ECE 308 -12

The One-Side z-Transform

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The One-Side z-Transform

The one-sided z-transform of a signal x(n) is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$x(n) \overset{z^+}{\leftrightarrow} X^+(z)$$

The one-sided z-transform has the following characteristics:

1. It does not contain information about the signal x(n) for negative values of time (i.e., for n<0)
2. It is unique only for causal signals, because only these signals are zero for n<0.
3. The one-sided z-transform $X^+(z)$ of x(n) is identical to the two-sided z-transform of the signal x(n)u(n).
4. ROC of $X^+(z)$ is always the exterior of the circle. So it is not necessary to refer to their ROC

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Example:

$$x_1(n)=\{\underset{\uparrow}{1},2,5,7,0,1\}\overset{x^+}{\longleftrightarrow}X_1^+(z)=1+2z^{-1}+5z^{-2}+7z^{-3}+z^{-5}$$

$$x_2(n)=\{1,2,\underset{\uparrow}{5},7,0,1\}\overset{z^+}{\longleftrightarrow}X_2^+(z)=5+7z^{-1}+z^{-3}$$

$$x_3(n)=\{\underset{\uparrow}{0},0,1,2,5,7,0,1\}\overset{x^+}{\longleftrightarrow}X_3^+(z)=1z^{-2}+2z^{-3}+5z^{-4}+7z^{-5}+z^{-7}$$

$$x_4(n)=\delta(n)\overset{x^+}{\longleftrightarrow}X_4^+(z)=1$$

$$x_5(n)=\delta(n-k)\overset{x^+}{\longleftrightarrow}X_5^+(z)=z^{-k}$$

$$x_6(n)=\delta(n+k)\overset{x^+}{\longleftrightarrow}X_6^+(z)=0$$

$$x_4(n)=a^n u(n)\overset{x^+}{\longleftrightarrow}X_4^+(z)=\frac{1}{1-az^{-1}}$$

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Shifting Property

Case 1:Time Delay

if $x(n)\overset{x^+}{\longleftrightarrow}X^+(z)$

Then $x(n-k)\overset{x^+}{\longleftrightarrow}z^{-k}\left[X^+(z)+\sum_{n=1}^{k}x(-n)z^{n}\right]\quad k>0$

In case x(n) is causal, then $x(n-k)\overset{x^+}{\longleftrightarrow}z^{-k}X^+(z)\quad k>0$

Example: $x(n)=a^n u(n)\overset{x^+}{\longleftrightarrow}X^+(z)=\dfrac{1}{1-az^{-1}}$

The z-transform of $x_1(n)=x(n-2)$

$$x(n-2)=a^{n-2}u(n-2)\overset{x^+}{\longleftrightarrow}z^{-2}\left[\frac{1}{1-az^{-1}}+x(-1)z+x(-2)z^{2}\right]$$

$$=\frac{z^{-2}}{1-az^{-1}}+x(-1)z^{-1}+x(-2)$$

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Case 2: Time advance

if $x(n) \overset{x^+}{\leftrightarrow} X^+(z)$

Then $x(n+k) \overset{x^+}{\leftrightarrow} z^k \left[X^+(z) - \sum_{n=0}^{k-1} x(n) z^{-n} \right] \quad k > 0$

Example: $x(n) = a^n u(n) \ \overset{x^+}{\leftrightarrow} X^+(z) = \frac{1}{1 - az^{-1}}$

The z-transform of $x_1(n) = x(n+2)$

$$x(n+2) = a^{n+2} u(n+2) \ \overset{x^+}{\leftrightarrow} z^2 \left[\frac{1}{1 - az^{-1}} + x(0) + x(1) z^{-1} \right]$$

$$= \frac{z^2}{1 - az^{-1}} + x(0) z^2 + x(1) z$$

$$= \frac{z^2}{1 - az^{-1}} + z^2 + az$$

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Final Value Theorem:

if $x(n) \overset{x^+}{\leftrightarrow} X^+(z)$

Then $\lim_{n \to \infty} x(n) \ \overset{x^+}{\leftrightarrow} \lim_{z \to 1} (z-1) X^+(z)$

This is exits if the ROC of $(z-1)X^+(z)$ includes the unit circle.

Example: $x(n) = u(n)$ $h(n) = \alpha^n u(n)$ $y(n) = h(n) * x(n)$

$Y(z) = H(z)X(z) = \frac{1}{1 - \alpha z^{-1}} \frac{1}{1 - z^{-1}} = \frac{z^2}{(z-\alpha)(z-1)}$ ROC $|z| > |\alpha|$

$(z-1)Y(z) = (z-1) \frac{z^2}{(z-\alpha)(z-1)} = \frac{z^2}{(z-\alpha)}$ ROC $|z| > |\alpha|$

$$\lim_{n \to \infty} x(n) \ \overset{x^+}{\leftrightarrow} \lim_{z \to 1} \frac{z^2}{z - \alpha} = \frac{1}{1 - \alpha}$$

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Solution of Difference Equation:

The one-sided z-transform is a very efficient tool for the solution of difference equation.

Example: $y(n) = y(n-1) + y(n-2)$

The initial condition $y(-1) = 0$ $y(-2) = 1$

$$Y^+(z) = z^{-1}\left[Y^+(Z) + y(-1)z\right] + z^{-2}\left[Y^+(z) + y(-1)z + y(-2)z^2\right]$$

$$Y(z) = z^{-1}Y^+(Z) + y(-1) + z^{-2}Y^+(z) + z^{-1}y(-1) + y(-2)$$

$$Y(z)(1 - z^{-1} - z^{-2}) = 1$$

$$Y(z) = \frac{1}{1 - z^{-1} - z^{-2}} = \frac{z^2}{z^2 - z - 1}$$

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Example:(cont)

$$Y(z) = \frac{1}{\left(1 - 1.618z^{-1}\right)\left(1 + 0.618z^{-1}\right)} = \frac{A_1}{\left(1 - 1.618z^{-1}\right)} + \frac{A_2}{\left(1 - 1.618z^{-1}\right)}$$

$$A_1 = \left(1 - 1.618z^{-1}\right)Y(z)\Big|_{z^{-1} = \frac{1}{1.618}} = \frac{1}{\left(1 + .618z^{-1}\right)}\Big|_{z^{-1} = \frac{1}{1.618}}$$

$$= \frac{1}{\left(1 + .618\frac{1}{1.618}\right)} = 0.7236$$

$$A_2 = \left(1 + 0.618z^{-1}\right)Y(z)\Big|_{z^{-1} = -\frac{1}{0.618}} = \frac{1}{\left(1 - 1.618z^{-1}\right)}\Big|_{z^{-1} = -\frac{1}{0.618}}$$

$$= \frac{1}{\left(1 - 1.618(-\frac{1}{0.618})\right)} = 0.2764$$

Finally

$$y(n) = \left[0.7236\left(1.618\right)^n - 0.2764\left(0.618\right)^n\right]u(n)$$

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Example: Find the step response of the system

$y(n) = \alpha y(n-1) + x(n)$ $\quad -1 < \alpha < 1$

The initial condition $y(-1) = 1$

Taking one-sided z-transform

$$Y^+(z) = \alpha\left[z^{-1}Y^+(Z) + y(-1)\right] + X^+(z)$$

$$Y^+(z) = \alpha z^{-1}Y^+(Z) + \alpha y(-1) + \frac{1}{1-z^{-1}}$$

$$Y^+(z) = \frac{\alpha}{1-\alpha z^{-1}} + \frac{1}{(1-\alpha z^{-1})(1-z^{-1})}$$

The inverse transform

$$y(n) = \alpha^{n+1}u(n) + \frac{1-\alpha^{n+1}}{1-\alpha}u(n)$$
$$= \frac{1}{1-\alpha}(1-\alpha^{n+2})u(n)$$

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Analysis of LTI Systems in the z-Domain

Response of System with Rational System Function

Let us assume that the input signal x(n) and the corresponding system function h(n) have rational z-transform X(z) and H(z) of the form

$X(z) = \frac{N(z)}{D(z)}$ and $H(z) = \frac{B(z)}{A(z)}$

If the system initially relaxed (initial conditions for difference equation are zero), the z-transform of system output has

$$Y(z) = H(z)X(z) = \frac{B(z)N(z)}{A(z)D(z)}$$

Partial fraction expansion of Y(z) will be in the following form if no pole-zero cancellation

$$Y(z) = \sum_{k=1}^{N}\frac{A_k}{1-p_x z^{-1}} + \sum_{k=1}^{L}\frac{Q_k}{1-q_x z^{-1}}$$

where $p_1, p_2, \ldots p_N$ system poles, $q_1, q_2, \ldots q_L$ the input signal poles, and $p_k \neq q_m$

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The inverse transform of Y(z) yields

$$y(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n) + \sum_{k=1}^{L} Q_k (q_k)^n u(n)$$

where A_k and Q_k are functions of both sets of poles p_k and q_k.

We can separate the y(n) into two parts

$$y_{nr}(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n)$$ ⇨ natural response

Natural response is different than zero-input response. If X(z) is zero, then Y(z) is zero

$$y_{fr}(n) = \sum_{k=1}^{L} Q_k (q_k)^n u(n)$$ ⇨ forced response of the system

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Response of Pole-zero System with Nonzero Initial Condition.

The input signal x(n) is assumed to be causal. The effect of all previous input signals to the system are reflected in the initial conditions y(-1), y(-2), …..y(-N). We will look at the one-sided z-transform

$$Y^+(z) = -\sum_{k=1}^{N} a_k z^{-k} \left[Y^+(z) + \sum_{k=1}^{k} y(-n) z^n \right] + \sum_{k=1}^{M} b_k z^{-k} X^+(z)$$

Since x(n) is causal, we can set $X^+(z) = X(z)$

$$Y^+(z) = \frac{\sum_{k=1}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} X(z) - \frac{\sum_{k=1}^{N} a_k z^{-k} \sum_{k=1}^{k} y(-n) z^n}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$Y^+(z) = H(z)X(z) - \frac{N_0(z)}{A(z)}$$ $$N_0(z) = \sum_{k=1}^{N} a_k z^{-k} \sum_{k=1}^{k} y(-n) z^n$$

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The output of the system can subdivide into two parts.

$Y_{zs}(z) = H(z)X(z)$ The zero-state response

$Y_{zi}^{+}(z) = \frac{N_0(z)}{A(z)}$ the zero-input response

Therefore, the total of the inverse z-transform of responses gives us y(n)

$$y(n) = y_{zs}(n) + y_{zi}(n)$$

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Example:

Determine the unit step response of the system of the following equation.

$$y(n) = 0.9y(n-1) - 0.81y(n-2) + x(n)$$

Condition: a. $y(-1) = y(-2) = 0$

b. $y(-1) = y(-2) = 1$

Solution:

$$Y(z) = 0.9\left[z^{-1}Y(z) + y(-1)\right] - 0.81\left[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)\right] + X(z)$$

$$Y(z)(1 - 0.9z^{-1} + 0.81z^{-2}) = 0.9y(-1) - 0.81z^{-1}y(-1) + 0.81y(-2) + X(z)$$

$$Y(z) = \frac{1}{(1 - 0.9z^{-1} + 0.81z^{-2})}X(z)$$

The system function is

$$H(z) = \frac{1}{(1 - 0.9z^{-1} + 0.81z^{-2})}$$

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Example: (cont)

The system poles are

$p_1 = 0.9e^{j\frac{\pi}{3}}$ and $p_2 = 0.9e^{-j\frac{\pi}{3}}$

The z-transform of input sequence is

$$X(z) = \frac{1}{(1-z^{-1})}$$

Therefore,

$$Y(z) = \frac{1}{(1-0.9e^{j\frac{\pi}{3}}z^{-1})(1-0.9e^{j\frac{\pi}{3}}z^{-1})(1-z^{-1})}$$

$$Y(z) = \frac{0.544e^{-j95.2}}{(1-0.9e^{j\frac{\pi}{3}}z^{-1})} + \frac{0.544e^{-j95.2}}{(1-0.9e^{j\frac{\pi}{3}}z^{-1})} + \frac{1.099}{(1-z^{-1})}$$

The zero-state response is

$$y_{zs}(n) = \left[1.099 + 1.088(0.9)^n \cos(\frac{\pi}{3}n - 95.2^\circ)\right]u(n)$$

Since the initial condition are zero $y(n) = y_{zs}(n)$

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Example(cont)

Using initial condition

$$Y(z)(1-0.9z^{-1}+0.81z^{-2}) = 0.9 - 0.81z^{-1} + 0.81 + X(z)$$

$$Y(z) = \frac{1}{(1-0.9z^{-1}+0.81z^{-2})}X(z) - \frac{0.81-0.81z^{-1}}{(1-0.9z^{-1}+0.81z^{-2})}$$

$$Y_{zi}(z) = \frac{N_0(z)}{A(z)} = \frac{0.81-0.81z^{-1}}{(1-0.9z^{-1}+0.81z^{-2})}$$

$$Y_{zi}(z) = \frac{0.4956e^{j84.8}}{1-0.9e^{j\frac{\pi}{3}}z^{-1}} + \frac{0.4956e^{-j84.8}}{1-0.9e^{j\frac{\pi}{3}}z^{-1}}$$

The zero input response

$$y_{zi}(n) = \left[0.99(0.9)^n \cos(\frac{\pi}{3}n + 84.8^\circ)\right]u(n)$$

The total response $Y(z) = Y_{zs}(z) + Y_{zi}(z)$

$$y_{zs}(n) = \left[1.099 + 1.44(0.9)^n \cos(\frac{\pi}{3}n + 38^\circ)\right]u(n)$$

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Transient and Steady-State Response

The response of a system to a given input can be separated into two components

Natural response $y_{nr}(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n)$

where $p_k \;\; i = 1, 2, ..., k$, are the poles of the system and A_k are scale factor.

If $|p_k| < 1$ for all *k*, $y_{nr}(n)$ goes zero for n approaches infinity. In this case, natural response of the system accept as *transient response*.

The forced response: $y_{fr}(n) = \sum_{k=1}^{L} Q_k (q_k)^n u(n)$

where q_k, $i = 1, 2, ..., k$ are the poles in the forcing function and are Q_k scale factor.

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If $|q_k| < 1$ for all k, $y_{fr}(n)$ goes zero for *n* approaches infinity
If the causal input signal is sinusoidal, then the forced signal will be sinusoidal for all $n \geq 0$. In the case, the forced response is called the steady-state response

Example: Find the transient and steady-state response of

$y(n) = 0.5y(n-1) + x(n)$

when the input signal is $x(n) = 10\cos(\frac{\pi n}{4})u(n)$

The system is initially relaxed.

Solution:

$Y(z) = 0.5z^{-1}Y(z) + X(z)$

$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.5z^{-1}}$

The poles at $z = 0.5$

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Example: (cont)

The z-transform of input signal

$$X(z)=10\frac{1-z^{-1}\cos\frac{\pi}{4}}{1-2z^{-1}\cos\frac{\pi}{4}+z^{-2}}=\frac{10(1-\frac{1}{\sqrt{2}}z^{-1})}{1-\sqrt{2}z^{-1}+z^{-2}}$$

$$X(z)=\frac{10(1-\frac{1}{\sqrt{2}}z^{-1})}{(1-e^{j\pi/4}z^{-1})(1-e^{-j\pi/4}z^{-1})}$$

$$Y(z)=H(z)X(z)=\frac{10(1-\frac{1}{\sqrt{2}}z^{-1})}{(1-0.5z^{-1})(1-e^{j\pi/4}z^{-1})(1-e^{-j\pi/4}z^{-1})}$$

$$Y(z)=\frac{A_1}{(1-0.5z^{-1})}+\frac{A_2}{(1-e^{j\pi/4}z^{-1})}+\frac{A_3}{(1-e^{-j\pi/4}z^{-1})}$$

$$A_1=\frac{10(1-\frac{1}{\sqrt{2}}z^{-1})}{(1-\sqrt{2}z^{-1}+z^{-2})}\Big|_{z^{-1}=2}=\frac{10(1-\frac{1}{\sqrt{2}}2)}{(1-\sqrt{2}2+2^{-2})}=-1.907$$

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Example (cont)

$$A_2=\frac{10(1-\frac{1}{\sqrt{2}}z^{-1})}{(1-0.5z^{-1})(1-e^{-j\pi/4}z^{-1})}\Big|_{z^{-1}=e^{-j\pi/4}}=6.78e^{-j28.7}$$

$$A_3=\frac{10(1-\frac{1}{\sqrt{2}}z^{-1})}{(1-0.5z^{-1})(1-e^{j\pi/4}z^{-1})}\Big|_{z^{-1}=e^{j\pi/4}}=6.78e^{j28.7}$$

$$Y(z)=\frac{-1.907}{(1-0.5z^{-1})}+\frac{6.78e^{-j28.7^\circ}}{(1-e^{j\pi/4}z^{-1})}+\frac{6.78e^{j28.7^\circ}}{(1-e^{-j\pi/4}z^{-1})}$$

The natural or transient response is

$$y_{nr}(n)=-1.907(0.5)^n u(n)$$

And the forced or steady-state response is

$$y_{fr}(n)=[6.78e^{-j28.7^\circ}(e^{j\pi n/4})+6.78e^{j28.7^\circ}(e^{-j\pi n/4})]u(n)$$

$$y_{fr}(n)=13.56\cos(\frac{\pi}{4}n-28.7^\circ)u(n)$$