ECE 308 -7

# LTI Described by Difference Equations


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## Discrete Time Systems Described by Difference Equations

### Recursive and Nonrecursive Discrete-Time Systems

If a system output y(n) at time n depends on any number of past output value y(n-1), y(n-2),… , it is called a recursive system.

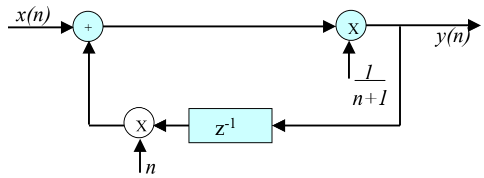
Let's have a DTS that gives the cumulative average

$$y(n) = \frac{1}{n+1}\sum_{k=0}^{n} x(k)$$

We can find y(n) more efficient by utilizing the output y(n-1).

$$\begin{aligned} y(n) &= \frac{1}{n+1}\left(\sum_{k=0}^{n-1} x(k) + x(n)\right) \\ &= \frac{1}{n+1}\left(ny(n-1) + x(n)\right) \\ &= \frac{n}{n+1} y(n-1) + \frac{1}{n+1} x(n) \end{aligned}$$

## Recursive and Nonrecursive Discrete-Time Systems




This system requires two multiplication, one addition, and one memory location. This is a recursive system which means the output at time n depends on any number of a past output values. So, a recursive system has feed back output of the system into the input. This feed back loop contains a delay element.

$$y(0) = x(0)$$

$$y(1) = \frac{1}{2}y(0) + \frac{1}{2}x(1)$$

$$y(2) = \frac{2}{3}y(1) + \frac{1}{3}x(2)$$

$$y(n_0) = \frac{n_0}{n_0+1}y(n_0-1) + \frac{1}{n_0+1}x(n_0)$$

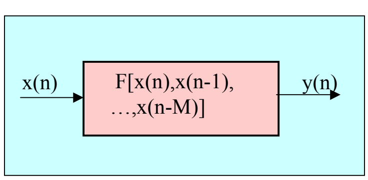


## Recursive and Nonrecursive Discrete-Time Systems

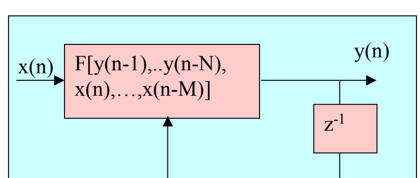
If y(n) depends only on the present and past input, it is called nonrecursive.

For the causal FIR systems

$$\begin{aligned} y(n) &= \sum_{k=0}^{M} h(k)x(n-k) \\ &= h(0)x(n) + h(1)x(n-1) + \ldots + h(M)x(n-M) \\ &= F\left[x(n), x(n-1), \ldots, x(n-M)\right] \end{aligned}$$




Nonrecursive system

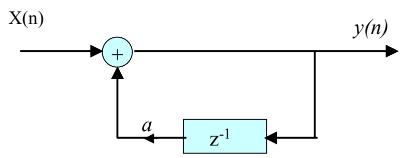



Recursive system

## Constant-Coefficient Difference Equations

LTI Systems can be described by Constant-Coefficient Difference Equations to represent the input-output relations

Let's have a recursive system that is first-order difference




$$y(n) = ay(n-1) + x(n)$$

where  is a constant and system is time invariant. We assume that we have initial condition y(-1).

For , y(n) can be obtained

$$y(0) = ay(-1) + x(0)$$

$$y(1) = ay(0) + x(1) = a^2 y(-1) + ax(0) + x(1)$$

$$y(n) = a^{n+1} y(-1) + a^n x(0) + a^{n-1} x(1) + \ldots + x(n)$$

$$y(n) = a^{n+1} y(-1) + \sum_{k=0}^{n} a^k x(n-k)$$

$n \geq 0$



## Constant-Coefficient Difference Equations

The response y(n) of the system depends on
- initial condition y(-1) of the system and
- the system response to the input signal.

If the system is initially relaxed at time n=0, its memory should be zero. So, y(-1)=0.

Then, system is at zero state and the corresponding output is called *zero-state response* or *forced response*.

$$y_{zs}(n) = \sum_{k=0}^{n} a^k x(n-k)$$ $n \geq 0$

If system is initially nonrelaxed (y(-1)≠0) and the input x(n)=0 for all n.

The corresponding output is called *zero-input response* or *natural response*.

$$y_{zi}(n) = a^{n+1} y(-1)$$ $n \geq 0$

## Constant-Coefficient Difference Equations

The total system response is

$$y(n) = y_{zs}(n) + y_{zi}(n)$$

General form of linear constant-coefficient difference equation is

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

In order to find y(n), we need to know initial conditions y(n-1), y(-2),…,y(n-N) and the input x(n) for all n>=0.

Recursive system may be relaxed or non-relaxed, depending on the initial condition.



## Constant-Coefficient Difference Equations

A system is linear if satisfy the following requirements:

1. The total response is equal to the sum of the zero-input and zero-state responses

$$y(n) = y_{zs}(n) + y_{zi}(n)$$

2. The principles of superposition applies to the zero-state response (zero-state linear)

3. The principles of superposition applies to the zero-input response (zero-input linear)

If a system does not satisfy all three separate requirement, system is called nonlinear.