ECE 308 -6

ECE 308
Discrete-Time Signals and Systems

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Analysis of Linear Time Invariant Systems (LTI)

Intoduction

Two basic methods for analyzing the response of LTI:

- The direct solution of the input-output equation for the linear system

$$y(n) = F\left[y(n-1), y(n-2), \ldots, y(n-N), x(n), x(n-1), \ldots, x(n-M)\right]$$

In general form of the input-output relationship is (called *a difference equation*), which is given

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

where a_k and b_k are constant parameters

Analysis of Linear Time Invariant Systems (LTI)

The second methods is to resolve the input signal into sum of elementary signals. Then determine the response of each input elementary signals. We can use linearity property to find total response

Let's have arbitrary discrete-time signal . We can represent in the following form

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

Example: A finite-duration sequence

$$x(n) = \{2, \underset{\uparrow}{4}, 0, 3\}$$

Using the previous equation, $x(n)$ can be written as

$$x(n) = 2\delta(n+1) + 4\delta(n) + 3\delta(n-2)$$

Analysis of Linear Time Invariant Systems (LTI)

The convolution Sum

The response of the system to x(n) is

$$y(n) = \tau[x(n)] = \tau\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] = \sum_{k=-\infty}^{\infty} x(k)\tau[\delta(n-k)]$$

The response of the LTI system to the unit sample sequence is denoted as $h(n)$, and it is called the impulse response of a linear time invariant system

$$h(n) = \tau[\delta(n)]$$

So the output sequence *y(n)* is found as

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

This is called *convolution sum*.

The response y(n) at $n=n_0$ is $$y(n_0) = \sum_{k=-\infty}^{\infty} x(k)h(n_0-k)$$

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The convolution Sum

The process of computing the convolution between *x(n)* and *h(k)* involves the following steps.

1. *Folding*. Fold $h(k)$ about $k=0$ to obtain $h(-k)$
2. *Shifting*. Shift $h(-k)$ by n_0 to the right (left) if n_0 is positive (negative), to obtain $h(n_0-k)$.
3. *Multiplication*. Multiply $x(k)$ by $h(n_0-k)$ to obtain the product sequence
4. *Summation*. Sum all the values of the product sequence to obtain the value of the output at time $n=n_0$.
5. Step 2 through 4 must be repeated, for all possible time shifts . $-\infty < n < \infty$

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Example:

The impulse response of a linear time invariant system is

$$h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$$

The input signal is

$$x(n) = \{1, 2, \underset{\uparrow}{3}, 1\}$$

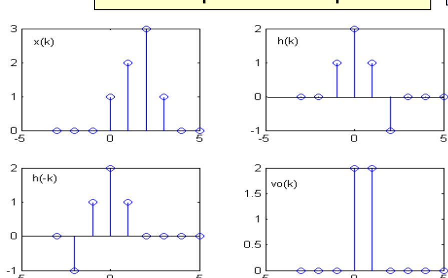
Solution:

The output at n=0 is

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k)$$

and the product sequence

$$v_0(k) = x(k)h(-k)$$



$$y(0) = \sum_{k=-\infty}^{\infty} v_0(k) = 2+2=4$$

Analysis of Linear Time Invariant Systems (LTI)

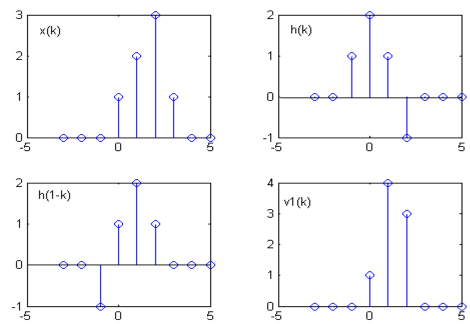
Solution (cont)

Similarly,The output at n=1 is

$$y(1)=\sum_{k=-\infty}^{\infty} x(k)h(1-k)$$

and the product sequence

$$v_1(k)=x(k)h(1-k)$$



$$y(1)=\sum_{k=-\infty}^{\infty} v_1(k)=1+4+3=8$$

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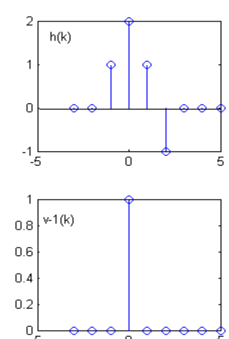
Solution (cont)

The output y(n) at n=-1 is

$$y(-1)=\sum_{k=-\infty}^{\infty} x(k)h(-1-k)$$

and the product sequence

$$v_{-1}(k)=x(k)h(-1-k)$$



$$y(-1)=\sum_{k=-\infty}^{\infty} v_{-1}(k)=1$$

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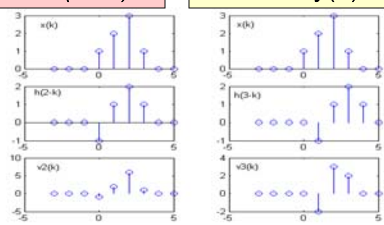
Solution (cont)

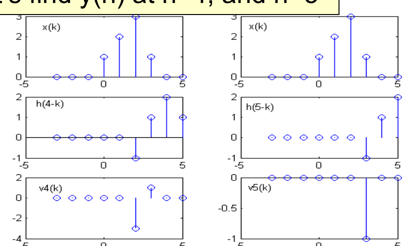
Let's find y(n) at n=2, and n=3



$$y(2)=\sum_{k=-\infty}^{\infty} v_2(k) = -1+2+6+1=8$$

$$y(3)=\sum_{k=-\infty}^{\infty} v_3(k) = -2+3+2=3$$

Let's find y(n) at n=4, and n=5



$$y(4)=\sum_{k=-\infty}^{\infty} v_4(k) = -3+1=-2$$

$$y(5)=\sum_{k=-\infty}^{\infty} v_5(k) = 1=1$$

$$y(n)=0,\ n\geq 6$$

The entire system response to x(n) is

$$y(n)=\{\ldots,0,0,1,\underset{\uparrow}{4},8,8,3,-2,-1,0,0,\ldots\}$$

Analysis of Linear Time Invariant Systems (LTI)

MatLab

Save the following m file as conv_m.m

```
function [y,ny] = conv_m(x,nx,h,nh)
% [y,ny] : convolution result
% [x,nx] : input signal, enter it as x=[2,4,3,5,4]; nx=[-2:2];
% [h,nh] : second signal, enter it as h=[3,2,4,2,2,2]; nx=[-1:4];

% to run this program
% Enter x,nx, h,nh and run the program as
% >> x=[2,4,3,5,4]; nx=[-2:2]; h=[3,2,4,2,2,2]; nh=[-1:4];
% >> [y,ny]=conv_m(x,nx,h,nh)

nylower = nx(1)+nh(1);
nyhigher = nx(length(x))+nh(length(h));
ny = [nylower:nyhigher];
y = conv(x,h);
stem(ny,y); title ('y[n]'); xlabel('n')
```

```
% To run this program
% Enter x,nx, h,nh values and run the program as
% Example:

>> x=[2,4,3,5,4];
>> nx=[-2:2];
>> h=[3,2,4,2,2,2];
>> nh=[-1:4];
>> [y,ny]=conv_m(x,nx,h,nh)
```

Analysis of Linear Time Invariant Systems (LTI)

Properties of Convolution

We show the convolution operation with asterisk

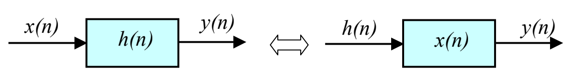
$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

Commutative Law

$$x(n) * h(n) = h(n) * x(n)$$

which means

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

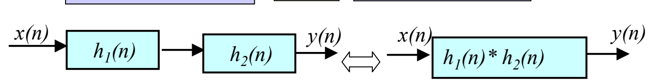


Analysis of Linear Time Invariant Systems (LTI)

Associative Law

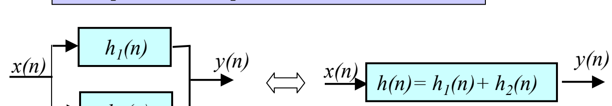
$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$$

$h(n) = h_1(n) * h_2(n)$ and $y(n) = x(n) * h(n)$



Distributed Law

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$$



$h(n) = h_1(n) + h_2(n)$ and $y(n) = x(n) * h(n)$

Properties of Convolution

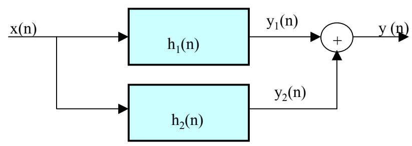
Example

Find y(n) of the following system:

$$x(n) = \{\ldots 0, 2, \underset{\uparrow}{3}, 2, 1, 0, \ldots\}$$

$$h_1(n) = (2n+1)[u(n+1) - u(n-3)]$$

$$h_2(n) = 2(n+1)\left[u(n+2) - u(n-2)\right]$$



MatLab

```
>> x=[2,3,2,1]; nx=[-1:2]; h=[-2,-1,3,7,5]; nh=[-2:2];
>> [y,ny]=conv_m(x,nx,h,nh)
```


Causal LTI Systems

Causal LTI Systems

Causal system is the output is depends only on present and past input signal

The output for LTI systems is given by

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

For causal system $h(n) = 0 \text{ for } n < 0)$ the output becomes

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

$$= \sum_{k=-\infty}^{n} x(k)h(n-k)$$

If the input is a causal sequence $(x(n) = 0 \text{ for } n < 0)$too, convolution becomes

$$y(n) = \sum_{k=0}^{n} x(k)h(n-k) = \sum_{k=0}^{n} h(k)x(n-k)$$

Causal LTI Systems

System with Finite-Duration and Infinite-Duration Impulse Response

Finite-Duration Impulse Response (FIR) system has an impulse response that is zero outside of some finite time interval.

$$h(n) = 0, \quad n < 0 \text{ and } n \geq M$$

The convolution is

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

FIR system has a finite memory of length-M samples

Causal LTI Systems

Infinite-Duration Impulse Response (IIR) of LTI has an infinite-duration impulse response that is not finite time interval.

The convolution is

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

where causality has been assumed.