INTRINSIC OBSERVER-BASED STABILIZATION FOR SIMPLE MECHANICAL SYSTEMS ON LIE GROUPS*

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Abstract. This paper presents a dynamic observer for a class of simple mechanical systems on Lie groups, that provides velocity estimates based on configuration measurements. The observer is *intrinsic*, so its performance does not depend on the choice of coordinates, and it is *coordinate free*, in the sense that the equations may be written explicitly without specifying coordinates for the configuration space. Our main result is obtained by specializing a previous result of Aghannan and Rouchon concerning velocity estimation of simple mechanical systems on Riemannian manifolds to such systems on Lie groups. This specialization is non-obvious, and extremely powerful. Further we extend the original result to include velocity dependent external forces. This estimator, combined with a coordinate-free formulation of passivity-based state-feedback control, allows the construction of a coordinate-free, intrinsic dynamic output feedback compensator. This is, to our knowledge, the first time such a result has been reported. Explicit expressions are computed for the Lie groups $SO(3)$ and $SE(3)$, allowing easy specialization to practical problems of rigid body motion. The theory is illustrated via application to the axi-symmetric top and to a six degree of freedom microelectromechanical system.

Key words. Nonlinear Observers, Mechanical Systems, Lie groups, MEMS

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1. Introduction. The traditional approach to nonlinear control has been to extend the extremely successful concepts developed for linear systems. This tactic has led to notable success, but it is inherently limited by the great variety of nonlinear phenomena. An alternative is to exploit the structural properties of specific classes of nonlinear systems. In particular, the systematic geometric study of *mechanical control systems* has received much attention. Formally, a holonomic simple mechanical system consists of (i) a smooth manifold, corresponding to the configuration space of the system, (ii) a smooth Lagrangian corresponding to kinetic energy minus potential energy and (iii) a set of external forces or one-forms [1]. When some of these forces may be used for control, we refer to a simple mechanical *control* system [6]. The study of mechanical systems from a modern geometric point of view can be found, for example, in the excellent texts of Abraham and Marsden [1], Arnold [5], Bloch [12], Bullo and Lewis [15] and Marsden and Ratiu [29]. Certain important nonlinear optimal control problems naturally lead to the consideration of such systems. The relationship between optimal control and mechanics is explored in essays by Bloch and Crouch [8] and by Jurdjevic [22]. Work by Bloch and Leonard [9, 11], Bloch [12], and the references therein, develops the notion of controlled Lagrangian systems, in which Lagrangian systems are stabilized by symmetry-preserving kinetic energy shaping and damping injection in such a way that the closed-loop system retains a Lagrangian structure [9, 11]. These results have been extended to the case where the uncontrolled mechanical system has no underlying symmetry [10]. A parallel development for controller synthesis in Hamiltonian systems is the Port Controlled

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Hamiltonian approach [33, 34, 40]. Both the Lagrangian and Hamiltonian methods make extensive use of structural features such as the Riemannian metric or the Poisson structure of the system. Symmetry preserving tracking controls have been developed recently for general control systems admitting symmetry by Martin et al. [30] using the geometric notion of Cartan's Moving frame method. The underlying Riemannian structure is exploited by Bullo and Murray [14] to derive intrinsic tracking controls for fully actuated simple mechanical systems on a general Riemannian manifold. The tools of passivity-based control have also been extended, through a generalization of LaSalle's invariance theorem, to a class of simple mechanical systems on Riemannian manifolds [32]. In a recent paper, Aghannan and Rouchon present an intrinsic observer that, given measurements of the configuration variables of a simple mechanical system on a Riemannian manifold, provides estimates of the states (both configurations and velocities) [2]. This is accomplished with a reformulation of the Luenberger observer, where the observation error is defined intrinsically by the geodesic distance between the actual and estimated configuration variables. These powerful results concerning the dynamics and control of systems on manifolds are intrinsic, implying that the performance will not depend on the choice of coordinates. However, on a general Riemannian manifold, their explicit expression requires coordinates. On a Lie group however, due to the natural identification of tangent spaces and neighborhoods of a point by left-translation, coordinate-free explicit expression may be feasible.

Simple mechanical control systems on Lie groups provide a rich source of control problems. Some examples include underwater vehicles, satellites, surface vessels, airships, hovercraft, and robots [3, 4, 11, 13, 41, 42]. Simple mechanical systems on Lie groups are also interesting as *subsets* of more complex interconnected systems. An example is a model of an electrostatically-actuated microelectromechanical system (MEMS) with a mechanical subsystem represented as a simple mechanical system on the Lie group $SE(3)$, and an electrical subsystem without such additional structure [25]. Systems on Lie groups exhibiting symmetry can be further exploited. When the Lagrangian of the system is invariant under the left action of the group on itself, and the external forces acting on the system are left-invariant, a reduction of dynamics to the Lie algebra of the Lie group is immediate [11, 13, 29]. In this case no coordinates need to be introduced on the Lie group to express the system and control strategies. Open-loop, coordinate free, motion planning algorithms using small amplitude forcing have been developed for underactuated systems with such symmetry[13]. One of the main contributions of the present work is to show how, with left-invariant kinetic energy but without additional symmetry constraints, explicit formulas for intrinsic, dynamic output feedback controllers may be obtained without introducing coordinates on the Lie group. For example, there is a wide class of mechanical systems on Lie groups for which the forces are dissipative or non-left invariant. Examples include MEMS, robot manipulators, land vehicles, and general three dimensional motion in a gravitational field with or without damping.

Specifically we specialize to Lie groups two intrinsic results for control [32] and estimation [2] of simple mechanical systems on general Riemannian manifolds. We give coordinate-free explicit expressions of these results valid on any simple mechanical system on a Lie group with left-invariant kinetic energy. For a given Lie group the key computations required are the Levi-Civita connection, the Riemannian curvature, and an approximation to the associated distance function and parallel transport. The computation of these quantities only require a choice of coordinates for the Lie algebra of the Lie group. Once these quantities are computed they can be used for any simple mechanical system with left-invariant kinetic energy, merely by specifying the particular kinetic energy tensor and external forces. Unless the external forces are left-invariant, the expression of the force terms may require coordinates on the Lie group.

In section 2 below we briefly review the necessary mathematical background. In section 3 we first employ the results of [32] to derive passivity-based full state feedback control for uncoupled simple mechanical systems on Lie groups. We then derive similar results for simple mechanical systems on Lie groups that are coupled to a system on a general manifold by generalizing the results of [16].

Often in application the case that configuration and velocity are not both easily measured. In some cases the velocities are available [4, 31, 36], while in others, it is the configuration [3, 37, 39]. In section 4 we consider the latter case, where the configuration variables are measured, and the velocities must be estimated. Section 4 presents a coordinate free explicit expression of the results of [2]. We apply feedback passivation followed by damping injection. Since the original result in [2] assumes all forces to be only configuration dependent we require an extension of that result to accommodate velocity- and configuration-dependent forces. The extension is presented in the appendix.

Section 4 concludes with a statement of a separation principle for the dynamic output feedback law resulting from the combination of this estimator with the uncoupled passivity based controller of section 3.

In section 5 we specifically compute and apply the dynamic output feedback controller to representative systems on two Lie groups of special interest, namely the rotation group $SO(3)$ and the Euclidian motion group $SE(3)$. The expressions given in this section may be applied to many problems of practical significance arising from rigid body motion by specializing only the inertia tensor and the external force terms. The classical axi-symmetric top problem is used to demonstrate the construction and performance of the observer on $SO(3)$. An example on $SE(3)$ models setpoint control in the presence of a saddle-node bifurcation for an electrostatically-actuated microelectromechanical system (MEMS). Simulation results show excellent performance.

2. Mathematical Background.

This section briefly describes the notations and several geometric notions that will be used in the rest of the paper. For additional details the reader is referred to the texts of [1, 15, 12, 18, 20, 21, 29, 35]. Let G be a connected Lie group and let $\mathcal{G} \simeq T_eG$ be its Lie algebra. The left translation of $\zeta \in \mathcal{G}$ to T_gG will be denoted $g \cdot \zeta = DL_g\,\zeta$. The Lie bracket on $\mathcal{G}$ for any two $\zeta, \eta \in \mathcal{G}$ will be denoted $[\zeta, \eta] = ad_\zeta\,\eta$ and the dual of the ad operator will be denoted ad^*. Any smooth vector field $X(g)$ on G has the form $g \cdot \zeta(g)$ for some smooth $\zeta : G \mapsto \mathcal{G}$. Let $\{e_i\}$ be any basis for the Lie algebra $\mathcal{G}$ and let $\{E_i(g) = g \cdot e_i\}$ be the associated left invariant basis vector field on G. Now $[e_i, e_j] = C^k_{ij} e_k$, where C^k_{ij} are the structure constants of the Lie algebra $\mathcal{G}$ ($C^k_{ij} = -C^k_{ji}$), and $[E_i, E_j] = C^k_{ij} E_k$.

2.1. The Riemannian Structure.

Consider a left-invariant metric $\langle\langle \cdot, \cdot \rangle\rangle$ on G. Such a metric induces a unique inner product $\langle\langle \cdot\,, \cdot \rangle\rangle_{\mathcal{G}}$ on $\mathcal{G}$ by the restriction of $\langle\langle \cdot\,, \cdot \rangle\rangle$ to T_eG. Define the isomorphism, $I : \mathcal{G} \mapsto \mathcal{G}^*$ by the relation $\langle I\zeta\,, \eta \rangle = \langle\langle \zeta\,, \eta \rangle\rangle_{\mathcal{G}}$. Here $\langle \cdot, \cdot \rangle$ denotes the usual pairing between a vector and a co-vector. Let the matrix I be defined by $I_{ij} = \langle\langle e_i\,, e_j \rangle\rangle_{\mathcal{G}}$ and let I^{ij} be its inverse. I is symmetric and positive definite. In similar fashion such an I induces a unique left-invariant metric on G by the relation $\langle\langle g \cdot \zeta\,, g \cdot \eta \rangle\rangle = \langle I\zeta\,, \eta \rangle$.

The presentation that follows is based on the texts of [20, 21, 35]. Associated with any metric there exists a unique connection that is torsion free and metric called the Levi-Civita connection. For a vector field $X = X^k E_k$ and a vector $v = v^k E_k$ the Levi-Civita connection is given by

$$\nabla_v X = (dX^k(v) + \omega^k_{ij}(g) v^i X^j) E_k \,, \tag{2.1}$$

where $\omega^k_{ij}(g)$ are the connection coefficients in the frame $\{E_k\}$. If the metric is left invariant then the connection coefficients are constant, given by

$$\omega^k_{ij} = \frac{1}{2}\left(C^k_{ij} - I^{ks}(I_{ir}C^r_{js} + I_{jr}C^r_{is})\right). \tag{2.2}$$

Note that since in general E_k are not coordinate vector fields, ω^k_{ij} are not the Christoffel symbols. In the case of a left invariant metric, the coefficients of the Riemannian curvature two forms R^k_{jab} are also constant and can be shown to be [26],

$$R^k_{jab} = (-\omega^k_{rj}C^r_{ab} + 2\omega^k_{ar}\omega^r_{bj}). \tag{2.3}$$

We remark that they are in general different from the usual curvature coefficients that one would obtain in a coordinate frame field. The Riemannian curvature is then

$$R(\zeta\,,\,\eta)\xi = \{R^k_{jab}\xi^j(\zeta^a\,\eta^b - \zeta^b\,\eta^a) - \omega^k_{ij}C^i_{ab}\zeta^a\,\eta^b\,\xi^j\}e_k. \tag{2.4}$$

These derivations are based on Cartan's structural equations as presented in the sections 9.3b - 9.3e of [20].

2.1.1. The Local Distance Function on a Lie Group. Given any two points g and $\tilde{g}$ on a Riemannian manifold $(G, \langle\langle\cdot,\cdot\rangle\rangle)$, define the set of curves,

$$\Lambda(g,\tilde{g}) := \{\gamma : [0\;1] \mapsto G \;\mid\; \gamma \text{ is piecewise smooth and } \gamma(0) = g, \gamma(1) = \tilde{g}\}. \tag{2.5}$$

Then the distance between g and $\tilde{g}$ is defined as,

$$d(g,\tilde{g}) := \inf\{l(\gamma) : \gamma \in \Lambda(g,\tilde{g})\}, \tag{2.6}$$

and defines a metric on the Riemannian manifold $(G, \langle\langle\cdot,\cdot\rangle\rangle)$ [21, 35]. If a C^1 curve $\gamma \in \Lambda(g,\tilde{g})$ exists such that $d(g,\tilde{g}) = l(\gamma)$ then it is referred to as a *segment*. It is known that segments are always geodesics and that any two sufficiently close points can be connected by a unique segment. In fact, since Lie groups are geodesically complete from the Hopf-Rinow theorem [21, 35] it follows that any two points on a Lie group can be joined by a geodesic.

For g and $\tilde{g}$ sufficiently close there exists a unique $\zeta_e \in \mathcal{G}$ such that

$$e := g^{-1}\tilde{g} = \exp \zeta_e. \tag{2.7}$$

Recall that $\exp s\zeta_e = e(s)$ is the one parameter subgroup generated by ζ_e with respect to left translation with $e(0) = id$ and $e(1) = e$. The inverse of this exponential map (2.7) defines local coordinates around g, commonly referred to as logarithmic coordinates [18]. For a fixed $g \in G$, define the function

$$f(\tilde{g}) := ||\zeta_e||_{\mathcal{G}}. \tag{2.8}$$

Since $e(s) = \exp s\zeta_e$ is a one-parameter subgroup, $f(\tilde{g})$ is the length of this curve and hence for g fixed $d(g,\tilde{g}) \le f(\tilde{g})$. Equality holds if the metric is bi-invariant. In logarithmic coordinates, $f(\tilde{g}) = \sqrt{\zeta_e^T I \zeta_e}$ and up to order-two terms in ζ_e, $d(g, \tilde{g}) = \sqrt{\zeta_e^T I \zeta_e}$. Thus up to order-two terms, the geodesic distance between g and $\tilde{g}$ is explicitly given by the function (2.8) and is referred to as a local distance function.

The function $F(\tilde{g}) := \frac{1}{2} d^2(g, \tilde{g})$ plays a crucial role in the observer to be presented in section 4. From the above discussion it follows that up to third order $F(\tilde{g}) = \frac{1}{2} f^2(\tilde{g})$. Thus in logarithmic coordinates, up to second order, it follows that $grad\ F(\tilde{g}) = \tilde{g} \cdot \zeta_e$. The approximation arguments are intrinsic since smooth coordinate changes will not reduce the order of the neglected higher order terms.

2.2. Simple Mechanical Control Systems on Lie Groups. A simple mechanical control system evolving on a Lie group G, equipped with a left invariant metric $\langle\langle \cdot\, , \cdot \rangle\rangle$, is defined as a system with kinetic energy $E(\dot{g}) = \frac{1}{2} \langle\langle \dot{g}\, , \dot{g} \rangle\rangle$, and Lagrangian $L(g, \dot{g}) = E(\dot{g}) - U(g)$ for some smooth function $U(g)$ on G, [15, 32]. A function with all non-degenerate critical points is referred to as a Morse function. For convenience we in this paper assume that $U(g)$ is a globally defined Morse function. Let $I : \mathcal{G} \mapsto \mathcal{G}^*$ be the isomorphism associated with the kinetic energy metric. Then the Euler-Lagrange equations of motion are given by,

$$\dot{g} = g \cdot \zeta, \tag{2.9}$$

$$\nabla_{\dot{g}}\, \dot{g} = g \cdot I^{-1} \left(f^c(g) + f^d(g, \zeta) + \sum_i^m u_i f^i(g) \right) = g \cdot S(g, \zeta), \tag{2.10}$$

where $f^c(g)$, $f^d(g, \zeta)$, $f^i(g) \in \mathcal{G}^*$ and $u_i \in \mathcal{R}$. The conservative force $f^c(g)$ and damping force $f^d(g, \zeta)$ satisfy the conditions $\langle dU, g \cdot \xi \rangle = -\langle f^c(g), \xi \rangle$ and $\langle f^d(g, \xi), \xi \rangle \le 0$ for any $\xi \in \mathcal{G}$. Here the $f^i(g)$ denote the control directions and are assumed to be linearly independent. The u_i are the magnitude of the forces and are the controls of the system. If $m < dim(G)$ then the system is said to be *underactuated*, and if $m = dim(G)$ the system is said to be *fully actuated*.

Equation (2.9) is the kinematic equation and (2.10) is the Euler-Lagrange equation of the system. These equations can also be expressed as,

$$\dot{g} = g \cdot \zeta\, , \tag{2.11}$$

$$\dot{\zeta} = I^{-1} \left(ad_\zeta^*\, I\zeta + f^c(g) + f^d(g, \zeta) + \sum_i^m u_i f^i(g) \right), \tag{2.12}$$

where now (2.11) – (2.12) defines a dynamical system on $G \times \mathcal{G}$ the tangent bundle to the left trivialization of TG. This formulation does not require coordinates on the Lie group G. In the case where the forcing terms do not depend on the configuration variable g, (2.12) represents a complete reduction of dynamics to $\mathcal{G}$ and is referred to as the Euler-Poincare equations. The kinematic equation (2.11) can be integrated to recover the configuration once the velocities have been solved for and hence are referred to as the reconstruction equations.

3. Passivity-Based Control for Simple Mechanical Systems.

3.1. Uncoupled Simple Mechanical Systems. The uncontrolled equilibrium points of the system are of the form $(\bar{g}, 0)$ where $\bar{g}$ is a critical point of $U(g)$. Assume that $\bar{g}$ is a local minimum. Since $U(g)$ is assumed to be a Morse function, $\bar{g}$ is in fact an isolated local minimum. This implies that the equilibrium $(\bar{g}, 0)$ of the uncontrolled system is stable. If any of these criteria are not satisfied by the natural potential energy, for example if the desired equilibrium is not a local minima of $U(g)$, the methods of [10, 32] may be used to shape the potential energy. If the damping forces satisfy $\langle f^d(g,\zeta),\zeta\rangle < 0$ then the equilibrium $(\bar{g},0)$ is locally asymptotically stable. If this inequality is not strict then the equilibrium is guaranteed only to be stable. Here we wish to enforce convergence to the equilibrium via "damping injection" control. Similarly, if the natural damping is insufficient we may use this strategy to augment it.

For convenience assume that the damping force $f^d(g,\zeta)$ is of Rayleigh type. That is, $f^d(g,\zeta) = -R(g)\zeta$ where $R(g) : \mathcal{G} \mapsto \mathcal{G}^*$ is a map smooth in g so that the relation $\langle R(g)\zeta,\eta\rangle = \langle\langle \zeta,\eta\rangle\rangle_D$ defines a degenerate inner product on $\mathcal{G}$. This means that in a matrix representation $R(g)$ is symmetric and positive semi-definite. Let $Im(B(g)) := span\{f^1(g),\cdots,f^m(g)\}$.

To be *passive* a system must have a storage function satisfying the dissipation inequality with supply rate y^Tu, where y is the system output [40]. For simple mechanical control systems the output that is compatible with passivity is completely determined and is given intrinsically by $y_i = \langle f^i(g)\ ,\ \zeta\rangle$ or in a matrix representation as $y = B(g)^T\zeta$. Consider the storage function

$$H(g,\zeta) = \frac{1}{2}\langle\langle \zeta\ ,\ \zeta\rangle\rangle_{\mathcal{G}} + U(g). \tag{3.1}$$

$$\dot{H} = -\langle R(g)\zeta,\zeta\rangle + \sum_i^m u_i\langle f^i(g)\ ,\ \zeta\rangle \le \sum_i^m u_i\langle f^i(g)\ ,\ \zeta\rangle = y^Tu. \tag{3.2}$$

Thus (2.11) – (2.12) is passive with storage function H. Now consider the *damping injection* control

$$u_i = -y_i = -\langle f^i(g)\ ,\ \zeta\rangle. \tag{3.3}$$

In matrix representations (3.3) can also be written as $u = -B(g)^T\zeta$ and (3.2) as $\dot{H} = -\zeta^T R(g)\zeta - \zeta^T B(g)B(g)^T\zeta$.

Recall that Lie groups are complete metric spaces [35]. Thus, using this control, the generalized LaSalle invariance theorem of [32] guarantees that the trajectories of (2.11) – (2.12) converge to the largest invariant set of (2.11) – (2.12) contained in $\mathcal{S} := \{(g,\zeta)\mid \dot{H}=0\}$. Let $\mathcal{N}(R(g))$ be the null space of the degenerate inner product $\langle\langle\cdot,\cdot\rangle\rangle_D$ and define $[Im(B(g))]^\perp := \{\zeta\in\mathcal{G}\mid \langle f^i(g),\zeta\rangle = 0 \ \text{ for } \ i=1,\cdots,m\}$. If at every $g\in G$, $\mathcal{N}(R(g))\cap[Im(B(g))]^\perp = \{0\}$ then $\mathcal{S} = \{(g,\zeta)\mid \zeta=0\}$ and the largest invariant set contained in $\mathcal{S}$ consists of only the equilibrium points of the system. The equilibrium points of the system are given by the critical points of the potential energy function $U(g)$, and by assumption $(\bar{g},0)$ is a local non-degenerate minimum. Thus the damping control (3.3) locally asymptotically stabilizes $(\bar{g},0)$. In terms of a matrix representation the condition $\mathcal{N}(R(g))\cap[Im(B(g))]^\perp = \{0\}$ implies that the symmetric matrix $R(g)+B(g)B^T(g)$ is positive definite. From this point on we will assume this condition is satisfied. This is trivially the case for fully-actuated systems.

We say an equilibrium is almost globally asymptotically stable if its region of attraction is an open and dense set. In particular the stabilization results are almost-global if the potential energy function $U(g)$ is a globally defined smooth proper Morse function with a unique minimum at the desired equilibrium configuration $\bar{g}$ [23]. We return to this point in section 4 when we consider almost global performance of the dynamic output feedback compensator.

3.2. Coupled Simple Mechanical Systems. Consider the product space $\mathcal{M} = \mathcal{Q} \times TG$ for some smooth manifold Q, and the class of systems on $\mathcal{M}$ of the form,

$$\dot{q} = s^0(q, g, \zeta) + \sum_{i=1}^{m} s^i(q, g, \zeta)u_i, \tag{3.4}$$

$$\dot{g} = g \cdot \zeta \ , \tag{3.5}$$

$$\dot{\zeta} = I^{-1}\left(ad^*_\zeta I\zeta + f^c(g) + f^d(g, \zeta) + \sum_{i=m}^{m} f^i(g, \zeta, y)y_i\right), \tag{3.6}$$

$$y_i = h_i(q) \quad \text{for} \quad i = 1, 2, \cdots m, \tag{3.7}$$

where $q \in \mathcal{Q}$ and $s^i : \mathcal{M} \mapsto T\mathcal{Q}$ are smooth maps such that $\pi_Q \circ s^i = \pi_M$, where $\pi_Q : T\mathcal{Q} \mapsto \mathcal{Q}$ is the projection of $T\mathcal{Q}$ onto $\mathcal{Q}$ and $\pi_M : \mathcal{M} \mapsto \mathcal{Q}$ is the projection of $\mathcal{M}$ on to $\mathcal{Q}$. Furthermore let $y := [y_1\, y_2\, \cdots\, y_m]^T = [h_1\, h_2\, \cdots\, h_m]^T := h \in \mathcal{R}^m$ and $s := [s^1\, s^2\, \cdots\, s^m]$ be such that $D\,h$ is onto at every $q \in \mathcal{Q}$ and $dim(span\{s^i\}_{i=1}^m) = m$ uniformly. If the matrix $L_s\,h := (L_{s^i}h_j)$ for $i, j = 1, 2, 3 \cdots, m$ is nonsingular for all $q \in \mathcal{Q}, g \in G, \zeta \in \mathcal{G}$, then the interconnected system (3.4) – (3.7) has uniform relative degree $\{1, 1, \cdots 1\}$ with respect to the outputs y_i. The uniform relative degree of the system implies that the feedback law

$$u = -[L_s\,h]^{-1}(L_{s^0}\,h - \nu), \tag{3.8}$$

is globally smooth. Thus the state feedback (3.8) input-output linearizes the system. Since $D\,h$ is full rank at each $q \in \mathcal{Q}$, the set $h^{-1}(y) \subset \mathcal{Q}$ is a smooth embedded submanifold of $\mathcal{Q}$ for each $y \in \mathcal{R}^m$. Introducing local coordinates (y, z) on $\mathcal{Q}$, the system (3.4) – (3.7) together with the control (3.8) can be expressed as,

$$\dot{y} = \nu, \tag{3.9}$$

$$\dot{g} = g \cdot \zeta \ , \tag{3.10}$$

$$\dot{\zeta} = I^{-1}\left(ad^*_\zeta I\zeta + f^c(g) + f^d(g, \zeta) + \sum_{i=m}^{m} f^i(g, \zeta, y)y_i\right), \tag{3.11}$$

$$\dot{z} = N(z, g, \zeta, \nu, y). \tag{3.12}$$

The zero dynamics of the system are given by,

$$\dot{g} = g \cdot \zeta \ , \tag{3.13}$$

$$\dot{\zeta} = I^{-1}\left(ad^*_\zeta I\zeta + f^c(g) + f^d(g, \zeta)\right), \tag{3.14}$$

$$\dot{z} = N(z, g, \zeta, 0, 0). \tag{3.15}$$

Consider the following candidate storage function for the input-output linearized system (3.9) – (3.11)

$$V(y, g, \zeta) = \frac{1}{2}\langle\langle \zeta, \zeta \rangle\rangle_{\mathcal{G}} + U(g) + \frac{1}{2}\sum_{i=1}^{m} {y_i}^2, \tag{3.16}$$

where the potential energy of the mechanical system is $U(g)$ and $U(g) \geq 0$ is Morse. Then specializing the results of [16, 38, 40] on passivity of interconnected subsystems, it can be shown that the control $\nu_i = -\langle f^i(g, \zeta, y)\ ,\ \zeta\rangle + w_i$ renders the input-output linearized system (3.9) – (3.11) passive with respect to the input-output pair (w, y)

and storage function V. Thus if $\bar{g}$ is a nondegenerate local minimum of $U(g)$ and $\langle f^d(g,\zeta)\ ,\ \zeta\rangle < 0$ for all $\zeta \in \mathcal{G}$, the control $w = -y$ locally asymptotically stabilizes the equilibrium $(0,\bar{g},0)$ of the input-output linearized system (3.9) – (3.11). Explicitly this control is given by

$$\nu_i = -\langle f^i(g,\zeta,y)\ ,\ \zeta\rangle - y_i. \tag{3.17}$$

Furthermore if the equilibrium of (3.15) is locally asymptotically stable with $g \equiv \bar{g}$ and $\zeta = 0$ then the equilibrium $(0,\bar{g},0)$ of the whole system (3.4) – (3.6) is locally asymptotically stable. The stability result of the composite system (3.4) – (3.6) is almost-global if additionally $U(g)$ is a smooth proper Morse function with a unique minimum at the equilibrium configuration $\bar{g}$ and N is vacuous or satisfies some additional requirements given by Theorem 4.7 of [38]. Observe that if $dim(\mathcal{Q}) = m$ then N is vacuous.

4. Intrinsic Observer for Velocity Estimation. The controls (3.3) and (3.17) involve the feedback of both the configuration g and the velocity ζ. In the case where the configuration is available for measurement, but the velocity is not, we propose an intrinsic observer to estimate the velocity variable. This is based on the work reported in [2]. There a velocity estimate based on configuration measurement is presented for a general Riemannian manifold, and expressed in coordinates. We specialize that results to a Lie group equipped with a left invariant metric. Our re-formulation avoids the need to introduce coordinates on the Lie group and includes a proof that the external forcing can exhibit velocity dependence in addition to configuration dependence .

Consider the system given by (2.9) – (2.10). Let $(\tilde{g},\tilde{\zeta})$ be the estimated value of (g,ζ) and let $\alpha,\beta > 0$ be constant. Then it is shown in [2] that the following observer converges locally exponentially if the initial observer configuration error is sufficiently small.

$$\dot{\tilde{g}} = \tilde{g}\cdot\tilde{\zeta} - 2\alpha\, grad\ F(\tilde{g}), \tag{4.1}$$

$$\nabla_{\dot{\tilde{g}}}\ \tilde{g}\cdot\tilde{\zeta} = \tilde{g}\cdot\Gamma(S) - \tilde{g}\cdot R(\tilde{\zeta},\tilde{g}^{-1}grad\ F(\tilde{g}))\tilde{\zeta} - \beta\, grad\, F(\tilde{g}), \tag{4.2}$$

where $F(\tilde{g}) = \frac{1}{2}d(g,\tilde{g})^2$ and $\Gamma(S)$ is the parallel transport of the resultant external force S at g to $\tilde{g}$ along the geodesic joining the two points. In [2] it is pointed out that replacing $\Gamma(S)$ and $grad\, F$ by their respective first order approximations will not affect the local convergence properties of the observer.

Although convergence is proved in [2] assuming that $S = S(g)$, the same basic argument holds when $S = S(g,\zeta)$, where we now use $\tilde{\zeta}$ instead of ζ in the parallel transport term $\Gamma(S(g,\tilde{\zeta}))$. In [2], the first variation of the observer dynamics is constructed, then contraction analysis is used to prove local exponential convergence of the observer. In the appendix we show that the first variation of the observer dynamics does not change when S is allowed to depend on the velocity ζ. Thus the contraction argument of [2] applies without modification and the local exponential convergence of the observer (4.1) – (4.2) follows even when $S = S(g,\zeta)$.

It was shown in section 2.1.1 that up to second order $grad\ F = \tilde{g}\zeta_e$. Therefore the first order approximation of the observer (4.1) – (4.2) can be expressed as

$$\dot{\tilde{g}} = \tilde{g}\cdot(\tilde{\zeta} - 2\alpha\zeta_e), \tag{4.3}$$

$$\nabla_{\dot{\tilde{g}}}\ \tilde{g}\cdot\tilde{\zeta} = \tilde{g}\cdot\Gamma(S) - \tilde{g}\cdot R(\tilde{\zeta},\zeta_e)\tilde{\zeta} - \beta\,\tilde{g}\cdot\zeta_e. \tag{4.4}$$

Expanding (4.4) the first order approximation of the observer is explicitly given by,

$$\dot{\tilde{g}} = \tilde{g} \cdot (\tilde{\zeta} - 2\alpha\zeta_e), \tag{4.5}$$

$$\dot{\tilde{\zeta}} = I^{-1}\Big(ad^*_{\tilde{\zeta}} I\tilde{\zeta} - \alpha(ad^*_{\zeta_e} I\tilde{\zeta} + ad^*_{\tilde{\zeta}} I\zeta_e)\Big) + \alpha[\zeta_e, \tilde{\zeta}]_{\mathcal{G}} + \Gamma(S) - R(\tilde{\zeta}, \zeta_e)\tilde{\zeta} - \beta\zeta_e. \tag{4.6}$$

where up to order-two terms

$$\Gamma(S) = \Big(S^k(g, \tilde{\zeta}) - \omega^k_{ij} S^i(g, \tilde{\zeta})\zeta^j_e\Big)\ e_k. \tag{4.7}$$

This coordinate-free formulation of the observer clearly shows its structure. For instance the terms in (4.6) involving the gain α are the corrections to the inertial forces of the observer that are needed to compensate for curvature effects, $\Gamma(S)$ is the intrinsic model of the external forces of the observed system, and $R(\tilde{\zeta}, \zeta_e)\tilde{\zeta}$ is the curvature term that is needed to correct for the effects of possible divergence of nearby geodesics. The $2\alpha\zeta_e$, and $\beta\zeta_e$ terms are the error feedback that ensure convergence. The coordinate-free formulation also shows the versatilty of the expressions (4.5) – (4.6). Specifically it is readily applicable to any simple mechanical system on the Lie group G. Depending on the specific problem all the control designer needs to do is specify the kinetic energy tensor I and the external forces S.

Using the observer (4.5) – (4.6), the control (3.3) can be implemented with velocity estimates replacing velocity measurements as

$$u_i = -\langle f^i(g)\ ,\ \tilde{\zeta}\rangle. \tag{4.8}$$

It is natural to ask whether the dynamic output feedback control (4.8) preserves the stability properties of the state feedback control (3.3)— that is, whether a separation principle holds. In the appendix we show using results of [28] that it does. In particular, if (3.3) is almost globally stabilizing (resp. asymptotically stabilizing) then so is (4.8).

5. Examples. In this section we demonstrate the preceding constructions for two cases of practical significance in which the configuration space of the simple mechanical systems are Lie groups—first $SO(3)$, and then $SE(3)$. Since these groups arise in many practical problems involving rigid body motions we include here explicit expressions for the Riemannian connection, Riemannian curvature, and the approximate local distance functions. To implement the observer in a specific application, now only the inertia tensor I and the external force S need to be changed. The effectiveness of the observer is demonstrated in $SO(3)$ for the axi-symmetric top and in $SE(3)$ for a model of an electrostatically actuated MEMS.

5.1. The Rotation Group $SO(3)$. The rotation group, $SO(3)$, is the group of matrices $R \in GL(3, \mathcal{R})$ that satisfy the conditions $R\,R^T = R^T\,R = I$ and $det(R) = 1$. Euler's theorem states that any given $R \in SO(3)$ is a rotation about some axis n by an angle ψ, that is, $R = \exp(\psi\hat{n})$, where

$$\psi\hat{n} = \frac{\psi}{2\sin\psi}(R - R^T), \tag{5.1}$$

and, $\cos\psi = (tr(R) - 1)/2$, for $|\psi| < \pi$.

The Lie algebra $so(3)$ of $SO(3)$ is the set of traceless skew symmetric 3×3 matrices. The Lie algebra $so(3)$ is identified with $\mathcal{R}^3$ by the isomorphism,

$$\xi \in \mathcal{R}^3 \mapsto \hat{\xi} = \begin{bmatrix} 0 & -\xi^3 & \xi^2 \\ \xi^3 & 0 & -\xi^1 \\ -\xi^2 & \xi^1 & 0 \end{bmatrix} \in so(3), \tag{5.2}$$

where $\xi = [\xi^1 \ \xi^2 \ \xi^3]^T$. We will use both ξ and $\hat{\xi}$ to mean the same element of $so(3)$.

The isomorphism $I : so(3) \simeq \mathcal{R}^3 \mapsto so(3)^* \simeq \mathcal{R}^3$ defined by the positive definite matrix I induces a left invariant metric on $SO(3)$ by the relation, $\langle\langle R \cdot \xi \, , \, R \cdot \psi \rangle\rangle = \langle\langle \xi \, , \, \psi \rangle\rangle_{so(3)} = I\xi \cdot \psi$, for any two elements $R \cdot \xi, R \cdot \psi \ \in T_R SO(3)$. The Lie bracket on $so(3)$ is $[\xi \, , \, \psi]_{so(3)} = ad_\xi \, \psi = \xi \times \psi$ and the dual of the ad operator is given by $ad^*_\xi \, \Pi = \Pi \times \xi$, where $\Pi \in so(3)^* \simeq \mathcal{R}^3$.

From (2.11) – (2.12), a simple mechanical control system on $SO(3)$ takes the form,

$$\dot{R} = R\hat{\zeta}, \tag{5.3}$$

$$\dot{\zeta} = I^{-1}\left(I\zeta \, \times \, \zeta + \tilde{S}(R,\zeta)\right). \tag{5.4}$$

where $\tilde{S}(R,\zeta) = f^c(R) + f^d(R,\zeta) + \sum_i^m u_i f^i(R)$. The passivity-based damping injection (3.3) takes the form,

$$u_i = -\langle f^i(R) \, , \, \zeta \rangle. \tag{5.5}$$

The intrinsic observer (4.5) – (4.6) takes the form,

$$\dot{\tilde{R}} = \tilde{R}(\hat{\tilde{\zeta}} - 2\alpha\hat{\zeta}_e), \tag{5.6}$$

$$\dot{\tilde{\zeta}} = I^{-1}\left(I\tilde{\zeta} \times \tilde{\zeta} - \alpha\left(I\tilde{\zeta} \times \zeta_e + I\zeta_e \times \tilde{\zeta}\right)\right) + \alpha\zeta_e \times \tilde{\zeta} + \Gamma(S) - R_c(\tilde{\zeta},\zeta_e)\tilde{\zeta} - \beta\zeta_e, \tag{5.7}$$

where ζ_e satisfies $exp(\zeta_e) = R^T\tilde{R}$ and is given by (5.1) as,

$$\zeta_e = \frac{\psi}{2\sin\psi}(R^T\tilde{R} - \tilde{R}^T R), \tag{5.8}$$

where, $\cos\psi = (tr(R^T\tilde{R}) - 1)/2$, for $|\psi| < \pi$. The parallel transport term $\Gamma(S)$ is calculated from (4.7) where $S(R,\zeta) = I^{-1}\tilde{S}(R,\zeta)$ and the curvature term $R_c(\tilde{\zeta},\zeta_e)\tilde{\zeta}$ is calculated from (2.4).

If the potential energy $U(R)$ of the mechanical system is a globally defined smooth Morse function with a unique minimum at the equilibrium configuration $\bar{R}$ then the control (5.5) almost globally stabilizes the equilibrium $(\bar{R}, 0)$. Furthermore since $SO(3)$ is compact from Corollary A.2 it also follows that (5.5) implemented with the velocity observer also almost globally stabilizes the equilibrium $(\bar{R}, 0)$ if the initial observer configuration error is sufficiently small.

In the canonical basis the nonzero structure constants C_{ij}^k on $so(3) \simeq \mathcal{R}^3$ are,

$$C_{12}^3 = 1, \, C_{13}^2 = -1, \, C_{23}^1 = 1.$$

In the special case of axisymmetric rigid bodies, $I = diag(I_x, I_y, I_z)$. For such examples using equation (2.2) the nonzero connection coefficients ω_{ij}^k are calculated to be

$$\omega_{23}^1 = \frac{I_x - I_y + I_z}{2I_x} \, , \omega_{32}^1 = \frac{-I_x - I_y + I_z}{2I_x} \, , \, \omega_{13}^2 = \frac{I_x - I_y - I_z}{2I_y} \, , \, \omega_{31}^2 = \frac{I_x + I_y - I_z}{2I_y} \, ,$$

$$\omega_{12}^3 = \frac{-I_x + I_y + I_z}{2I_z}, \ \omega_{21}^3 = \frac{-I_x + I_y - I_z}{2I_z}.$$

The nonzero curvature coefficients are too many to be listed here.

5.1.1. Angular Velocity Estimation for the Axi-symmetric Top. In this section we demonstrate the effectiveness of the observer (5.6)–(5.7) by means of simulation. Consider the classical problem of a axi-symmetric top in a gravitational field. Let $P = \{P_1, P_2, P_3\}$ be an inertial frame fixed at the fixed point of the top and let $e = \{e_1, e_2, e_3\}$ be a body fixed orthonormal frame with the origin coinciding with that of P. At $t = 0$, the two frames coincide. Then let the coordinates of a point p in the inertial frame P be given by x and in the body frame e be given by X. They are related by $x(t) = R(t)X$ where $R(t) \in SO(3)$. Let $-P_3$ be the direction of gravity and let I be the inertia matrix of the axi-symmetric top about the fixed point. The kinetic energy of the top is $K = I\zeta \cdot \zeta/2$, where ζ is the body angular velocity and the potential energy is $U(R) = mgl\, Re_3 \cdot P_3$. Here m is the mass of the top, g is the gravitational constant, l is the distance along the e_3 axis to the center of mass. For simplicity we assume the top to be symmetric about the e_3 axis. The generalized potential forces $f^c(R)$ in the body frame will be given by the relation $\langle f^c(R), \zeta\rangle = -\langle dU, R \cdot \zeta\rangle = -mgl\, R\hat{\zeta} e_3 \cdot P_3$ for any $\zeta \in so(3)$, which yields that $f^c(R) = mgl\ R^T P_3 \times e_3$. The metric induced on $SO(3)$ by the kinetic energy is left invariant, and the system is a simple mechanical system on $SO(3)$. Thus the equations of motion on $SO(3) \times so(3)$ are given by, (5.3) – (5.4) where $\tilde{S}(R) = mgl\ R^T P_3 \times e_3$. Since it is assumed that the top is symmetric about the e_3 axis, the inertia matrix is diagonal with $I_1 = I_2$, that is, $I = diag(I_1, I_1, I_3)$.

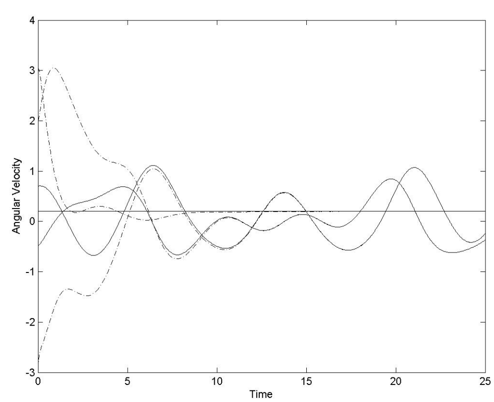


FIG. 5.1. *Angular velocity estimates versus true values in axi-symmetric top simulation. The true values are the solid lines while the dotted lines are the estimated values*

Substituting $I = diag(1, 1, 2)$ and $S = I^{-1}(R^T P_3 \times e_3)$ in the observer (5.6) – (5.7) with $\alpha = \beta = 10$, we estimate the angular velocities of the axi-symmetric top. The simulation results are shown in Figure 5.1. The initial body angular velocities of the axi-symmetric top are $[.7 \ \ -.5 \ \ .2]$ and the initial observer angular velocity is $[-3 \ \ 2 \ \ 3]$ while the initial observer configuration error corresponds to a $\pi/10$ radian rotation about the $P_2 = [0 \ 1 \ 0]^T$ axis.

5.2. The Special Euclidian Motion Group $SE(3)$. The special Euclidian motion group $SE(3)$ is the semi-direct product $SO(3) \times_s \mathcal{R}^3$. As a matrix group, an element $A \in SE(3)$ and its inverse A^{-1} can be represented by,

$$A = \begin{bmatrix} R & b \\ 0 & 1 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} R^T & -R^T b \\ 0 & 1 \end{bmatrix}.$$

where $R \in SO(3)$ and $b \in \mathcal{R}^3$.

The Lie algebra of $SE(3)$, denoted by $se(3)$, is the set of matrices,

$$\zeta = \begin{bmatrix} \hat{\xi} & v \\ 0 & 0 \end{bmatrix},$$

where $\hat{\xi} \in so(3)$ and $v \in \mathcal{R}^3$. Then $se(3) \simeq \mathcal{R}^3 \times \mathcal{R}^3$ by identifying $\zeta \in se(3)$ with $(\xi\,,\,v) \in \mathcal{R}^3 \times \mathcal{R}^3$.

Let the inner product between the two elements $(\xi, v), (\psi, u) \in se(3)$ on $se(3)$, $\langle\langle\cdot\,,\,\cdot\rangle\rangle_{se(3)}$ be defined as, $\langle\langle(\xi, v), (\psi, u)\rangle\rangle_{se(3)} = I_b\xi \cdot \psi + Mv \cdot u$, where I_b is a positive definite matrix. This inner product on $se(3)$ defines a left invariant metric on $SE(3)$ in the usual way. The Lie bracket on $se(3)$ is given by,

$$[(\xi\,,\,v), (\psi\,,\,u)]_{se(3)} = ad_{(\xi\,,\,v)}(\psi\,,\,u) = (\xi \times \psi\,,\,\xi \times u - \psi \times v). \tag{5.9}$$

and the dual of the ad operator is given by,

$$ad^*_{(\xi\,,\,v)} \begin{bmatrix} \Pi \\ \mu \end{bmatrix} = \begin{bmatrix} \Pi \times \xi + \mu \times v \\ \mu \times \xi \end{bmatrix}, \tag{5.10}$$

where $(\Pi, \mu) \in se(3)^* \simeq \mathcal{R}^3 \times \mathcal{R}^3$.

From (2.11) – (2.12), a simple mechanical control system on $SE(3)$ takes the form,

$$\begin{bmatrix} \dot{R} & \dot{b} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\xi} & v \\ 0 & 0 \end{bmatrix} \tag{5.11}$$

$$\begin{bmatrix} \dot{\xi} \\ \dot{v} \end{bmatrix} = I^{-1}\left(\begin{bmatrix} I_b\xi \times \xi \\ Mv \times \xi \end{bmatrix} + \tilde{S}(R, b, \xi, v) \right), \tag{5.12}$$

where $\tilde{S}(R, b, \xi, v) = f^c(R, b) + f^d(R, b, \xi, v) + \sum_i^m u_i f^i(R, b)$.

The passivating control (3.3) takes the form,

$$u_i = -\langle f^i(R, b)\,,\,(\xi, v)\rangle. \tag{5.13}$$

The intrinsic observer (4.5) – (4.6) takes the form,

$$\begin{bmatrix} \dot{\tilde{R}} & \dot{\tilde{b}} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \tilde{R} & \tilde{b} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\tilde{\xi}} - 2\alpha\,\hat{\xi}_e & \tilde{v} - 2\alpha\, v_e \\ 0 & 0 \end{bmatrix} \tag{5.14}$$

$$\begin{bmatrix} \dot{\tilde{\xi}} \\ \dot{\hat{v}} \end{bmatrix} = \begin{bmatrix} I_b^{-1}\left(I_b\tilde{\xi} \times \tilde{\xi} - \alpha(I_b\tilde{\xi} \times \xi_e + I_b\xi_e \times \tilde{\xi})\right) + \alpha\,\xi_e \times \tilde{\xi} \\ \tilde{v} \times \tilde{\xi} - 2\alpha\,\tilde{v} \times \xi_e \end{bmatrix}$$
$$+\Gamma(S) - R_c(\tilde{\zeta}, \zeta_e)\tilde{\zeta} - \beta\zeta_e, \tag{5.15}$$

where $\zeta_e = (\Omega_e,\, V_e)$ satisfies $\exp(\zeta_e) = A^{-1}\tilde{A}$ and is explicitly given by,

$$\Omega_e = \frac{\psi}{2\sin\psi}(R^T\tilde{R} - \tilde{R}^T R), \tag{5.16}$$

$$V_e = W^{-1}(R^T\tilde{b} - \tilde{R}^T b), \tag{5.17}$$

where, $\cos\psi = (tr(R^T\tilde{R}) - 1)/2$, for $|\psi| < \pi$ [29] and

$$W = I_{3\times 3} + \frac{(1-\cos\psi)}{\psi^2}\Omega_e + \frac{(\psi - \sin\psi)}{\psi^3}\Omega_e^2.$$

The parallel transport term $\Gamma(S)$ is calculated from (4.7) where $S(R,b,\xi,v) = I^{-1}\tilde{S}(R,b,\xi,v)$ and the curvature term $R_c(\tilde{\zeta},\zeta_e)\tilde{\zeta}$ is calculated from (2.4). In the canonical basis the nonzero structure constants C_{ij}^k on $se(3) \simeq \mathcal{R}^6$ are,

$C_{12}^3 = 1$, $C_{13}^2 = -1$, $C_{15}^6 = 1$, $C_{16}^5 = -1$, $C_{23}^1 = 1$, $C_{24}^6 = -1$, $C_{26}^4 = 1$, $C_{34}^5 = 1$, $C_{35}^4 = -1$

In the case where $I_b = diag(I_x, I_y, I_z)$ and M is a positive scalar, using equation (2.2) the nonzero connection coefficients ω_{ij}^k are shown to be,

$$\omega_{23}^1 = \frac{I_x - I_y + I_z}{2I_x}, \omega_{32}^1 = \frac{-I_x - I_y + I_z}{2I_x}, \omega_{13}^2 = \frac{I_x - I_y - I_z}{2I_y}, \omega_{31}^2 = \frac{I_x + I_y - I_z}{2I_y},$$

$$\omega_{12}^3 = \frac{-I_x + I_y + I_z}{2I_z}, \omega_{21}^3 = \frac{-I_x + I_y - I_z}{2I_z}.$$

$$\omega_{15}^6 = \omega_{26}^4 = \omega_{34}^5 = 1, \quad \omega_{16}^5 = \omega_{24}^6 = \omega_{35}^4 = -1.$$

The coefficients of the curvature tensor R_{jab}^k can be calculated using equation (2.3).

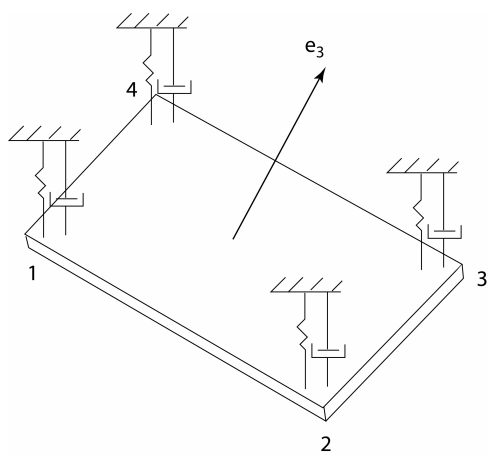


FIG. 5.2. *Schematic diagram of a Rigid 3D microactuator.*

5.2.1. Stabilization of an Electrostatically-Actuated 6-DOF Micromirror. In this section, our observer-based stabilization approach is applied to an electrostatically-actuated MEMS. An intrinsic and geometric model of a 6-DOF, electrostatically-actuated micromirror is developed in [25]. Such devices may be used as steerable micromirrors for example to guide an optical beam into one of a number of output fiberoptics, or to simultaneously correct the phase and amplitude of an optical signal. For further physical motivation and applications the reader is referred to [25, 27] and the references therein. The device consists of a fixed bottom plate and a rigid top plate. The top plate is free to rotate and translate, subject to the constraint that each side is connected to a support structure through flexible cantilevers. The bottom plate is segmented into four drive electrodes. The system is actuated by a voltage difference between each electrode and the grounded top plate. Figure 5.2 is a schematic of the movable top plate. The points q_i denotes the locations at which the external spring and damping forces act on the system.

The device is modeled as a mechanical subsystem coupled to an electrical subsystem via the electrostatic actuation forces. The configuration space of the mechanical system is the group of three dimensional Euclidian motions represented by $SE(3)$. Let $e = (e_1, e_2, e_3)$ be a body-fixed orthonormal coordinate frame centered at the center of mass of the top plate, and let $P = (P_1, P_2, P_3)$ be an inertially fixed orthonormal coordinate frame coinciding with e when the system is in equilibrium with no actuation. At a given time t the orientation of the top plate with respect to the inertial frame is given by $R(t)$, while the displacement of the center of mass of the top plate with respect to the inertial frame is given by $b(t)$. The body angular velocity of the top plate is denoted by $\xi(t)$. The velocity of the center of mass of the top plate in the inertial coordinates is denoted by $\dot{b} = R(t)v(t)$, where $v(t)$ is the velocity of the center of mass in the body frame. The non-actuated equilibrium gap between the center of mass of the top plate and the bottom plate is d. The resistance in the p^{th} capacitor circuit is r_p. The permittivity of the dielectric medium between the electrodes is ϵ. The electrode areas are each assumed to be equal, and denoted A. The charge stored in the p^{th} capacitor is Q_p. The voltage control supplied to the p^{th} capacitor is u_p. The position vector in the body fixed coordinates of the i^{th} point is q_i. The inertia matrix of the top plate is denoted by I. The total mass of the top plate is m.

The spring forces exerted by the cantilever beams are assumed to be linear in the absolute displacement. This assumption implies that the spring forces $F_p^C(R, b)$ are given by

$$\begin{bmatrix} F_i^C(R,b) \\ 0 \end{bmatrix} = -\begin{bmatrix} K_i & 0 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} R & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_i \\ 1 \end{bmatrix} - \begin{bmatrix} q_i \\ 1 \end{bmatrix} \right).$$

The structural dissipative forces $F_i^D(R, b, \xi, v)$ of the system is assumed to be of Rayleigh type and is given by

$$\begin{bmatrix} F_i^D(R,b,\xi,v) \\ 0 \end{bmatrix} = -\begin{bmatrix} C_i & 0 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} R & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\xi} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q_i \\ 1 \end{bmatrix} \right).$$

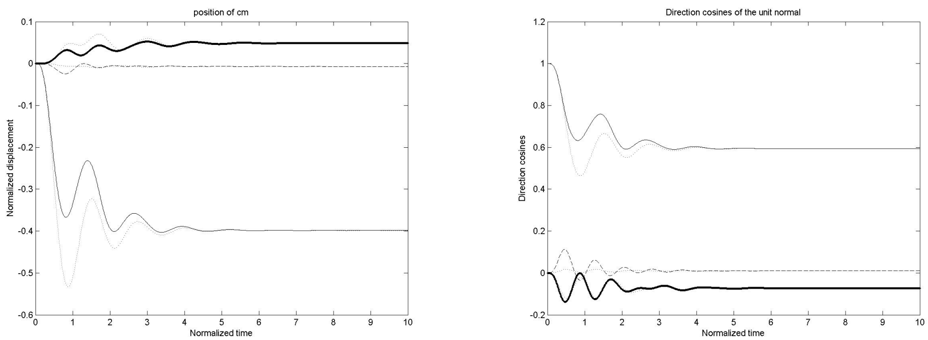
The 3×3 matrices K_i and C_i are positive semidefinite. Computing the torques about the center of mass of the movable electrode in the body coordinates the generalized body forces due to the stiffness and structural damping of the cantilever are expressed by

$$f_i^c = \begin{bmatrix} \hat{q}_i R^T F_i^c \\ R^T F_i^C \end{bmatrix} \quad f_i^d = \begin{bmatrix} \hat{q}_i R^T F_i^D \\ R^T F_i^D \end{bmatrix},$$

Neglecting parasitics, allowing fringing but assuming that the electrostatic field generated by each individual electrode does not interact with the others (these are standard assumptions in the modeling of multi-electrode electrostatic devices [19]) the generalized body electrostatic forces are given by $BW(Q)$ where $Q = [Q_1 \;\; Q_2 \;\; Q_3 \;\; Q_4]^T$

$$W(Q) = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \frac{1}{4\epsilon A} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} Q_1^2 \\ Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{bmatrix},$$

$$B = \begin{bmatrix} -l_y\, e_1 & -l_x\, e_2 & 0 \\ 0 & 0 & 2e_3 \end{bmatrix}.$$



(a)Center of Mass (b) Direction Cosines

FIG. 5.3. *The position of the center of mass and the direction cosines of the unit normals of the movable electrode versus time. In the case of dynamic output feedback the solid curve corresponds to the P_3 direction, the thick solid curve corresponds to the P_1 direction and the dashed curve corresponds to the P_2 direction. The dotted curves correspond to full state feedback.*

Using these generalized forces, in [25] the governing equation of the MEMS is shown to be,

$$\dot{Q}_p = -\frac{1}{r_p}\left(V_{d_p} - u_p\right) \qquad \text{for} \quad p = 1,2,3,4, \tag{5.18}$$

$$\begin{bmatrix} \dot{R} & \dot{b} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\xi} & v \\ 0 & 0 \end{bmatrix} \tag{5.19}$$

$$\begin{bmatrix} \dot{\xi} \\ \dot{v} \end{bmatrix} = I^{-1}\left(\begin{bmatrix} I_b \xi \times \xi \\ M v \times \xi \end{bmatrix} + \sum_{i=1}^{4} f_i^c(R,b) + \sum_{i=1}^{4} f_i^d(R,b,\xi,v) + B\, w(Q) \right), \tag{5.20}$$

$$y_{m_p} = V_{d_p} \qquad \text{for} \quad p = 1,2,3,4, \tag{5.21}$$

$$y_p = Q_p \qquad \text{for} \quad p = 1,2,3,4. \tag{5.22}$$

where g, y_{m_p} and y_p are the measured outputs. The system has relative degree $\{1,1,1,1\}$ with respect to the output $y = Q$, and stable zero dynamics.

For a given $\bar{Q}$, the corresponding equilibrium points of (5.19) – (5.20) are given by $\bar{\xi} = 0$, $\bar{v} = 0$ and

$$0 = \sum_{i=1}^{4} f_i^c(\bar{R}, \bar{b}) + Bw(\bar{Q}). \tag{5.23}$$

Using the results of section 3.2 it can be shown that the feedback law

(5.24)

$$u = \begin{bmatrix} V_{d_1} \\ V_{d_2} \\ V_{d_3} \\ V_{d_4} \end{bmatrix} + \begin{bmatrix} r_1(l_y\xi_1 - l_x\xi_2 + 2v_3)(Q_1 + \bar{Q}_1)/4\epsilon A \\ r_2(-l_y\xi_1 - l_x\xi_2 + 2v_3)(Q_2 + \bar{Q}_2)/4\epsilon A \\ r_3(-l_y\xi_1 + l_x\xi_2 + 2v_3)(Q_3 + \bar{Q}_3)/4\epsilon A \\ r_4(l_y\xi_1 + l_x\xi_2 + 2v_3)(Q_4 + \bar{Q}_4)/4\epsilon A \end{bmatrix} - \alpha \begin{bmatrix} r_1(Q_1 - \bar{Q}_1) \\ r_2(Q_2 - \bar{Q}_2) \\ r_4(Q_3 - \bar{Q}_3) \\ r_4(Q_4 - \bar{Q}_4)) \end{bmatrix},$$

with $\alpha > 0$ locally asymptotically stabilizes the corresponding given equilibrium $(0, \bar{Q}, \bar{R}, \bar{b}, 0, 0)$.

The control (5.24) involves angular and linear velocity measurements. Making these measurements *in situ* on the MEMS is infeasible. Thus assuming that the configuration variables (R, b) are available for measurement (see [25] for a discussion of how to do this), we estimate the angular and linear velocities of the mirror in the body frame using the intrinsic observer of section 5.2, where now $S(R, b, \xi, v) = I^{-1}(\sum_{i=1}^{4} f_i^c(R, b) + \sum_{i=1}^{4} f_i^d(R, b, \xi, v) + B\ w(Q))$. MATLAB simulation results are shown in figure 5.3. For comparison purposes the performance of both the full state feedback controller and the dynamic output feedback controller are plotted on the same figures for identical initial conditions of the MEMS. In the case of the dynamic output feedback control the initial observer configuration error corresponds to a $\pi/10$ rotation about the [1 1 0] axis and a translation of [3 3 3]. The initial body angular velocity error is $[9\ 12\ -9]$ and the initial velocity error of the center of mass in the body coordinates is $[6\ 9\ -3]$.

6. Conclusion. We present an intrinsic observer-based approach to the stabilization of a class of simple mechanical control systems on a Lie group. Specifically, we consider systems with left-invariant kinetic energy and measured configuration variable. The result is obtained by specializing two general formulations on arbitrary Riemannian manifolds, namely passivity-based control and intrinsic velocity estimation. This specialization is noteworthy because it results in drastically simplified explicit expressions for the controller that can be readily applied once the kinetic energy tensor and the external forces are specified. The observer is explicitly computed for two special cases of particular importance, namely the rotation group $SO(3)$ and the Euclidian motion group $SE(3)$. The expressions given in those sections may be applied to the many problems of practical significance arising from rigid body motion by specializing only the inertia tensor and the external forces. Here the results are applied to estimation of the velocities of the axi-symmetric top and to the stabilization of an electrostatically-actuated MEMS model. Simulations show excellent performance.

Appendix A.

A.1. First Variation Equations of the Observer. If a separation principle is to hold in the presence of velocity dependent control then the observer must converge even when velocity terms are allowed in the external forces. Following [2] we first construct the first variation of the observer and then use contraction analysis to prove local exponential convergence of the observer. In what follows we show that including such velocity terms in S does not change the first variation equations obtained in [2]. Thus the contraction analysis of [2] holds without modification, and the observer converges when $S = S(q, v)$.

Consider the simple mechanical system on a Riemannian manifold $(M, \langle\langle\cdot,\cdot\rangle\rangle)$ given by,

$$\dot{q} = v, \tag{A.1}$$

$$\nabla_{\dot{q}}\, v = S(q, v). \tag{A.2}$$

and the observer

$$\dot{\tilde{q}} = X(\tilde{q}, \tilde{v}) \;=\; \tilde{v} - \alpha\, grad\, F(\tilde{q}), \tag{A.3}$$

$$\nabla_{\dot{\tilde{q}}}\, \tilde{v} = Y(\tilde{q}, \tilde{v}) \;=\; \Gamma(S(q, \tilde{v})) - R(\tilde{v}, grad\, F(\tilde{q}))\tilde{v} - \beta\, grad\, F(\tilde{q}), \tag{A.4}$$

where $R(\cdot,\cdot)\cdot$ is the curvature and $\Gamma(S(q,\tilde{v}))$ is the parallel transport of the external forces $S(q,\tilde{v})$ to $\tilde{q}$ along the unique geodesic joining q to $\tilde{q}$ (for q and $\tilde{q}$ sufficiently close). Observe that in the parallel transport term of $\Gamma(S(q,\tilde{v}))$ we use $\tilde{v}$ instead of v.

Let $(q(t),v(t))$ a solution of (A.3) – (A.4) with initial condition (q_0,v_0) and $(\tilde{q}(t),\tilde{v}(t))$ be a solution of (A.3) – (A.4) with initial conditions $(\tilde{q}(0),\tilde{v}(0)) \neq (q_0,v_0)$. Let $s \mapsto \gamma(s) \in M$ be a smooth curve on M such that $\gamma(0) = q_0$ and $\gamma(1) = \tilde{q}_0$ and let $s \mapsto \tau(s) \in T_{\gamma(s)}M$ be a smooth vector field defined along $\gamma(s)$ such that $\tau(0) = v_0$ and $\tau(1) = \tilde{v}_0$. Let $(\tilde{q}(s,t),\tilde{v}(s,t))$ be a solution of (A.3) – (A.4) with initial conditions $(\gamma(s),\tau(s))$. Define

$$\frac{\partial \tilde{q}}{\partial s}(s,t) = J_q(s,t) \in T_{\tilde{q}(s,t)}M, \tag{A.5}$$

$$\frac{\partial \tilde{v}}{\partial s}(s,t) = \nabla_{J_q}\tilde{v} = J_v(s,t) \in T_{\tilde{q}(s,t)}M. \tag{A.6}$$

In coordinates these equations correspond exactly to those given by (6) in [2]. By construction it follows that $[J_q,\dot{\tilde{q}}] = 0$ and thus the first variation of (A.3) – (A.4) can be intrinsically computed as follows.

$$\frac{\partial \dot{\tilde{q}}}{\partial s} = \nabla_{J_q}\dot{\tilde{q}} = \nabla_{\dot{\tilde{q}}}J_q = \nabla_{J_q}X, \tag{A.7}$$

$$\frac{\partial \nabla_{\dot{\tilde{q}}}\,\tilde{v}}{\partial s} = \nabla_{J_q}\nabla_{\dot{\tilde{q}}}\,\tilde{v} = \nabla_{\dot{\tilde{q}}}\nabla_{J_q}\,\tilde{v} + R(J_q,\dot{\tilde{q}})\tilde{v} = \nabla_{J_q}Y, \tag{A.8}$$

where the second equality in (A.7) follows from $[J_q,\dot{\tilde{q}}] = 0$ and the second equality in (A.8) follows from

$$\nabla_{J_q}\nabla_{\dot{\tilde{q}}}\,\tilde{v} - \nabla_{\dot{\tilde{q}}}\nabla_{J_q}\,\tilde{v} = R(J_q,\dot{\tilde{q}})\tilde{v}.$$

Thus from (A.6), (A.7) and (A.8) we have the first variation equations of (A.3) – (A.4) as

$$\nabla_{\dot{\tilde{q}}}J_q = \nabla_{J_q}X, \tag{A.9}$$

$$\nabla_{\dot{\tilde{q}}}J_v = -R(J_q,\dot{\tilde{q}})\tilde{v} + \nabla_{J_q}Y. \tag{A.10}$$

When $X(\tilde{q},\tilde{v}) = \tilde{v}$ and $Y(\tilde{q},\tilde{v}) = 0$ we recover the Jacobi's equation of geodesic variation. $\nabla^2_{\dot{\tilde{q}}}J_q = -R(J_q,\tilde{v})\tilde{v}$.

Substituting for X and Y from (A.3) – (A.4) in (A.9) and (A.10) we have

$$\nabla_{\dot{\tilde{q}}}J_q = J_v - \alpha\nabla_{J_q} grad\ F, \tag{A.11}$$

$$\nabla_{\dot{\tilde{q}}}J_v = -R(J_q,\dot{\tilde{q}})\tilde{v} + \nabla_{J_q}\Gamma - \beta\nabla_{J_q} grad\ F - \nabla_{J_q}R(\tilde{v}, grad\ F(\tilde{q}))\tilde{v}. \tag{A.12}$$

From [35]

$$\begin{aligned}\nabla_{J_q}R(\tilde{v}, grad\ F(\tilde{q}))\tilde{v} = (\nabla_{J_q}R)\left((\tilde{v}, grad\ F(\tilde{q}))\tilde{v}\right) + R(\nabla_{J_q}\tilde{v}, grad\ F(\tilde{q}))\tilde{v}\\ +R(\tilde{v},\nabla_{J_q} grad\ F(\tilde{q}))\tilde{v} + R(\tilde{v}, grad\ F(\tilde{q}))\nabla_{J_q}\tilde{v}.\end{aligned}$$

Then the first variation equations of the observer (A.3) – (A.4) are

$$\nabla_{\dot{\tilde{q}}}J_q = J_v - \alpha\nabla_{J_q} grad\ F, \tag{A.13}$$

$$\begin{aligned}\nabla_{\dot{\tilde{q}}}J_v = {} & -R(J_q,\dot{\tilde{q}})\tilde{v} + \nabla_{J_q}\Gamma - \beta\nabla_{J_q} grad\ F\\ & -(\nabla_{J_q}R)\left((\tilde{v}, grad\ F(\tilde{q}))\tilde{v}\right) - R(\nabla_{J_q}\tilde{v}, grad\ F(\tilde{q}))\tilde{v}\\ & -R(\tilde{v},\nabla_{J_q} grad\ F(\tilde{q}))\tilde{v} - R(\tilde{v}, grad\ F(\tilde{q}))\nabla_{J_q}\tilde{v}.\end{aligned} \tag{A.14}$$

These equations correspond exactly with (9) of [2]. Note our curvature convention follows [20, 21, 35] which results in a sign difference from [2].

When $\tilde{q}$ and q are sufficiently close (ie. up to second order)

$$grad\, F = 0, \quad \nabla_{J_q} grad\, F = J_q, \quad \nabla_{J_q}\Gamma(S(q,\tilde{v})) = 0.$$

Thus for sufficiently close $\tilde{q}$ and q the first variation equations reduce to

$$\nabla_{\dot{\tilde{q}}} J_q = J_v - \alpha J_q, \tag{A.15}$$

$$\nabla_{\dot{\tilde{q}}} J_v = -\beta J_q. \tag{A.16}$$

These equations correspond exactly with equation (10) of [2].

A.2. Separation Principle. The following lemma proved in [28] provides the basis for a separation principle when the Lie group G is compact. Consider the system

$$\dot{g} = g \cdot \zeta, \tag{A.17}$$

$$\dot{\zeta} = I^{-1}\left(ad^*_{\zeta}\tilde{I}\zeta + f(g,\zeta)\right) + \psi(g,\zeta,q), \tag{A.18}$$

with $(g,\zeta) \in G \times \mathcal{G}$ and $q \in \mathcal{R}^n$ and $V = U(g) + \frac{1}{2}\langle\langle \zeta,\zeta\rangle\rangle_{\mathcal{G}}$ where $U(g)$ is a smooth globally defined Morse function and is the potential energy of the system. Also consider the following assumptions.

Assumption 1. *The point $(\bar{g},0)$ is a almost globally stable equilibrium of (A.17)–(A.18) with $\psi \equiv 0$ and further more $\langle\langle g^{-1}grad\, U,\zeta\rangle\rangle_{\mathcal{G}} + \langle\langle I^{-1}f,\zeta\rangle\rangle_{\mathcal{G}} \le 0$.*

The condition $\langle\langle g^{-1}grad\, U,\zeta\rangle\rangle_{\mathcal{G}} + \langle\langle I^{-1}f,\zeta\rangle\rangle_{\mathcal{G}} \le 0$ is satisfied by any simple mechanical system with potential energy $U(g)$ and Rayleigh type dissipation. The equilibrium $(\bar{g},0)$ is an almost globally stable equilibrium if $\bar{g}$ is a unique minimum of $U(g)$.

Assumption 2. *The function $q(t) \in \mathcal{R}^n$ satisfies*

$$||q(t)|| \le c||q(0)||e^{-\lambda t}, \tag{A.19}$$

for some $c > 0, \lambda > 0$ and all $t > 0$.

Assumption 3. *The interconnection term satisfies $\psi(g,\zeta,0) \equiv 0$ and the following linear growth conditions,*

$$||\psi|| \le \gamma_1(||q||)||\zeta|| + \gamma_2(||q||), \tag{A.20}$$

for two class $\mathcal{K}_\infty$ functions $\gamma_1(\cdot), \gamma_2(\cdot)$.

Lemma A.1. *If the Lie group G is compact and if assumptions 1 – 3 are satisfied, then the equilibrium $(\bar{g},0)$ of the system (A.17) – (A.18) is almost globally stable. Convergence is asymptotic if the inequality in assumption 1 is strict.*

Corollary A.2. *For a compact Lie group G if the state feedback control (3.3) almost globally stabilizes $(\bar{g},0)$ then the dynamic output feedback control (4.8) almost globally stabilizes $(\bar{g},0)$. Convergence is asymptotic if the control (3.3) ensures asymptotic convergence.*

Proof of Corollary 1. *Define the observation error of the velocities by $\zeta_{oe} := \tilde{\zeta} - \zeta$. From Theorem 1 of [2] we have that for sufficiently small initial observer configuration error*

$$\left(d(g(t),\tilde{g}(t)) + ||\zeta_{oe}(t)||_{\mathcal{G}}\right) < \left(d(g(0),\tilde{g}(0)) + ||\zeta_{oe}(0)||_{\mathcal{G}}\right) e^{-\lambda t} \tag{A.21}$$

for some $\lambda > 0$. From (A.21) it follows that for nonzero initial velocity error

$$||\zeta_{oe}(t)||_{\mathcal{G}} < c\, ||\zeta_{oe}(0)||_{\mathcal{G}}\; e^{-\lambda t}, \tag{A.22}$$

for some $c > 0$.

Substituting $\tilde{\zeta} = \zeta + \zeta_{oe}$ in the controls (4.8) we obtain a simple mechanical system in the form of (A.17) – (A.18) where the interconnection term $\psi(g, \zeta, \zeta_{oe}) = B(g)B(g)^T \zeta_{oe}$ with $B(g) = [f^1(g)\ \ f^2(g)\ \ \cdots\ \ f^m(g)]$. If the potential energy $U(g)$ of the system is a globally defined smooth Morse function with a unique minimum at $\bar{g}$ and the damping injection $u = -B^T\zeta$ almost globally stabilizes the equilibrium $(\bar{g}, 0)$ then it can be verified that assumption 1 is satisfied. From (A.22) assumption 2 is satisfied. Since $\psi(g, \zeta, \zeta_{oe}) = B(g)B(g)^T\zeta_{oe}$, if G is compact, it can be easily verified that assumption 3 is satisfied. Thus from Lemma A.1 the dynamic feedback control (4.8) almost globally stabilizes the equilibrium $(\bar{g}, 0)$ provided that the initial observer configuration error is sufficiently small. Convergence is asymptotic if the inequality in assumption (1) is strict. This is guaranteed if (3.3) ensures asymptotic convergence.

REFERENCES

[1] R. Abraham and J. E. Marsden, *Foundations of Mechanics, Second Ed.* Westview, 1978.

[2] N. Aghannan and P. Rouchon, "An Intrinsic Observer for a Class of Lagrangian Systems," *IEEE Transactions on Automatic Control*, (48) No. 6, June 2003, pp 936–945.

[3] M. R. Akella, "Rigid Body Attitude Tracking without Angular Velocity Feedback," *Systems and Control Letters*, (42), 2001, pp 321–326.

[4] M. R. Akella, J. T. Halbert and G. R. Kotamraju, "Rigid Body Attitude Control with Inclinometer and Low-cost Gyro Measurements," *Systems and Control Letters*, (49), 2003, pp 151–159.

[5] V. I. Arnold, *Mathematical Methods of Classical Mechanics, Second Ed.*, Springer-Verlag, New York 1989.

[6] J. Baillieul, "The Geometry of Controlled Mechanical Systems," *Mathematical Control Theory*, Springer-Verlag, New York 1999, pp 322–354.

[7] A. M. Bloch and J. E. Marsden, "Stabilization of Rigid Body Dynamics by the Energy-Casimir Method," *Systems and Control Letters*, (14), 1991, pp 341–346.

[8] A. M. Bloch and P. E. Crouch, "Optimal Control, Optimization and Analytical Mechanics," *Mathematical Control Theory*, Springer-Verlag, New York 1999, pp 268–321.

[9] A. M. Bloch, N. E. Leonard and J. E. Marsden, "Controlled Lagrangians and the Stabilization of Mechanical Systems I: The First Matching Theorem," *IEEE Transactions on Automatic Control*, (45) No. 12, Dec 2000, pp 2253 – 2270.

[10] A. M. Bloch, D. E. Chang, N. E. Leonard and J. E. Marsden, "Controlled Lagrangians and the Stabilization of Mechanical Systems II: Potential Shaping," *IEEE Transactions on Automatic Control*, (46) No. 10, Oct 2001, pp 1556 – 1571.

[11] A. M. Bloch and N. E. Leonard, "Symmetries Conservation Laws, and Control," *Geometry, Mechanics and Dynamics*, Springer-Verlag, New York 2002, pp 431–460.

[12] A. M. Bloch, J. Baillieul, P. Crouch and J. E. Marsden, *Nonholonomic Mechanics and Control,* Springer-Verlag, New York 2003.

[13] F. Bullo, N. E. Leonard and A. D. Lewis, "Controllability and Motion Algorithms for Underactuated Lagrangian Systems on Lie Groups," *IEEE Transactions on Automatic Control*, (45) No. 8, Aug 2000, pp 1437–1454.

[14] F. Bullo and R. M. Murray, "Tracking for Fully Actuated Mechanical Systems: A Geometric Framework," *Automatica*, (35), 1999, pp 17–34.

[15] F. Bullo and A. D. Lewis, *Geometric Control of Mechanical Systems: Modeling, Analysis, and Design for Simple Mechanical Control Systems,* Springer-Verlag, New York 2004.

[16] C. I. Byrnes, A. Isidori and J. C. Willems, "Passivity, Feedback Equivalence, and the Global Stabilization of Minimum Phase Nonlinear Systems," *IEEE Transactions on Automatic Control*, (36) No. 11, Nov 1991, pp 1228 – 1240.

[17] D. E. Chang, A. M. Bloch, N. E. Leonard, J. E. Marsden and C. A. Woolsey, "The Equivalence of Controlled Lagrangian and Controlled Hamiltonian Systems," *ESIAM: Control,Optimization and Calculus of Variations*, Vol. 8, June 2002, pp. 393-422.

[18] J. J. Duistermaat and J. A. C. Kolk, *Lie Groups* Springer-Verlag, Berlin Heidelberg 2000.
[19] D. Elata, O. Bochobza-Degni and Y. Nemirovsky, "Analytical Approach and Numerical alpha-line method for Pull-In Hyper-surface Extraction of Electrostatic Actuators with Multiple Uncoupled Voltage Sources," *Journal of Microelectromechanical Systems,* Vol. 12, No. 5, Oct 2003, pp. 681–691.
[20] T. Frankel, *The Geometry of Physics an Introduction,* Cambridge University Press, 1997.
[21] J. Jost, *Riemannian Geometry and Geometric Analysis,* Springer-Verlag, Berlin 2002.
[22] V. Jurdjevic, "Optimal Control, Geometry and Mechanics," *Mathematical Control Theory*, Springer-Verlag, New York 1999, pp 268–321.
[23] D. E. Koditschek, "The Application of Total Energy as a Lyapunov Function (for) Mechanical Control Systems," *In J. E. Marsden, P. S. Krishnaprasad and J. C. Simo (Eds) Dynamics and Control of Multi Body Systems*,Vol 97, Providence, RI:AMS, 1989, pp 131–157.
[24] A. D. Lewis and R. Murray, "Configuration Controllability of Simple Mechanical Control Systems," *SIAM Journal of Optimization and Control*, (35), 1997, pp 766–790.
[25] D. H. S. Maithripala, R. O. Gale, M. W. Holtz, J. M. Berg and W. P. Dayawansa, "Nano-precision control of micromirrors using output feedback," *Proc. of the 2003 CDC*, Maui, HW, 2003.
[26] D. H. S. Maithripala, J. M. Berg and W. P. Dayawansa, "An Intrinsic Observer for a Class of Simple Mechanical Systems on Lie Groups," *Proc. of the 2004 ACC*, Boston, 2004.
[27] D. H. S. Maithripala, J. M. Berg and W. P. Dayawansa, "Control of an Electrostatic MEMS Using Static and Dynamic Output Feedback," *Accepted for publication by the ASME Journal of Dynamical Systems Measurement and Control*, 2004.
[28] D. H. S. Maithripala, J. M. Berg and W. P. Dayawansa, "Almost-Global Tracking of Simple Mechanical Systems on a General Class of Lie Groups," *Accepted for publication by the IEEE Transactions on Automatic Control*, 2005.
[29] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry, Second Ed.* Springer-Verlag, New York 1999.
[30] Martin F., Rouchon P. and Rudolph J., "Invariant Tracking," *ESIAM: Control,Optimization and Calculus of Variations*, Vol. 10, Jan 2004, pp. 1–13.
[31] F. Mazenc and A. Astolfi, "Robust Output Feedback Stabilization of the Angular Velocity of a Rigid Body," *Systems and Control Letters*, (39), 2000, pp 203–210.
[32] M. C. Munoz-Lecanda and F. J. Yaniz-Fernandez, "Dissipative Control of Mechanical Systems," *SIAM Journal of Control and Optimization*, 40 (5), 2002, pp 1505–1516.
[33] R. Ortega, A. J. van der Schaft, I. Mareels and B. Maschke, "Putting Energy Back in Control," *IEEE Control Systems Magazine*, April 2001, pp 18-33.
[34] R. Ortega, A. J. van der Schaft, B. Maschke and G. Escobar "Interconnection and Damping Assignment Passivity-Bases Control of Port-Controlled Hamiltonian Systems," *Automatica*, 2002, pp 585–596.
[35] P. Petersen, *Riemannian Geometry,* Springer-Verlag, New York 1998.
[36] H. Rehbinder and X. Hu, "Nonlinear State Estimation for Rigid Body Motion with Low Pass Sensors," *Systems and Control Letters*, 40 (3), 2000, pp 183–191.
[37] S. Salcudean, "A Globally Convergent Angular Velocity Observer for Rigid Body Motion," *IEEE Transactions on Automatic Control*, 36 (12), December 1991, pp 1493–1497.
[38] R. Sepulchre, M. Jankovic and P. Kokotovic, *Constructive Nonlinear Control,* Springer-Verlag, London 1997.
[39] G. V. Smirnov, "Attitude Determination and Stabilization of a Spherically Symmetric Rigid Body in a Magnetic Field," *Ineternational Journal of Control*, (74) No. 4, 2001, pp 341–347.
[40] A. J. van der Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control,* Springer-Verlag, London 2000.
[41] C. A. Woolsey and N. E. Leonard, "Moving Mass Control for Underwater Vehicles," *Proc. of the American Control Conference, Anchorage, Ak*, 2001, pp 2824–2830.
[42] M. Zefran, V. Kumar and C. B. Croke, "On the Generation of Smooth Three-Dimensional Rigid Body Motions," *IEEE Transactions on Robotics and Automation*, (14) No. 4, August 1998, pp 576–589.