ECE 308 -9

Solution of Linear Constant-Coefficient Difference Equations

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Solution of Linear Constant-Coefficient Difference Equations

Example:

Determine the response $y(n),\ n \geq 0$ of the system described by the second-order difference equation

$$y(n) = 0.7y(n-1) - 0.1y(n-2) + 2x(n) - x(n-2)$$

to the input $x(n) = 4^n u(n)$

The homogenous solution is

$$y(n) - 0.7y(n-1) + 0.1y(n-2) = 0$$

$$\lambda^n - 0.7\lambda^{n-1} + 0.1\lambda^{n-2} = 0$$

$$\lambda^{n+2}\left(\lambda^2 - 0.7\lambda + 0.1\right) = 0$$ → $\lambda_1 = 0.5$ → and $\lambda_2 = 0.2$

$$y_h(n) = c_1 0.5^n + c_2 0.2^n$$

Solution of Linear Constant-Coefficient Difference Equations

Particular Solution:

$$y_p(n) = K4^n u(n)$$

$$K4^n u(n) - 0.7K4^{n-1}u(n-1) + 0.1K4^{n-2}u(n-2) = (2)4^n u(n) - 4^{n-2}u(n-2)$$

n=2

$$K4^2 - 0.7K4^1 + 0.1K4^0 = 2(4)^2 - 4^0$$

$$16K - 2.8K + 0.1K = 32 - 1$$

$$y_p(n) = 2.33(4)^n u(n) \quad \Leftarrow \quad K = \frac{31}{13.3} = 2.33$$

The total solution

$$y(n) = \left[c_1 0.5^n + c_2 0.2^n + 2.33(4)^n\right]u(n)$$

Solution of Linear Constant-Coefficient Difference Equations

To find c1 and c2
For n=0:

From difference equation,

$$y(0) = 0.7y(0-1) - 0.1y(0-2) + 2x(0) - x(0-2)$$
$$y(0) = 2$$

From the total solution,

$$y(0) = c_1 + c_2 + 2.33$$

For n=1:

From difference equation,

$$y(1) = 0.7y(1-1) - 0.1y(1-2) + 2x(1) - x(1-2)$$
$$y(1) = 1.4 + 8 = 9.4$$

From difference equation,

$$y(1) = 0.5c_1 + 0.2c_2 + 9.32$$

Therefore,

$$2 = c_1 + c_2 + 2.33 \qquad 9.4 = 0.5c_1 + 0.2c_2 + 9.32$$

$$c_2 = -0.807 \qquad c_1 = 0.466$$

Total Solution

$$y(n) = \left[0.466(0.5)^n - 0.807(0.2)^n + 2.33(4)^n\right]u(n)$$

The Impulse Response of a LTI recursive system

In general case

$$y_{zs}(n) = \sum_{k=0}^{n} h(k)x(n-k),\ n \geq 0$$

If the input $x(n) = \delta(n)$, then we obtain

$$y_{zs}(n) = h(n)$$

The impulse response can be obtained from the linear constant-coefficient difference equation. That is the solution of homogeneous equation and particular solution to the excitation function. In the case where the excitation function is an impulse function. The particular solution is zero $y_p(n) = 0$, since $x(n) = 0$ for n>0.

The Impulse Response of a LTI recursive system

Example: Find the impulse response form the following equation,

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$$

The homogenous solution is

$$y_h(n) = C_1(-1)^n + C_2 4^n$$

The particular solution is zero when $x(n) = \delta(n)$

To find $C_1$and C_2, we evaluate difference equation and homogenous solution for n=0 and n=1.($y(-1) = 0,\ y(-2) = 0,$ since the system must be relaxed)

$$y(0) = 1$$
$$y(1) = 3y(0) + 2 = 5$$

$$y(0) = C_1 + C_2$$
$$y(1) = -C_1 + 4C_2$$

$$C_2 = \frac{6}{5}$$

$$C_1 = -\frac{1}{5}$$

The impulse response is

$$h(n) = \left[-\frac{1}{5}(-1)^n + \frac{6}{5}4^n\right]u(n)$$

Implementation of Discrete-Time Systems

A system can be described by a linear constant-coefficient difference equation.

Let's consider the first order system

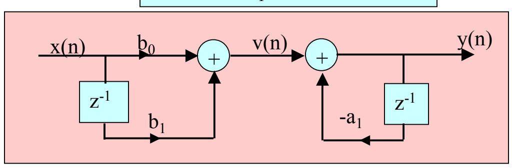
$$y(n) = -a_1 y(n-1) + b_0 x(n) + b_1 x(n-1)$$

The system can be described by two systems in cascade. The first is a nonrecursive system described by the equation

$$v(n) = b_0 x(n) + b_1 x(n-1)$$

The second part is recursive system

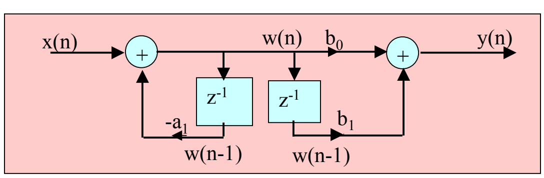
$$y(n) = -a_1 y(n-1) + v(n)$$



This is called a *direct form I structure*.

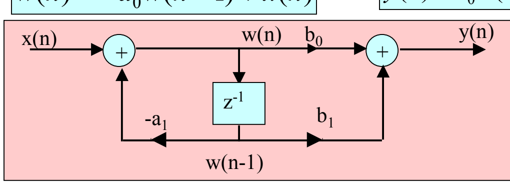
Implementation of Discrete-Time Systems

Using convolution properties, we can interchange the order of the recursive and nonrecursive system



$$w(n) = -a_0 w(n-1) + x(n)$$

$$y(n) = b_0 w(n) + b_1 w(n-1)$$



Using only one delay. It is more efficient in terms of memory requirement. It is called the *direct form II structure*

Implementation of Discrete-Time Systems

In general form, The difference equation is given by

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{N} b_k x(n-k)$$

The nonrecursive system is

$$v(n) = \sum_{k=0}^{N} b_k x(n-k)$$

The recursive system is

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + v(n)$$

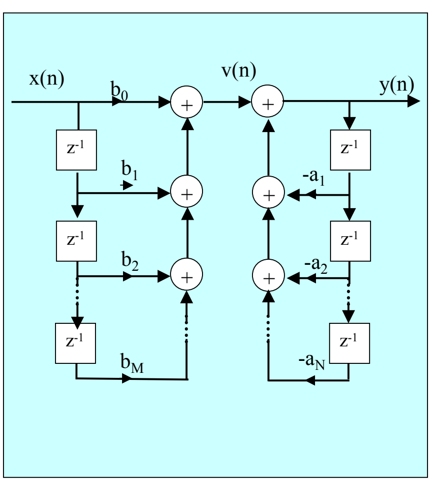
Direct form II structure , the recursive system is

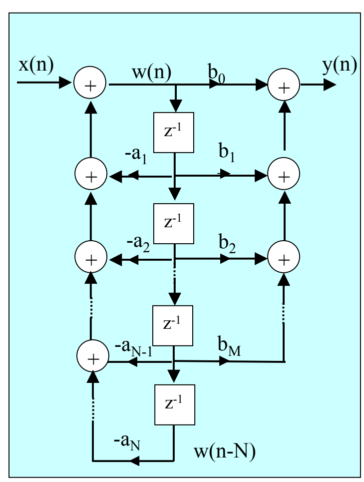
$$w(n) = -\sum_{k=1}^{N} a_k w(n-k) + x(n)$$

The nonrecursive system is

$$y(n) = \sum_{k=0}^{N} b_k w(n-k)$$

Implementation of Discrete-Time Systems





Step-1

Step-3

Implementation of Discrete-Time Systems

A special case of general case

$a_k = 0,\ k = 1,2,\ldots,N$ $\quad y(n) = \sum_{k=0}^{N} b_k x(n-k)$

which is a nonrecursive LTI system. Such a system coefficients b_k

$$h(k) = \begin{cases} b_k & 0 \le k \le M \\ 0 & \text{otherwise} \end{cases}$$

The second part, set M=0 to obtain the general case difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + b_0 x(n)$$

Crosscorrelation and Autocorrelation Sequence

The crosscorelation of x(n) and y(n) is s sequence $r_{xy}(l)$ is defined as

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l),\ l = 0,\pm 1,\pm 2,\ldots$$

Or

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n+l)y(n),\ l = 0,\pm 1,\pm 2,\ldots$$

The reverse crosscorrelation is

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)x(n-l),\ l = 0,\pm 1,\pm 2,\ldots$$

Or

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n+l)x(n),\ l = 0,\pm 1,\pm 2,\ldots$$

$$r_{xy}(l) = r_{yx}(-l)$$

Crosscorrelation and Autocorrelation Sequence

In special case , we have autocorrelation, which is defined as
So, that

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l)$$

Or

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n+1)x(n)$$

Example:

$$x(n) = \{\ldots 0,0,2,-1,3,7,\underset{\uparrow}{1},2,-3,0,0,\ldots\}$$

$$y(n) = \{\ldots 0,0,1,-1,2,-2,\underset{\uparrow}{4},1,-2,5,0,0,\ldots\}$$

$$r_{xy}(0) = \sum_{n=-\infty}^{\infty} x(n)y(n) = 0+0+2+6-14+4+2+6+0+0 = 7$$

$$r_{xy}(1) = \sum_{n=-\infty}^{\infty} x(n)y(n-1) = 0+0+-1-3+14-2+8-3+0+0 = 13$$

$$r_{xy}(l) = \{10,-9,19,36,-14,33,0,7,13,-18,16,-7,5,-3\}$$

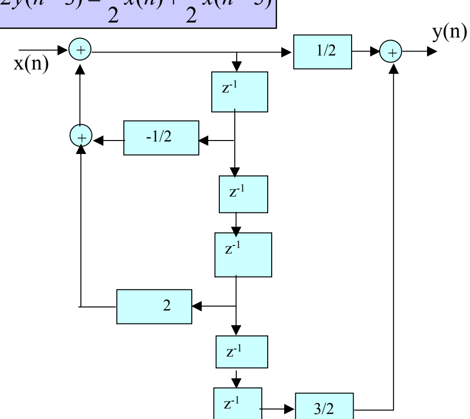
Problem Solutions

Problem 43.a

Determine the direct form II realization for the following LTI system

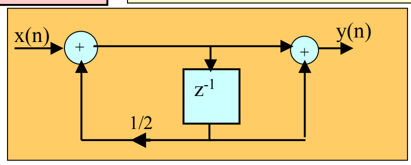
$$2y(n) + y(n-1) - 4y(n-3) = x(n) + 3x(n-5)$$

$$y(n) + \frac{1}{2}y(n-1) - 2y(n-3) = \frac{1}{2}x(n) + \frac{3}{2}x(n-5)$$



Problem Solutions

Problem 44.a Compute the first samples of its impulse response



$x(n) = \delta(n)$

$$x(n) = \{\underset{\uparrow}{1}, 0, 0, \ldots\}$$

$$y(n) = \frac{1}{2}y(n-1) + x(n) + x(n-1)$$

$$y(0) = x(0) = 1$$

$$y(1) = \frac{1}{2}y(0) + x(1) + x(0) = \frac{3}{2}$$

$$y(2) = \frac{1}{2}y(1) + x(2) + x(1) = \frac{3}{4}$$

$$y(3) = \frac{1}{2}y(2) + x(3) + x(2) = \frac{3}{8}$$

$$y(n) = \left\{1, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \frac{3}{32}, \frac{3}{64}, \ldots\right\}$$

Problem Solutions

Problem 44.b Find the input output relation

$$y(n) = \frac{1}{2}y(n-1) + x(n) + x(n-1)$$

Problem 44.c The input

$$x(n) = \{\underset{\uparrow}{1}, 1, 1, \ldots\}$$

$$y(n) = \frac{1}{2}y(n-1) + x(n) + x(n-1)$$

$$y(0) = x(0) = 1$$

$$y(1) = \frac{1}{2}y(0) + x(1) + x(0) = \frac{5}{2}$$

$$y(2) = \frac{1}{2}y(1) + x(2) + x(1) = \frac{13}{4}$$

$$y(3) = \frac{1}{2}y(2) + x(3) + x(2) = \frac{29}{8}$$

……

Problem Solutions

Problem 44.d

Use convolution

$$y(n) = u(n0 * h(n) = \sum_{k=0}^{\infty} u(k)h(n-k) = \sum_{k=0}^{n} h(n-k)$$

$$y(0) = h(0) = 1$$

$$y(1) = h(0) + h(1) = \frac{5}{2}$$

$$y(2) = h(0) + h(1) + h(2) = \frac{13}{4}$$

$$y(3) = h(0) + h(1) + h(2) + h(3) = \frac{29}{8}$$

Problem Solutions

Problem 54

Find the y(n) for the following equation

$$y(n) - 4y(n-1) + 4y(n-2) = x(n) - x(n-1)$$

$$x(n) = (-1)^n u(n)$$

The characteristic equation

$$\lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda_1 = 2, \quad \lambda_2 = 2$$

The homogenous solution is $y_h(n) = c_1 2^n + c_2 n 2^n$

The particular solution is $y_p(n) = k(-1)^n u(n)$

$$k(-1)^n u(n) - 4k(-1)^{n-1} u(n-1) + 4k(-1)^{n-2} u(n-2) = (-1)^n u(n) - (-1)^{n-1} u(n-1)$$

For n=2

$$k + 4k + 4k = 1 + 1 \rightarrow k = \frac{2}{9}$$

Problem Solutions

The total solution is

$$y(n) = y_h(n) + y_p(n) = \left[c_1 2^n + c_2 n 2^n + \frac{2}{9}(-1)^n \right] u(n)$$

Using the initial condition, $y(-1) = y(-2) = 0$

we can obtain from difference equation at

n=0 $$y(0) - 4y(-1) + 4y(-2) = x(0) - x(-1)$$ $$y(0) = 1$$

n=1 $$y(1) - 4y(0) + 4y(-1) = x(1) - x(0)$$ $$y(1) = 2$$

From the total solution

$$y(0) = c_1 + \frac{2}{9} = 1$$ $$y(1) = c_1 2 + c_2 2 - \frac{2}{9} = 2$$ $$y(1) = c_1 2 + c_2 2 - \frac{2}{9} = 2$$

$$c_1 = \frac{7}{9}$$ $$c_2 = \frac{1}{3}$$

The total solution

$$y(n) = \left[\frac{7}{9} 2^n + \frac{1}{3} n 2^n + \frac{2}{9}(-1)^n \right] u(n)$$

Problem Solutions

Problem 55

Find the impulse response h(n) for the following equation

$$y(n) - 4y(n-1) + 4y(n-2) = x(n) - x(n-1)$$

The homogenous solution is

$$y_h(n) = c_1 2^n + c_2 n 2^n$$

So system response $h(n) = \left[c_1 2^n + c_2 n 2^n \right] u(n)$

To find the constant

$$y(0) - 4y(-1) + 4y(-2) = \delta(0) - \delta(-1)$$ $$y(0) = 1$$

$$y(1) - 4y(0) + 4y(-1) = \delta(1) - \delta(0)$$ $$y(1) = 3$$

From the system response h(n)

$$y(0) = c_1 = 1$$ $$y(1) = c_1 2 + c_1 2 = 3$$

$$c_2 = \frac{1}{2}$$

$$h(n) = \left[2^n + \frac{1}{2} n 2^n \right] u(n)$$