ECE 308 -3

ECE 308
Sampling of Analog Signals
Quantization of Continuous-Amplitude Signals

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Sampling of Analog Signals

Example: $x_a(t) = 3\cos 100\pi t$

1. Find the minimum sampling rate required to avoid aliasing.
2. If $F_s = 200\ Hz$, What is the discrete-time signal after sampling?
3. If $F_s = 75 Hz$, What is the discrete-time signal after sampling?
4. What is the frequency F of a sinusoidal that yields sampling identical to obtained in part c?

Solution:

a $\Omega = 100\pi \longrightarrow F = 50\ \text{Hz}$

The minimum sampling rate is $F_s = 2F = 100\,\text{Hz}$

and the discrete-time signal is

$$x(n) = x_a(nT) = 3\cos\frac{100\pi}{100}n = 3\cos\pi n = 3\cos 2\pi\frac{1}{2}n$$

Sampling of Analog Signals

Solution:

b — If $F_s = 200\ Hz$, the discrete-time signal is

$$x(n) = 3\cos\frac{100\pi}{200}n = 3\cos\frac{\pi}{2}n = 3\cos 2\pi\frac{1}{4}n$$

c — If $F_s = 75\ \mathrm{Hz}$, the discrete-time signal is

$$x(n) = 3\cos\frac{100\pi}{75}n = 3\cos\frac{4\pi}{3}n = 3\cos\left(2\pi - \frac{2\pi}{3}\right)n = 3\cos 2\pi\frac{1}{3}n$$

c — For the sampling rate $F_s = 75\ \mathrm{Hz}$,

$F = fF_s = f75$ and $f = \frac{1}{3}$ in part in (c). Hence

$$F = \frac{75}{3} = 25\,\mathrm{Hz}$$

So, the analog sinusoidal signal is

$$\begin{aligned} y_a(t) &= 3\cos 2\pi Ft \\ &= 3\cos 50\pi t \end{aligned}$$

The Sampling Theorem

We must have some information about the analog signal especially the frequency content of the signal, to select the sampling period T or sampling rate F_s.

For example

A speech signal goes below around 20Khz.
A TV signal is up to 5Mhz.

Any analog signal can be represented as sum of sinusoids of different amplitudes, frequencies, and phases.

$$x_a(t) = \sum_{i=1}^{N} A_i \cos(2\pi F_i t + \theta_i)$$

where N the number of frequency components. Suppose that Nth frequency do not exceed the largest frequency $F_{\max}$

$$|F_i| < F_{\max}$$

The Sampling Theorem

To avoid the aliasing problem, is selected so that

$$F_s > 2F_{max}$$

The analog signal should be in the range of

$$-\frac{1}{2} \le f_i = \frac{F_i}{F_s} \le \frac{1}{2}$$

or in radians

$$-\pi \le \omega_i = 2\pi f_i \le \pi$$

The sampling rate $F_N = 2F_{max}$ is called the *Nyquist rate*.

The Sampling Theorem

Example: Consider an analog signal

$$x_a(t) = 3\cos 50\pi t + 10\sin 300\pi t + 3\cos 100\pi t$$

Solution

The frequencies in the analog signal

$F_1 = 25\,\text{Hz}$ $F_2 = 150\,\text{Hz}$ $F_3 = 50\,\text{Hz}$

The largest frequency is

$$F_{max} = F_2 = 150\,\text{Hz}$$

The Nyquist rate is

$$F_N = 2F_{max} = 300\ \text{Hz}$$

The Sampling Theorem

Example: The analog signal

$$x_a(t) = 3\cos 2000\pi t + 5\sin 6000\pi t - 10\cos 12000\pi t$$

1. What is the Nyquist rate for this signal?
2. Using a sampling rate $F_s = 5000$ samples/s . What is the discrete-time signal obtained after sampling?
3. What is the analog signal $y_a(t)$ we can reconstruct from the samples if we use ideal interpolation?

Solution

1. The frequencies of the analog signal are

$F_1 = 1$ KHz $\quad F_2 = 3$ KHz $\quad F_3 = 6$ KHz

The Nyquist rate is $F_N = 2F_{\max} = 12$ KHz

The Sampling Theorem

2. For $F_s = 5\text{KHz}$

$$x(n) = x_a(nT) = x_a\left(\frac{n}{F_a}\right)$$

$$= 3\cos 2\pi\left(\frac{1}{5}\right)n + 5\sin 2\pi\left(\frac{3}{5}\right)n - 10\cos 2\pi\left(\frac{6}{5}\right)n$$

$$= 3\cos 2\pi\left(\frac{1}{5}\right)n + 5\sin 2\pi\left(1-\frac{2}{5}\right)n - 10\cos 2\pi\left(1+\frac{1}{5}\right)n$$

$$= 3\cos 2\pi\left(\frac{1}{5}\right)n - 5\sin 2\pi\left(\frac{2}{5}\right)n - 10\cos 2\pi\left(\frac{1}{5}\right)n$$

$$= -7\cos 2\pi\left(\frac{1}{5}\right)n - 5\sin 2\pi\left(\frac{2}{5}\right)n$$

For $F_s = 5\text{KHz}$, the folding frequency is $F_{\max} = \frac{F_s}{2} = 2.5\text{KHz}$

Hence, $F_1 = 1$ KHz is not effected by aliasing

$F_2 = 3$ KHz is changed by the aliasing effect $\quad F_2' = F_2 - F_s = -2$ KHz

$F_3 = 6\text{KHz}$ is changed by the aliasing effect $\quad F_3' = F_3 - F_s = 1$ KHz

So that normalize frequencies are $f_1 = \frac{1}{5}$ $\quad f_2 = \frac{2}{5}$ $\quad f_3 = \frac{1}{5}$

The Sampling Theorem

Solution (cont)

c. The analog signal that we can recover is

$$y_a(t) = -7\cos 2000\pi t - 5\cos 4000\pi t$$

which is different than the original signal $x_a(t)$

Quantization of Continuous-Amplitude Signals

- Converting a discrete-time continuous-amplitude signal into a digital signal by expressing each sample value as a finite number of digits, is called *quantization*.
- The error between continuous-valued signal and a finite set of discrete value levels signal is called *quantization error* or *quantization noise*.

The output of quantizer is $x_q(n) = Q[x(n)]$

The quantizer error is $e_q(n) = x_q(n) - x(n)$

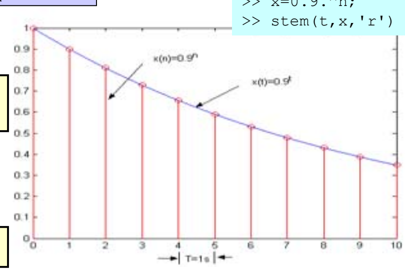
```
>> t=0:0.01:10;
>> x=0.9.^t;
>> plot (t,x)
>> hold on
>> n=0:10;
>> x=0.9.^n;
>> stem(t,x,'r')
```

Example:

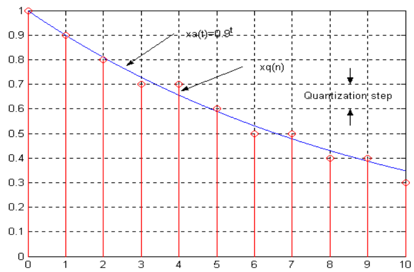
Let's consider the discrete-time signal as

$$x(n) = \begin{cases} 0.9^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

The sampling frequency is $F_s = 1Hz.$



Quantization of Continuous-Amplitude Signals




```
>> t=0:0.01:10;
>> x=0.9.^t;
>> plot (t,x)
>> hold on
>> t=0:10;
>> x=0.9.^t;
>> y=0.1*round(10*x);
>> stem(t,y,'r')
>> grid on
```

n	$x_q(n)$	$x(n)$	$e_q(n)$
0	1.0000	1.0000	0.0000
1	0.9000	0.9000	0.0000
2	0.8000	0.8100	-0.0100
3	0.7000	0.7290	-0.0290
4	0.7000	0.6561	0.0439
5	0.6000	0.5905	0.0095
6	0.5000	0.5314	-0.0314
7	0.5000	0.4783	0.0217
8	0.4000	0.4305	-0.0305
9	0.4000	0.3874	0.0126
10	0.3000	0.3487	-0.0487

Quantization of Continuous-Amplitude Signals

Using rounding process for quantization. The other method is truncation , which discards the excess digits.

- The values allowed in the digital signal are called *quantization level.*
- Distance Δ between two quantization level is called *quantization step size* or *resolution*
- If we use rounding process *the quantization error* is the range of

$$-\frac{\Delta}{2} \le e_q(n) \le \frac{\Delta}{2}$$

- If $x_{\min}$ and $x_{\max}$ represent the minimum and maximum value of $x(n)$ and L is number of *quantization level*, then

$$\Delta = \frac{x_{\max} - x_{\min}}{L-1}$$

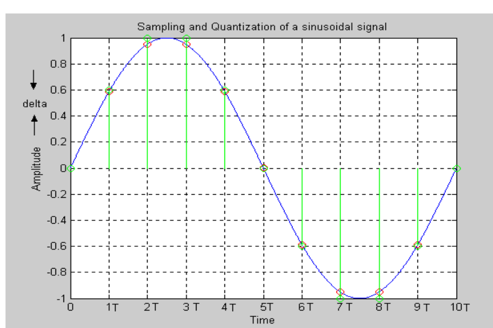
Quantization of Continuous-Amplitude Signals

In the example $x_{\min} = 0$, $x_{\max} = 1$, and, $L = 11$, which leads to $\Delta = 0.1$.

Note:

If L increases, Δ decreases. Hence, the quantization error $e_q(n)$ decreases and the accuracy of the quantizer increases.

Quantization of Sinusoidal Signal

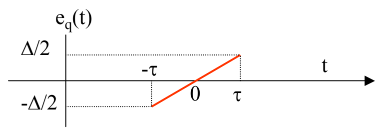


Let's look at the quantizer error by quantizing the analog sinusoidal signal $x_a(t)$.

Quantization of Sinusoidal Signal

The analog signal $x_a(t)$ is almost linear between quantization levels. The quantization error

$$e_q(t) = x_a(t) - x_q(t)$$



Here $e_q(t) = \frac{\Delta}{2\tau} t$ $\quad -\tau \le t \le \tau$

The mean-square error power P_q is

Find discrete time signal $x_1(n)$ and $x_2(n)$

$$P_q = \frac{1}{2\tau}\int_{-\tau}^{\tau} e_q^2(t)dt = \frac{1}{\tau}\int_0^{\tau} e_q^2(t)dt$$

$$P_q = \frac{1}{\tau}\int_0^{\tau}\left(\frac{\Delta}{2\tau}\right)^2 t^2 dt = \frac{1}{\tau}\left(\frac{\Delta}{2\tau}\right)^2 \frac{t^3}{3}\Big|_0^{\tau} = \frac{\Delta^2}{12}$$

Quantization of Sinusoidal Signal

For b bit the all range is 2A, then

$$\Delta = \frac{2A}{2^b}$$

Hence, the mean-square error power P_q for the signal $x_a(t)$ is

$$P_q = \frac{4A^2}{(12)2^{2b}} = \frac{A^2}{(3)2^{2b}}$$

The average power of the signal $x_a(t)$ is

$$P_x = \frac{1}{T_p}\int_0^{T_p}(A\cos\Omega t)^2 dt = \frac{A^2}{2}$$

The ratio of the signal average power to the noise power is *__the signal-quantization noise ratio__* (SQNR) gives

$$SQNR = \frac{P_x}{P_q} = \frac{3}{2}2^{2b}$$

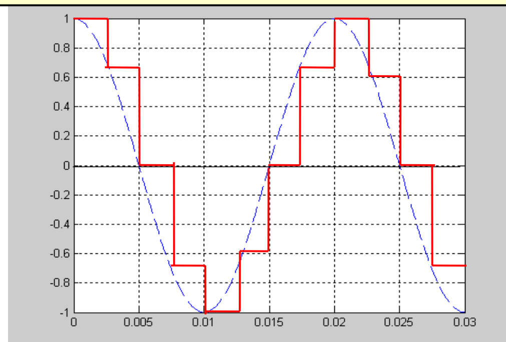
In dB, $SQNR(dB) = 10\log_{10} SQNR = 1.76 + 6.02b$

Digital-to-Analog Conversion

Some cases we may need to convert digital signal to analog signal again.

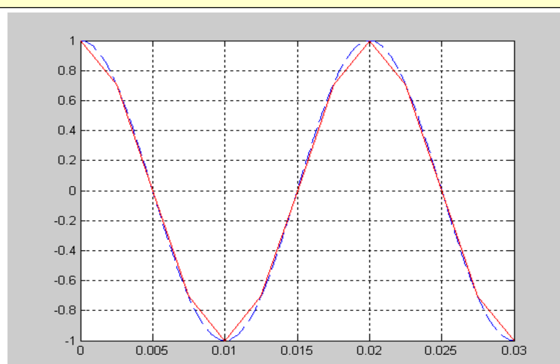
The process of converting a digital signal into an analog signal is called Digital-to-Analog (DAC).

All D/A converters use some kind of interpolation. A simple form of D/A conversion is zero-order hold or staircase approximation. Simply holds constant the value of one sample until the next one is received.



Digital-to-Analog Conversion

A Linear interpolation is connect successive samples with strait-line. It needs T second delay so that has knowledge about next sample values.



Better interpolation can be achieved by using more sophisticated high-order interpolation techniques.

Problem

Problem 1.7

- An analog signal contains frequencies up to 10Khz.

a. What range of sampling frequencies allows exact reconstruction of this signal from the samples?

b. Suppose that we sample this signal with a sampling frequency F_s=8 KHz. Examine what happens to the frequency F_1=5Khz.

c. Repeat part (b) for a frequency F_2=9Khz.

Solution 1.7

a $F_{\max} = 10Khz.$ $F_s \geq 2F_{\max} = 20Khz.$

b $F_s = 8Khz.$ $F_{fold} = \frac{F_s}{2} = 4Khz.$

So, $F = 5Khz$ will be alias of 3KHz

c $F = 9Khz$ will be alias of 1KHz.

Problem

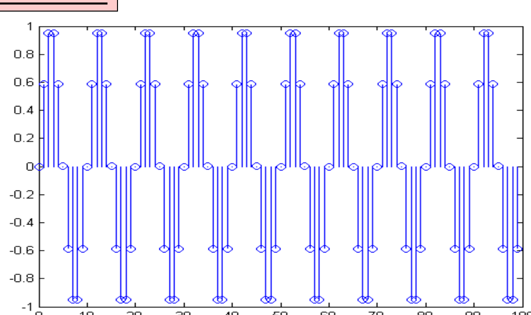
Problem 1.15

$x_a(t) = \sin 2\pi F_0 t$ $-\infty < t < \infty$

and $x(n) = x(nT) = \sin 2\pi \frac{F_0}{F_s} n$

$F_s = 5\,\text{Khz.}$ and $F_0 = 0.5\,\text{Khz.}$ $0 \leq n \leq 99$

Solution 1.15




```
>> n=0:99;
>> x=sin(2*pi*0.1*n);
>> stem (n,x)
```