Z Transform

Rational Z-Transform

The inverse of the z-transform

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Rational Z-Transform

Poles and Zeros

The *poles* of a z-transform are the values of z for which if X(z)=∞
The *zeros* of a z-transform are the values of z for which if X(z)=0

X(z) is in rational function form

$$X(z)=\frac{N(z)}{D(z)}=\frac{b_0+b_1z^{-1}+\ldots+b_Mz^{-M}}{a_0+a_1z^{-1}+\ldots+a_Nz^{-N}}=\frac{\sum_{k=0}^{M}b_kz^{-k}}{\sum_{k=0}^{M}a_kz^{-k}}$$

$$X(z)=\frac{N(z)}{D(z)}=\frac{b_0}{a_0}z^{-M+N}\frac{(z-z_1)(z-z_2)\ldots(z-z_M)}{(z-p_1)(z-p_2)\ldots(z-p_N)}$$

$$X(z)=\frac{N(z)}{D(z)}=Gz^{N-M}\frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)}$$

M finite zeros at $z=z_1,z_2,\ldots,z_M$

N finite poles at $z=p_1,p_2,\ldots,p_N$

And |N-M| zeros if N>M or poles if M>N at the origin z=0

Rational Z-Transform

Poles and Zeros

We can represent X(z) graphically by a pole-zero plot in complex plane.
Shows the location of poles by (x)
Shows the location of zeros by (o).
Definition of ROC of a z-transform should not contain any poles.

Example: Determine the pole-zero plot for the signal

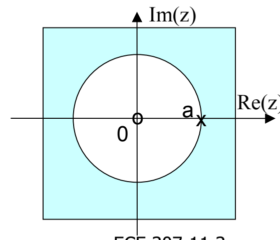
$$x(n) = a^n u(n)$$

The z-transform is

$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

One zero at $z_1=0$

One pole at $p_1=a$.
$p_1=a$ is not included in the ROC



Rational Z-Transform

Example.2: Determine the pole-zero plot for the signal

$$x(n) = \begin{cases} a^n & 0 \le n \le 7 \\ 0 & \text{elsewhere} \end{cases}$$

$a > 0$

The z-transform is

$$X(z) = \sum_{n=0}^{7} \left(az^{-1}\right)^n = \frac{1-\left(az^{-1}\right)^8}{1-az^{-1}} = \frac{z^8-a^8}{z^7(z-a)}$$

$$X(z) = \frac{(z-z_1)(z-z_2)\dots(z-z_7)}{z^7}$$

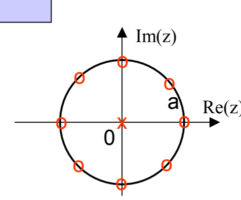
The zeros $z_k = ae^{j2\pi k/M}$

$z_1 = ae^{j2\pi/8}$ $z_2 = ae^{j4\pi/8}$

$z_7 = ae^{j14\pi/8}$

the poles at p=0

ROC: the entire z-plane except z=0



Rational Z-Transform

Pole Location and Time-Domain behavior for Causal Signals

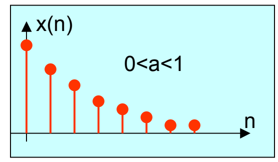
For the signal $x(n) = a^n u(n)$ $a > 0$

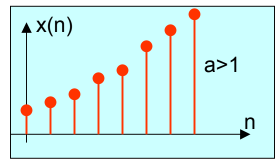
The z-transform is $X(z) = \dfrac{1}{1 - az^{-1}} = \dfrac{z}{z - a}$

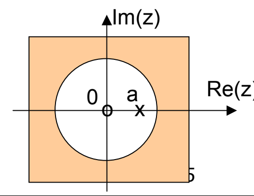
One zero at $z_1=0$. One pole at $p_1=a$.

The signal is decaying 0<a<1
The signal is fixed if a=1
The signal is growing if a>1
The signal alternates if a is negative

Causal signals with poles outside the unit circle become unbounded.



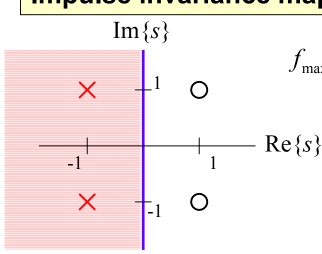




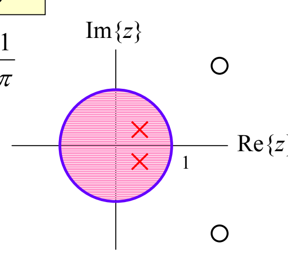
Rational Z-Transform

Pole Location and Time-Domain behavior for Causal Signals

Impulse invariance mapping is $z = e^{sT}$



$$f_{\max} = \frac{1}{2\pi} \Rightarrow f_s > \frac{1}{\pi}$$



$s = j2\pi f$

$s = -1 \pm j \quad \Rightarrow \quad z = 0.198 \pm j\,0.31 \ (T = 1\ \mathrm{s})$

$s = 1 \pm j \quad \Rightarrow \quad z = 1.469 \pm j\,2.287 \ (T = 1\ \mathrm{s})$

Laplace Domain	*Z Domain*
Left-hand plane	Inside unit circle
Imaginary axis	Unit circle
Right-hand plane	Outside unit circle

Rational Z-Transform

The system Function of a Linear Time –Invariant System

The input sequence x(n)
The output sequence y(n)

The relationship in the z-domain $Y(z) = H(z)X(z)$

H(z) is obtained $H(z) = \frac{Y(z)}{X(z)}$ is called *system function*

$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}$ We can take the inverse of z-transform to find h(n).

H(z) can be obtained from a linear constant-coefficient difference equation.

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{N} b_k x(n-k)$$

Rational Z-Transform

H(z) can be obtained from a linear constant-coefficient difference equation.

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{N} b_k x(n-k)$$

The z-transform

$$Y(z) = -\sum_{k=1}^{N} a_k Y(z)z^{-k} + \sum_{k=0}^{N} b_k X(z)z^{-k}$$

$$Y(z)\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right) = \sum_{k=0}^{N} b_k X(z)z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b_k z^{-k}}{\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right)}$$

Rational Z-Transform

Example: $y(n) = \frac{1}{2}y(n-1) + 2x(n)$

The z-transform from the difference equation

$$Y(z) = \frac{1}{2}Y(z)z^{-1} + 2X(z)$$

The system function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1-\frac{1}{2}z^{-1}}$$

The system has one pole at and one zero at
The inverse z-transform

$$h(n) = 2\left(\frac{1}{2}\right)^n u(n)$$

The inverse of the z-transform

The inverse z-transform is formally given by

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

There are three methods for evaluation of inverse z-transform

1. Direct evaluation by Contour integration (using Cauchy residue theorem)
2. Expansion into a series of term in the variable and
3. Partial-fraction expansion and table lookup.

The inverse of the z-transform

The Inverse z-transform by Power Series Expansion

$$X(z)=\sum_{n=-\infty}^{\infty} c_n z^{-n}$$

With given ROC:

Example: Using Power Series Expansion to determine the inverse z-transform of

$$X(z)=\frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

a. ROC: $|z|>1$

b. ROC: $|z|<0.5$

```
>> y=[1];
>> h=[1 –1.5 0.5];
>> n=7;
>> y=[y zeros(1,n-1)];
            //y={1,0,0,0,0,0,0}
>> [x,r]=deconv(y,h)
x = 1.0000   1.5000   1.7500
1.8750   1.9375   1.9688
r =     0          0          0
0          0          0
1.9844   -0.9844
```

$$x(0)=1,\ x(1)=1.5,\ x(2)=1.75,\ x(3)=1.875,\ x(4)=1.9375$$

The inverse of the z-transform

a. Since the ROC is the exterior of the circle, we can expect x(n) to be a causal.

$$\begin{array}{r|l}
 & 1+\frac{3}{2}z^{-1}+\frac{7}{4}z^{-2}+\frac{15}{8}z^{-3}+\frac{31}{16}z^{-4}+\ldots \\
\hline
1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2} & 1 \\
 & 1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2} \\
 & \frac{3}{2}z^{-1}-\frac{1}{2}z^{-2} \\
 & \frac{3}{2}z^{-1}-\frac{9}{4}z^{-2}+\frac{3}{4}z^{-3} \\
 & \frac{7}{4}z^{-2}-\frac{3}{4}z^{-3} \\
 & \frac{7}{4}z^{-2}-\frac{21}{8}z^{-3}+\frac{7}{8}z^{-4} \\
 & \frac{15}{8}z^{-3}-\frac{7}{8}z^{-4} \\
 & \frac{15}{8}z^{-3}-\frac{45}{16}z^{-4}+\frac{15}{16}z^{-5} \\
 & \frac{31}{16}z^{-4}-\frac{15}{16}z^{-5}
\end{array}$$

Therefore .

$$X(z)=1+\frac{3}{2}z^{-1}+\frac{7}{4}z^{-2}+\frac{15}{8}z^{-3}+\frac{31}{16}z^{-4}+\ldots\ldots$$

x(n) con be obtained as

$$x(n)=\{\underset{\uparrow}{1},\frac{3}{2},\frac{7}{4},\frac{15}{8},\frac{31}{16},\ldots\ldots\}$$

The inverse of the z-transform

b. ROC is the interior of a circle. The signal x(n) is anticausal.
The long division in the following way.

$$
\begin{array}{r l}
 & 2z^2+6z^3+4z^4+30z^5+\ldots \\
\frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1\,\big) & 1 \\
 & \underline{1-3z+2z^2} \\
 & 3z-2z^2 \\
 & \underline{3z-9z^2+6z^3} \\
 & 7z^2-6z^3 \\
 & \underline{7z^2-21z^3+14z^4} \\
 & 15z^3-14z^4 \\
 & \underline{15z^3-45z^4+30z^5} \\
 & 31z^4-30z^5
\end{array}
$$

Therefore,

$$X(z)=2z^2+6z^3+14z^4+30z^5+.....$$

x(n) can be obtained as

$$x(n)=\{\ldots 30,14,6,2,0,\underset{\uparrow}{0}\}$$

The inverse of the z-transform

The inverse z-transform by Partial Fraction Expansion

We usually have X(z) function in a rational function form

$$X(z)=\frac{N(z)}{D(z)}=\frac{b_0+b_1z^{-1}+\ldots+b_Mz^{-M}}{a_0+a_1z^{-1}+\ldots+a_Nz^{-N}}$$

A rational function is called proper if M<N.
A rational function is called improper if M≥N.
An improper rational function (M≥N) can be written as the sum of polynomial and a proper rational function.

$$X(z)=c_0+c_1z^{-1}+\ldots+c_{M-n}z^{-(M-N)}+\frac{N_1(z)}{D(z)}$$

Example:

$$X(z)=\frac{1+3z^{-1}+\frac{11}{6}z^{-2}+\frac{1}{3}z^{-3}}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$$

$$X(z)=1+2z^{-1}+\frac{\frac{1}{6}z^{-1}}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$$

The inverse of the z-transform

The inverse z-transform by Partial Fraction Expansion

Find the poles (denominator roots). We can have distinct real poles, distinct complex poles, multiple-order real poles, and multiple-order complex poles.

Distinct Poles: The poles are all different

$$X(z) = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \ldots + \frac{A_N}{1 - p_N z^{-1}}$$

We can determine coefficient using different methods. Let's give an example and use the of the methods

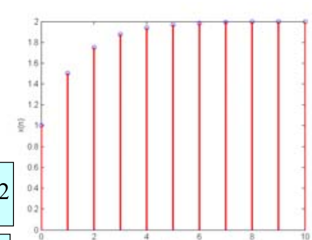
The inverse of the z-transform

Example:

Determine the partial fraction expansion the following function

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

$$X(z) = \frac{1}{(1 - 1z^{-1})(1 - 0.5z^{-1})} = \frac{A_1}{1 - 1z^{-1}} + \frac{A_2}{1 - 0.5z^{-1}}$$



To find coefficients

$$A_1 = (1 - 1z^{-1})X(z)\Big|_{z^{-1}=1} = \frac{1}{(1 - 0.5z^{-1})}\Big|_{z^{-1}=1} = \frac{1}{1 - 0.5} = 2$$

$$A_2 = (1 - 0.5z^{-1})X(z)\Big|_{z^{-1}=2} = \frac{1}{(1 - z^{-1})}\Big|_{z^{-1}=2} = \frac{1}{1 - 2} = -1$$

We have

$$X(z) = \frac{2}{1 - z^{-1}} - \frac{1}{1 - 0.5z^{-1}}$$

Assume that signal is causal
From the z-transform table

$$x(n) = \left[2 - (0.5)^n\right]u(n)$$

The inverse of the z-transform

Using MatLab to find inverse z-transform

```
>> syms z n
>> iztrans(1/(1-1.5*z^-1+0.5*z^-2))
 ans =
 2-(1/2)^n
```

```
>> syms z n
>> ztrans(2-(1/2)^n)
  ans =
   2*z/(z-1)-2*z/(2*z-1)
```


The inverse of the z-transform

Multiple – order poles:

If X(z) has repeated poles. This time we use a different expansion.
Let's say D(z) contains a repeated poles $(1-pz^{-1})^r$. Then

$$X(z)=\frac{N(z)}{(1-pz^{-1})^r D_1(z)}$$
$$=\frac{A_1}{(1-pz^{-1})}+\frac{A_2 z^{-1}}{(1-pz^{-1})^2}+\ldots+\frac{A_{r-1}z^{-r+2}}{(1-pz^{-1})^{r-1}}+\frac{A_r z^{-r+1}}{(1-pz^{-1})^r}+\frac{N_1(z)}{D_1(z)}$$

To find $A_1, A_2, \ldots, A_r$

$$A_1=\frac{1}{(r-1)!}\frac{d^{r-1}}{dz^{r-1}}\left[\frac{z^{r-1}N(z)}{D_1(z)}\right]\Bigg|_{z^{-1}=\frac{1}{p}}$$

$$A_2=\frac{d}{dz}\left[\frac{z^{r-1}N(z)}{D_1(z)}\right]\Bigg|_{z^{-1}=\frac{1}{p}}$$

$$A_r=\frac{z^{r-1}N(z)}{D_1(z)}\Bigg|_{z^{-1}=\frac{1}{p}}$$

The inverse of the z-transform

Example:

$$X(z)=\frac{1}{(1+z^{-1})(1-z^{-1})^2}$$

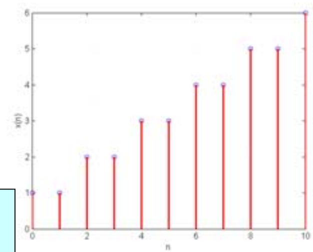
$$X(z)=\frac{1}{(1+z^{-1})(1-z^{-1})^2}=\frac{A_1}{(1+z^{-1})}+\frac{A_2}{(1-z^{-1})}+\frac{A_3z^{-1}}{(1-z^{-1})^2}$$

Determine the coefficients. A_1, A_2, and A_3

$$A_1=(1+z^{-1})X(z)\Big|_{z^{-1}=-1}=\frac{1}{(1-z^{-1})^2}\Big|_{z^{-1}=-1}=\frac{1}{4}$$

$$A_3=(1-z^{-1})^2zX(z)\Big|_{z^{-1}=1}=\frac{z}{(1+z^{-1})}\Big|_{z^{-1}=1}=\frac{1}{2}$$

$$A_2=\frac{d}{dz}\left[(1-z^{-1})^2zX(z)\right]_{z^{-1}=1}$$
$$=\frac{d}{dz}\left[\frac{z}{(1+z^{-1})}\right]_{z^{-1}=1}=\frac{(1+z^{-1})-z(-z^{-2})}{(1+z^{-1})^2}\Big|_{z^{-1}=1}=\frac{2+1}{2^2}=\frac{3}{4}$$



$$X(z)=\frac{1}{4}\frac{1}{(1+z^{-1})}+\frac{3}{4}\frac{1}{(1-z^{-1})}+\frac{1}{2}\frac{z^{-1}}{(1-z^{-1})^2}$$

$$x(n)=\left[\frac{1}{4}(-1)^n+\frac{3}{4}+\frac{1}{2}n\right]u(n)$$

The inverse of the z-transform

Distinct complex poles

Example

$$X(z)=\frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$$

Find inverse z-transform.

Partial Fraction

$$\frac{X(z)}{z}=\frac{1+z^{-1}}{(1-p_1z^{-1})(1-p_2z^{-1})}=\frac{1+z^{-1}}{\left[1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}\right]\left[1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}\right]}$$
$$=\frac{A_1}{\left[1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}\right]}+\frac{A_2}{\left[1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}\right]}$$

Determine the coefficients.

$$A_1=(1-p_1z^{-1})X(z)\Big|_{z=p_1}=\frac{1+z^{-1}}{\left[1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}\right]}\Bigg|_{z^{-1}=\frac{1}{\frac{1}{2}+j\frac{1}{2}}}=\frac{1+\frac{1}{\frac{1}{2}+j\frac{1}{2}}}{\left[1-\left(\frac{1}{2}-j\frac{1}{2}\right)\frac{1}{\frac{1}{2}+j\frac{1}{2}}\right]}=\frac{1}{2}-j\frac{3}{2}$$

The inverse of the z-transform

Example (cont)

$$A_2 = (1-p_2z^{-1})X(z)\Big|_{z=p_2} = \left.\frac{1+z^{-1}}{\left[1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}\right]}\right|_{z^{-1}=\frac{1}{\frac{1}{2}-j\frac{1}{2}}} = \frac{1+\frac{1}{\frac{1}{2}-j\frac{1}{2}}}{\left[1-\left(\frac{1}{2}+j\frac{1}{2}\right)\frac{1}{\frac{1}{2}-j\frac{1}{2}}\right]} = \frac{1}{2}+j\frac{3}{2}$$

$$X(z) = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}} = \frac{\frac{1}{2}-j\frac{3}{2}}{\left[1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}\right]} + \frac{\frac{1}{2}+j\frac{3}{2}}{\left[1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}\right]}$$

$$A_1 = \frac{1}{2}-j\frac{3}{2} = \frac{\sqrt{10}}{2}e^{-j71.565}$$ $$A_2 = \frac{1}{2}+j\frac{3}{2} = \frac{\sqrt{10}}{2}e^{+j71.565}$$ $$A_1 = A_2^*$$

$$p_1 = \frac{1}{2}+j\frac{1}{2} = \frac{1}{\sqrt{2}}e^{+j\pi/4}$$ $$p_1 = p_2^*$$

The inverse of the z-transform

Example (cont)

$$X(z) = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}} = \frac{\frac{\sqrt{10}}{2}e^{-j71.565}}{\left[1-\frac{1}{\sqrt{2}}e^{j\pi/4}z^{-1}\right]} + \frac{\frac{\sqrt{10}}{2}e^{j71.565}}{\left[1-\frac{1}{\sqrt{2}}e^{-j\pi/4}z^{-1}\right]}$$

Inverse z-transform.

$$x(n) = \frac{\sqrt{10}}{2}e^{-j71.565}\left(\frac{1}{\sqrt{2}}e^{j\pi/4}\right)^n u(n) + \frac{\sqrt{10}}{2}e^{j71.565}\left(\frac{1}{\sqrt{2}}e^{-j\pi/4}\right)^n u(n)$$

$$x(n) = \sqrt{10}\left(\frac{1}{\sqrt{2}}\right)^n\left(\frac{1}{2}e^{j\left(\frac{\pi n}{4}-71.565\right)} + \frac{1}{2}e^{-j\left(\frac{\pi n}{4}-71.565\right)}\right)u(n)$$

$$x(n) = \sqrt{10}\left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi n}{4}-71.565\right)u(n)$$

```
>> n=0:20;
>> x=sqrt(10)*(1/sqrt(2)).^n.*cos(pi*n/4-0.397*pi);
>> stem(n,x)
```