ECE 308 -13

Frequency Analysis of Signals and Systems

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Frequency Analysis of Signals and Systems

The Fourier Series for Discrete-Time Periodic Signals

The Fourier representation of signal maps the signal into frequency domain.

The Fourier transform provides a different way to interpret signals and systems.

It is useful for convolution operation in time domain maps into multiplication in frequency domain

It gives us information about system or signal characteristic of behavior

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The Fourier Series for Discrete-Time Periodic Signals

A given periodic sequence *x(n)* with period *N*, that is *x(n)=x(n+N)*

The Fourier representation of *x(n)* can be expressed as

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

where c_k ara the coefficients in the series representation. This equation is called the *Discrete-time Fourier Series* (DTFS)

The Fourier coefficients $\{c_k\},\ k = 1,2,\ldots,N-1$ provides the description of x(n) in the frequency domain

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

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c_k represents the amplitude and phase associated with the frequency component

$$s_k = e^{j2\pi kn/N} = e^{j\omega_k n}$$

where $\omega_k = 2\pi k / N$

The function S_k are periodic with period *N*. Hence

$$s_k(n) = s_k(n+N)$$

So, that $c_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j2\pi(k+N)n/N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} = c_k$

Therefore, c_k is periodic sequence with fundamental period N.

Instead of focusing in periodic range *k=0,1,…, N-1* in frequency domain $0 < \omega_k = \frac{2\pi k}{N} < 2\pi$ for *0<k<N-1*, We do in range of $-\pi < \omega_k = \frac{2\pi k}{N} < \pi$, which corresponds to $-\frac{N}{2} < k < \frac{N}{2}$

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Example: Determine the spectra of the following signal

$$x(n)=\cos\frac{\pi n}{3} \qquad \omega_0=\frac{\pi}{3} \qquad f_0=\frac{1}{6}$$

Hence x(n) is periodic with fundamental period N=6

$$c_k=\frac{1}{N}\sum_{n=0}^{N-1}x(n)e^{-j2\pi kn/N}=\frac{1}{6}\sum_{n=0}^{5}x(n)e^{-j2\pi kn/6} \quad ,k=0,1,\ldots,5$$

$$x(n)=\cos\frac{\pi n}{3}=\frac{1}{2}e^{j2\pi n/6}+\frac{1}{2}e^{-j2\pi n/6}$$

$$e^{j2\pi n/6}=e^{j2\pi(5-6)n/6}=e^{j2\pi 5n/6}$$ which means that $c_{-1}=c_5$

$$c_k=\frac{1}{6}\sum_{n=0}^{5}\cos(\frac{n\pi}{3})e^{-j2\pi kn/6} \quad ,k=0,1,\ldots,5$$

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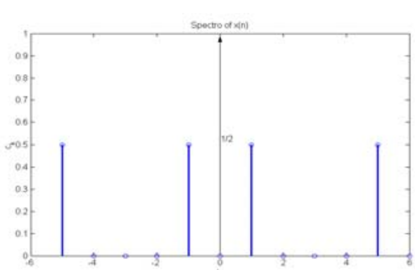
Example:(cont)

$$c_k=\frac{1}{6}(1+\cos\frac{\pi}{3}e^{-j2\pi k/6}+\cos\frac{2\pi}{3}e^{-j2\pi k2/6}+\cos\frac{3\pi}{3}e^{-j2\pi k3/6}$$
$$+\cos\frac{4\pi}{3}e^{-j2\pi k4/6}+\cos\frac{5\pi}{3}e^{-j2\pi k5/6}) \quad , \quad k=0,1,\ldots,5$$

$c_0=0$ $\quad c_1=\frac{1}{2}$ $\quad c_2=0$ $\quad c_3=0$ $\quad c_4=0$ $\quad c_5=\frac{1}{2}$




```
% Find the Fourier Series coefficients of x(n)=cos(pi n/3)

for k=1:6
    c(k)=0;
    for n=1:6
        c(k)=c(k)+cos(pi*(n-1)/3)*exp(-j*2*pi*(k-1)*(n-1)/6);
    end
    c(k)=c(k)/6;
end

for k=1:6;
    rats(c(k))
end
.
```

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Example: Determine the spectra of the following periodic signal with period N=4

$$x(n) = \{1,1,0,0\}$$

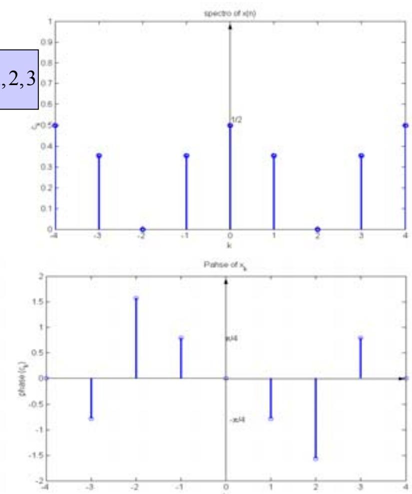
$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} = \frac{1}{4}\sum_{n=0}^{3} x(n)e^{-j2\pi kn/4} \quad ,k = 0,1,2,3$$

$$c_k = \frac{1}{4}(1+e^{-j2\pi k/4}) \quad , \quad k = 0,1,2,3$$

$$c_0 = \frac{1}{2} \quad c_1 = \frac{1}{4}(1-j) \quad c_2 = 0 \quad c_3 = \frac{1}{4}(1+j)$$

$$c_0 = \frac{1}{2} \quad |c_1| = \frac{\sqrt{2}}{4} \quad \measuredangle c_1 = -\frac{\pi}{4} \quad c_2 = 0$$

$$|c_3| = \frac{\sqrt{2}}{4} \quad \measuredangle c_3 = \frac{\pi}{4}$$



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Fourier Transform for DT

The Fourier transform of a finite-energy discrete time signal x(n) is defined as

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$X(\omega)$ signal is periodic with period 2π

$$X(\omega+2\pi k) = \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega+2\pi k)n}$$

$$X(\omega+2\pi k) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}e^{-j2\pi kn}$$

$$X(\omega+2\pi k) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = X(\omega)$$

The inverse Fourier transform of discrete-time signal is

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

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Properties of the Fourier Transform for DT

Properties	Time Domain	Frequency Domain
Notation	$x(n),\ x_1(n),\ x_2(n)$	$X(\omega),\ X_1(\omega),\ X_2(\omega)$
Linearity	$ax_1(n)+b\,x_2(n)$	$aX_1(\omega)+bX_2(\omega)$
Time Shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time Reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation	$r_{x_1x_2}(l)=x_1(l) * x_2(-l)$	$S_{x_1x_2}=X_1(\omega)X_2(-\omega)=X_1(\omega)X_2^*(\omega)$
Frequency Shifting	$e^{j\omega_0 n}x(n)$	$X(\omega-\omega_0)$
Modulation	$x_1(n)\cos\omega_0 n$	$\frac{1}{2}X(\omega+\omega_0)+\frac{1}{2}X(\omega-\omega_0)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi}X_1(\lambda)X_2(\omega-\lambda)d\lambda$
Differentiation in Frequency domain	$x^*(n)$	$j\frac{dX(\omega)}{d\omega}$
Conjugation	$nx(n)$	$X^*(-\omega)$
Parseval's Theorem	$\sum_{n=-\infty}^{\infty}x_1(n)x_2^*(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X_1(\omega)X_2^*(\omega)d\omega$	

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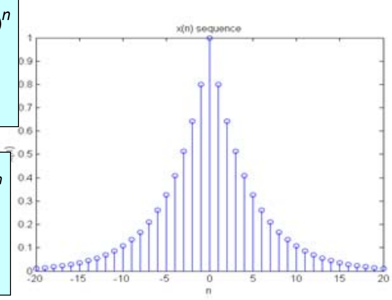
Example: $x(n)=0.8^{|n|}$ $\quad -\infty<n<\infty$ $\quad x(n)=x_1(n)+x_2(n)$

$$x_1(n)=\begin{cases}a^n & n\ge 0\\ 0 & n<0\end{cases}\qquad x_2(n)=\begin{cases}a^{-n} & n<0\\ 0 & n\ge 0\end{cases}$$

$$X_1(\omega)=\sum_{n=-\infty}^{\infty}x(n)e^{-j\omega n}=\sum_{n=0}^{\infty}0.8^n e^{-j\omega n}=\sum_{n=0}^{\infty}(0.8e^{-j\omega})^n$$
$$=\frac{1}{1-0.8e^{-j\omega}}=\frac{e^{j\omega}}{e^{j\omega}-0.8}$$

$$X_2(\omega)=\sum_{n=-\infty}^{\infty}x(n)e^{-j\omega n}=\sum_{n=-\infty}^{-1}0.8^{-n}e^{-j\omega n}=\sum_{n=-\infty}^{-1}(0.8e^{j\omega})^{-n}$$
$$=\sum_{n=1}^{\infty}(0.8e^{j\omega})^n=\frac{0.8e^{j\omega}}{1-0.8e^{j\omega}}$$



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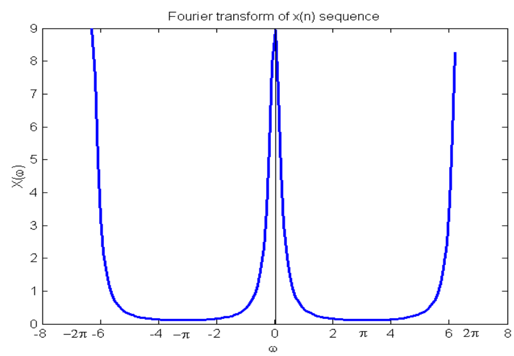
Example: (cont)

$$X(\omega) = X_1(\omega) + X_2(\omega) = \frac{1}{1-0.8e^{-j\omega}} + \frac{0.8e^{j\omega}}{1-0.8e^{j\omega}} = \frac{1-0.8^2}{1-2(0.8)\cos\omega + 0.8^2}$$




```
n=-20:20;
x=0.8.^abs(n);
stem (n,x)
xlabel ('n')
ylabel ('x(n)')
title ('x(n) sequence')

figure;
w=-2*pi:0.1:2*pi;
y=(1-0.8^2)./(1-2*0.8*cos(w)+0.8^2);
plot (w,y);
xlabel ('\omega')
ylabel ('y(\omega)')
title ('Fourier transform of x(n)
sequence')
```

(Fourier1.m)

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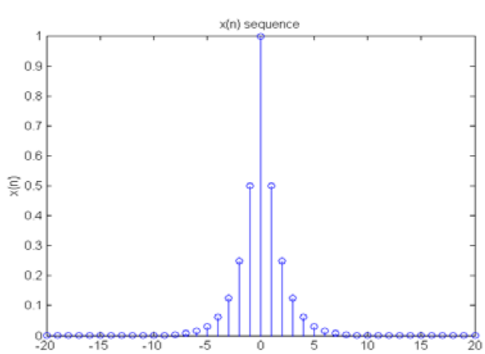
Example: $x(n) = 0.5^{|n|}$ $\quad -\infty < n < \infty$




```
n=-20:20;
b=0.5
x=b.^abs(n);
stem (n,x)
xlabel ('n')
ylabel ('x(n)')
title ('x(n) sequence')

figure;
w=-2*pi:0.1:2*pi;
y=(1-b^2)./(1-2*b*cos(w)+b^2);
plot (w,y);
xlabel ('\omega')
ylabel ('y(\omega)')
title ('Fourier transform of x(n) sequence')
```

(Fourier2.m)

$$X(\omega) = \frac{1-0.5^2}{1-\cos\omega + 0.5^2}$$

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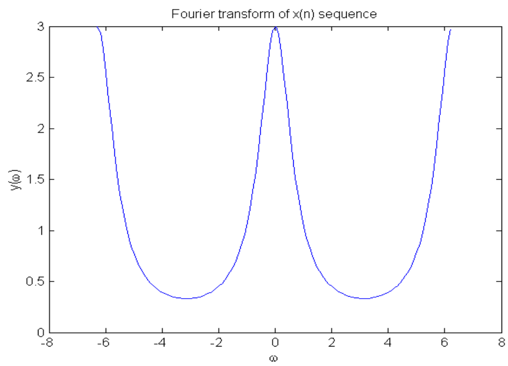
Example: (cont)

$$X(\omega)=\frac{1-0.5^2}{1-\cos\omega+0.5^2}$$



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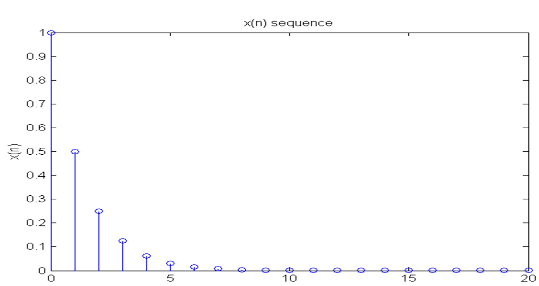
Example: $x(n)=0.5^n$ $\quad 0<n<\infty$

$$X(\omega)=\sum_{n=-\infty}^{\infty}x(n)e^{-j\omega n}=\sum_{n=0}^{\infty}0.5^n e^{-j\omega n}=\sum_{n=0}^{\infty}(0.5e^{-j\omega})^n$$
$$=\frac{1}{1-0.5e^{-j\omega}}=\frac{e^{j\omega}}{e^{j\omega}-0.5}$$



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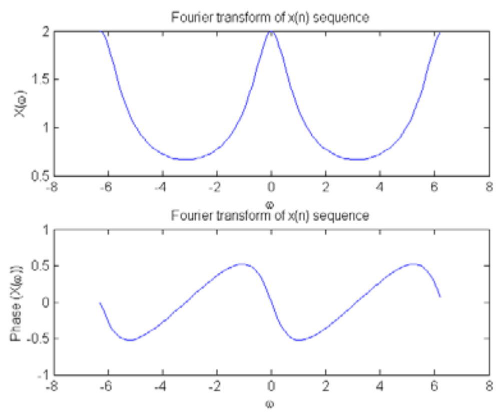
Example: (cont)




```
n=-0:20;
b=0.5
x=b.^abs(n);
stem (n,x)
xlabel ('n')
ylabel ('x(n)')
title ('x(n) sequence')

figure;
w=-2*pi:0.1:2*pi;
X=exp(j*w)./(exp(j*w)-b);
subplot (2,1,1)
plot (w,abs(X));
xlabel ('\omega')
ylabel ('X(\omega)')
title ('Fourier transform of x(n) sequence')
subplot (2,1,2);
plot (w,phase(X));
xlabel ('\omega')
ylabel ('Phase (X(\omega))')
title ('Fourier transform of x(n) sequence')
```

Fourier3.m

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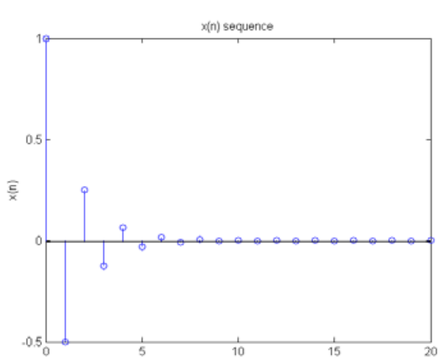
Example:

$x(n) = (-0.5)^n u(n)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=0}^{\infty}(-0.5)^n e^{-j\omega n} = \sum_{n=0}^{\infty}(-0.5e^{-j\omega})^n$$
$$= \frac{1}{1+0.5e^{-j\omega}} = \frac{e^{j\omega}}{e^{j\omega}+0.5}$$




```
n=0:20;
b=-0.5
x=b.^n;
stem (n,x)
xlabel ('n')
ylabel ('x(n)')
title ('x(n) sequence')

figure;
w=-2*pi:0.1:2*pi;
X=exp(j*w)./(exp(j*w)-b);
subplot (2,1,1)
wpi=w/pi;
plot (wpi,abs(X));
xlabel ('\omega/\pi')
ylabel ('X(\omega)')
title ('Fourier transform of x(n)
sequence')
subplot (2,1,2);
plot (wpi,phase(X));
xlabel ('\omega/\pi')
ylabel ('Phase (X(\omega))')
title ('Fourier transform of x(n)
sequence')
```

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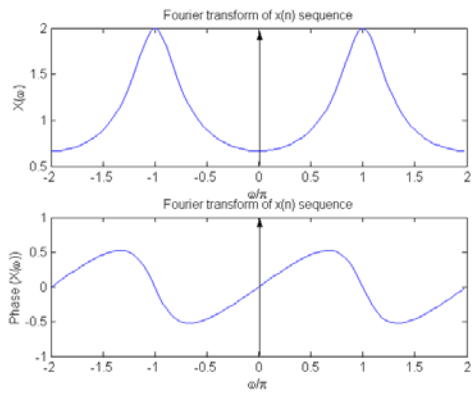
Example: (cont)



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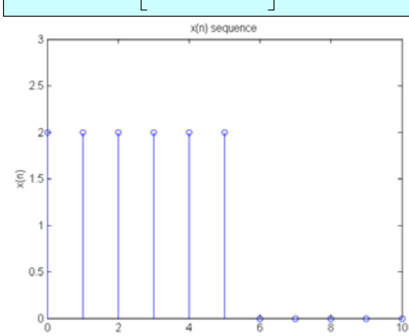
Example:

$$x(n) = \begin{cases} 2 & 0 \le n \le 5 \\ 0 & \text{otherwise} \end{cases}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=0}^{5} 2e^{-j\omega n} = 2\frac{1-e^{-j6\omega}}{1-e^{-j\omega}}$$

$$= 2\frac{e^{-j3\omega}\left[e^{j3\omega} - e^{-j3\omega}\right]}{e^{-j\omega/2}\left[e^{j\omega/2} - e^{-j\omega/2}\right]} = 2e^{-j\frac{5}{2}\omega}\frac{\sin 3\omega}{\sin \omega/2}$$




```
n=0:10;
x=2*(n>=0 & n<=5);
stem (n,x)
xlabel ('n')
ylabel ('x(n)')
title ('x(n) sequence')

figure;
w=-2*pi:0.01:2*pi;
X=2*(1-exp(-j*6*w))./(1-exp(-j*w));
subplot (2,1,1)
wpi=w/pi;
plot (wpi,abs(X));
xlabel ('\omega/\pi')
ylabel ('X(\omega)')
title ('Fourier transform of x(n)
sequence')
subplot (2,1,2);
plot (wpi,phase(X));
xlabel ('\omega/\pi')
ylabel ('Phase (X(\omega))')
title ('Fourier transform of x(n)
sequence')
```

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Example: (cont)