium $(id, 0)$ with $\psi \equiv 0$ is almost-globally asymptotically stable. Thus if the inequality in assumption 1 is strict the equilibrium $(id, 0)$ of (41) – (42) is almost-globally asymptotically stable. □

We now prove theorem 2.

Proof of Theorem 2: Consider the dynamic tracking control of (18) along with a velocity observer. Then we have the error dynamics

$$\dot{e} = e \cdot \eta_e, \tag{50}$$

$$\dot{\eta}_e = -\zeta_e - k\eta_e + \psi(e, \eta_e, \xi_e), \tag{51}$$

where $\xi_e = \zeta_o - \zeta$ and

$$\psi(e, \eta_e, \xi_e) = \mathrm{Ad}_g \left[\omega_{ij}^k(g)\xi_e^i\xi_e^i\, e_k + (\omega_{ij}^k(g) + \omega_{ji}^k(g))\zeta^i\xi_e^j\, e_k - [\zeta_r, \xi_e] - k\xi_e + I^{-1}f^d(g, \xi_e)\right],$$

and $g = e^{-1}g_r$, $\zeta = \zeta_r - \mathrm{Ad}_{g^{-1}}\eta_e$. Thus if Ad_g, $f^d(g, \xi_e)$ and $\omega_{ij}^k(g)$ are bounded in g, that is bounded for all $g \in G$, then ψ satisfies the linear growth condition (44) for bounded $\zeta_r(t)$. On a compact Lie group G the boundedness assumption is automatically satisfied. Thus from Lemma 2 we have that (18) tracks almost-globally for all initial observer errors for which the velocity error of the observer satisfies $||\xi_e|| \leq c||\xi_e(0)||e^{-\lambda t}$. □

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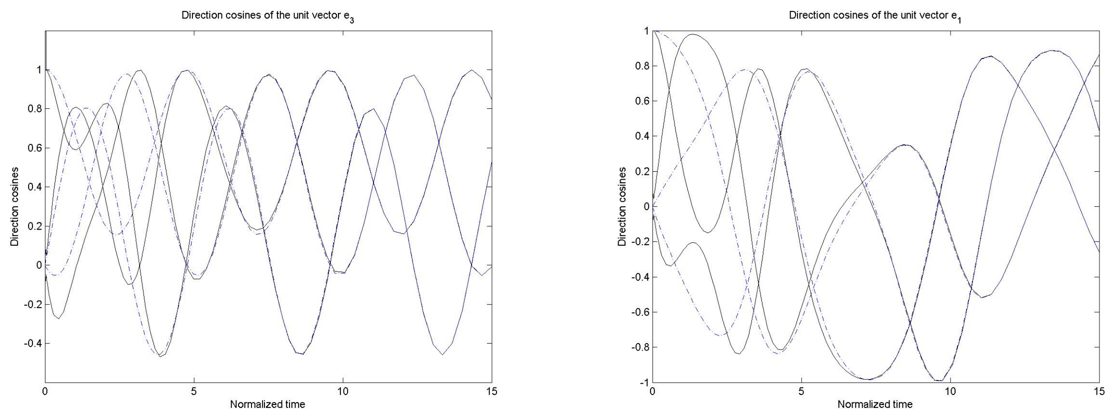
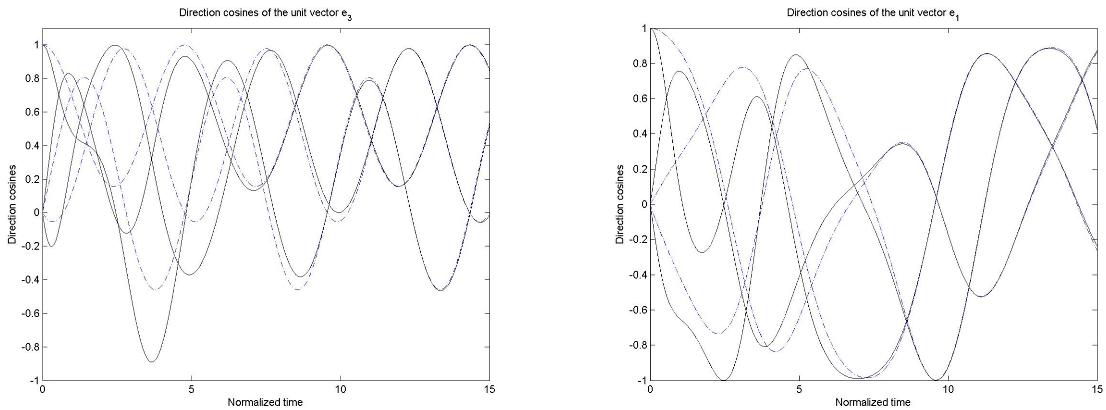


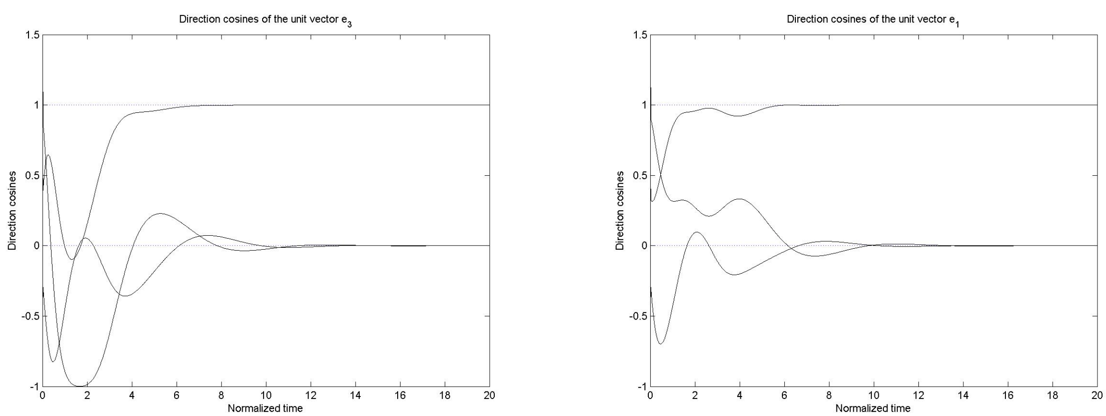
(a) Direction Cosines of e_3 (b) Direction Cosines of e_1

Fig. 1. Tracking with full state-feedback. The axi-symmetric top values are the solid lines while the dotted lines are the reference values.



(a) Direction Cosines of e_3 (b) Direction Cosines of e_1

Fig. 2. Tracking with dynamic configuration-feedback. The axi-symmetric top values are the solid lines while the dotted lines are the reference values.



(a) Direction Cosines of e_3 (b) Direction Cosines of e_1

Fig. 3. Stabilizing the vertical unstable equilibrium with dynamic configuration-feedback. The axi-symmetric top values are the solid lines while the dotted lines are the reference values.