

S –convexity

Extensional work of the results by Zachariades and Nikolova

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Abstract

In this extensional article, we extend the results of [14] so that they also apply to S –convexity, an ultimate extension of convexity. We define several necessary and basic concepts in terms of S –convexity aiming the logarithmic spaces $A_{\Theta}(\log A)_{b,p}$. Estimates are given aiming the module of S –convexity of $A_{\Theta}(\log A)_{b,p}$ in terms of the moduli of A_0 and A_1 , when one or both of them are uniformly S –convex. One of the most interesting findings in this work is that s_2 –convexity, previously thought, by us, to be the only proper extension of the concept of convexity, is actually the only proper ‘geometric’ extension of the concept. s_1 –convexity, on the other hand, is the only proper ‘analytical’ extension of the concept, for a few concepts, whilst s_2 takes its place in others, that is, there is an oscillatory behavior, in Analysis, as to who extends convexity better.

Key-words: Uniformity, smoothness, S –convexity, Modulus of convexity, Banach space, convex, S –convex, vector space.

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1 Introduction

In [14], the authors show the conditions under which uniform convexity can be preserved in the logarithm spaces as introduced by Edmunds and Triebel in [15]. In this paper, we extend their results to S –convexity as re-stated in [13].

2 Notations and Definitions

2.1 Notations

We use the symbology defined in [9], [13], and [14]:

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- K_s^1 for the class of s -convex functions in the first sense, some s ;
- K_s^2 for the class of s -convex functions in the second sense, some s ;
- K_0 for the class of convex functions;
- s_1 for the constant s , $0 < s \leq 1$, used in the first definition of s -convexity;
- s_2 for the constant s , $0 < s \leq 1$, used in the second definition of s -convexity. Thirdly, we pointed out that the class of 1-convex functions is just a restriction of the class of convex functions, that is, when $X = \mathfrak{R}_+$,

$$K_1^1 \equiv K_1^2 \equiv K_0;$$

- $A_0 \subset A_1$ for when A_0 and A_1 are two complex Banach spaces, A_0 is densely and continuously embedded in A_1 ;
- $0 \leq \theta \leq 1$ and $A_\theta = [A_0, A_1]_\theta$ are the complex interpolation space;
- If $1 \leq p < \infty$ and $b < 0$, $A_\theta(\log A)_{b,p}$ is the set of all $a \in \bigcap_{\theta < \eta < 1} A_\eta$ such that the norm of our spaces $A_\theta(\log A)_{b,p}$ is

$$\left(\sum_{j=J}^{\infty} 2^{jb} \|a\|_{A_{\theta(j)}}^p \right)^{\frac{1}{p}} < \infty$$

where J is a positive integer, such that $\theta(j) = \theta + 2^{-j} < 1$ for all $j \geq J$;

- If $1 \leq p < \infty$ and $b > 0$, $A_\theta(\log A)_{b,p}$ is the set of all $a \in A_1$ which can be represented as $a = \sum_{j=J}^{\infty} a_j$, $a_j \in A_{\eta(j)}$,

and

$$\left(\sum_{j=J}^{\infty} 2^{jb} \|a_j\|_{A_{\eta(j)}}^p \right)^{\frac{1}{p}} < \infty$$

where J is a positive integer, such that $\eta(j) = \theta - 2^{-j} > 0$ for all $j \geq J$. The infimum of the norm of $A_\theta(\log A)_{b,p}$, taken over all admissible representations of the sum of the a_j 's just mentioned, defines the norm of this space.

2.2 Definitions

We here use the definitions presented in [18]:

1. *S-convex sets*

From [23], we copy the following definitions:

- *s₁-convex sets*

Definition 1. Let V be a vector space over \mathfrak{R} . A subset $X \subset V$ is called s_1 -convex if every s_1 -convex curve, defined by $\lambda^s x_1 + (1 - \lambda^s)x_2, \forall x_1, x_2 \in X$, intersects X in an interval, that is:

$$(\lambda^s x_1 + (1 - \lambda^s)x_2) \subset X$$

when $0 < \lambda < 1$ and $x_1, x_2 \in X$.

- *s₂-convex sets*

Definition 2. Let V be a vector space over \mathfrak{R} . A subset $X \subset V$ is called s_1 -convex if every s_1 -convex curve, defined by $\lambda^s x_1 + (1 - \lambda)^s x_2, \forall x_1, x_2 \in X$, intersects X in an interval, that is:

$$(\lambda^s x_1 + (1 - \lambda)^s x_2) \subset X$$

when $0 < \lambda < 1$ and $x_1, x_2 \in X$.

and in [19] as mentioned in [20]:

2. *Locally convex topological vector spaces*

Definition 3. Let V be a topological vector space over a subfield of the complex numbers (usually taken to be \mathbb{R} or \mathbb{C}). If the topology of V has a basis where each member is a convex

set, then V is a locally convex topological vector space. Though most vector spaces occurring in practice are locally convex, the spaces L^p for $0 < p < 1$ are examples of spaces which are not locally convex.

and in [14]:

3. *Modulus of convexity of a topological vector space*

Definition 4. Let X be a normed space with $\dim X \geq 2$ and $B_X = \{x \in X : \|x\| \leq 1\}$ be the unit ball of X . The modulus of convexity $\delta_X(\epsilon)$ of X , for $0 \leq \epsilon \leq 2$, is defined by

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in B_X, \|x - y\| \leq \epsilon\right\}$$

4. *Modulus of smoothness of a space defined in terms of its convexity*

Definition 5. The modulus of smoothness $\rho_X(\tau)$ of X , for $\tau > 0$, is defined by

$$\rho_X(\tau) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : x \in B_X, \|y\| \leq \tau\right\}$$

Remark 1. The above definition is found in the mentioned literature. However, we do believe that it is missing something in the above definition to make it accurate and sensible. Basically, as stated in [26], x should be of precise size ‘one’, not less than or equal, so that the definition makes sense. With this, we re-write the above definition as:

$$\rho_X(\tau) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| \leq \tau\right\}.$$

5. *Uniform convexity in topological vector spaces*

Definition 6. The space X is uniformly convex (u.c.) if $\delta_X(\epsilon) > 0$ for every $\epsilon > 0$ and uniformly smooth (u.s.) if $\lim_{\tau \rightarrow 0} \left(\frac{\rho_X(\tau)}{\tau}\right) = 0$.

Definition 7. For the dual space of X , X^* , one has

$$\rho_{X^*}(\tau) = \sup\left\{\frac{\tau\epsilon}{2} - \delta_X(\epsilon) : 0 \leq \epsilon \leq 2\right\}$$

for every $\tau > 0$.

6. *Generalization remarks on moduli of convexity and smoothness of topological vector spaces*

Definition 8. A u.c.(u.s.) space X has modulus of convexity (respectively, smoothness) of power type p if there exists $c, 0 < c < +\infty$ such that $\delta_X(\epsilon) \geq c\epsilon^p$ (respectively, $\rho_X(\tau) \leq c\tau^p$). The modulus of convexity of X is of power type p if and only if the modulus of smoothness of X^* is of power type q , where $\frac{1}{p} + \frac{1}{q} = 1$.

2.3 Other introductory remarks

In [14] it is stated that [15] brings the following embedding properties for logarithmic spaces:

Proposition 2.1. *If $0 < \theta_0 < \theta < \theta_1 < 1$, $b_0 \leq b_1$, and $1 \leq p \leq \infty$, then*

$$A_{\theta_0} \subset A_{\theta}(\log A)_{b_1,p} \subset A_{\theta}(\log A)_{b_0,p} \subset A_{\theta_1}$$

Proposition 2.2. *If $0 < \theta < 1$, $b_0 < 0 < b_1$, and $1 \leq p \leq \tilde{p} \leq \infty$, then*

$$A_{\theta}(\log A)_{b_1,p} \subset A_{\theta}(\log A)_{b_1,\tilde{p}} \subset A_{\theta} \subset A_{\theta}(\log A)_{b_0,p} \subset A_{\theta}(\log A)_{b_0,\tilde{p}}$$

3 Discussion on what should be extended to S -convexity from the previous definitions and/or how good the previous definition are

- Convexity of a vector space has to do with continuity. A vector space is convex if it is both a convex set and a vector space. The properties that are ESSENTIAL to define a vector space are well mentioned in [21]:

Vector space

Definition 9. Let F be a field (or, more generally, a division ring). A vector space V over F is a set with two operations, $+$: $V \times V \longrightarrow V$ and \cdot : $F \times V \longrightarrow V$, such that

1. $(u + v) + \mathbf{w} = u + (v + \mathbf{w})$ for all $u, v, \mathbf{w} \in V$
2. $u + v = v + u$ for all $u, v \in V$
3. There exists an element $\mathbf{0} \in V$ such that $u + \mathbf{0} = u$ for all $u \in V$
4. For any $u \in V$, there exists an element $v \in V$ such that $u + v = \mathbf{0}$
5. $a \cdot (b \cdot u) = (a \cdot b) \cdot u$ for all $a, b \in F$ and $u \in V$
6. $1 \cdot u = u$ for all $u \in V$
7. $a \cdot (u + v) = (a \cdot u) + (a \cdot v)$ for all $a \in F$ and $u, v \in V$
8. $(a + b) \cdot u = (a \cdot u) + (b \cdot u)$ for all $a, b \in F$ and $u \in V$

Equivalently, a vector space is a module V over a ring F which is a field (or, more generally, a division ring).

The elements of V are called vectors, and the element $\mathbf{0} \in V$ is called the zero vector of V .

On whether there is any point in extending the concept of convex vector space to S -convex vector space

Basically, there is. Why? Because it is still a different and larger scope of points to work with. In the same sense we believed it was a progress to go from real to complex, why not extending it from convex to S -convex? This way we can work with worse and more demanding of work approximations. However, the graphical scope with mathematical properties gets far bigger. So, it is interesting. It is basically including larger sets as possible vector spaces besides the present ones. Basically, because the curve with fixed s spans the whole set on which the space is based, there is the increase, probably up to a limit, even if working with more than one dimension for that, still to be found out, of the area size in which the vectors may be born/may exist, so that it is still interesting as much as any geometrical new shape would be. There is also a chance that the fact that such a set is already more than one dimension and might be referred to as

one, might interest the topologist. It is basically two dimensions in one name that sounds like one dimension. This probably, at least, allows a simplification when dealing with complicated topological spaces in terms of number of dimensions and yet dimensions that may be matched as one without harming the results/observations made. Under any point of view, all we do here and we stop here once we are not topologists, is providing the initial tools for them to work or think about the concept.

On whether it is also necessary that we extend the concept of modulus of convexity of a space to S -convexity

- Modulus of convexity has got its use in terms of uniformity of the treatment of convex sets which form a several dimensioned vector space. It is about treating all component vector sets the same way, a common Language. This way, if the concept of S -convex topological vector spaces are ever interesting, so this one will be.
- Modulus of smoothness has to do with how long it takes for us to approach the initial unit ball considered for the calculi of the convexity. It is likely to suit progressive studies on approaching the end of the radius for the second element. The unit ball is kept fixed and we work with the upper bound limiting the distance between the furthest element of the basic fixed unit ball and the last element in terms of the radius where the second element is allowed.

Basically, an extension is obviously demanded once the topology involved has changed. If we want to use the same modulus of convexity here, perhaps there is no conflict once the objective is an unique measurement. However, to make things more uniform in terms of names, we have decided for updating the concept to also cope for the s element. If it is relevant in the description of the space, it should be relevant in the description of its moduli as well.

Because the difference between convexity and S -convexity was already studied by us in [22], and $\frac{1}{2}$ will appear as $\frac{1}{2^s-1}$, then the extension is obvious.

- The same remarks apply to the other concepts so that they must be all extended.

4 Extension of definitions from [14]

We then first extend the definitions of modulus and smoothness to S -convexity to afterwards extend the related results.

4.1 Modulus of S -convexity

A) For s_1 -convexity:

Definition 10. Let X be a normed space with $\dim X \geq 2$ and $B_X = \{x \in X : \|x\| \leq 1\}$ be the unit ball of X . The modulus of S -convexity $S\delta_X(\epsilon)$ of X , $s_1 \neq 1$, for $0 \leq \epsilon \leq 2$, is defined by

$$S\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2^{\frac{1}{s}}} : x, y \in B_X, \|x - y\| \leq \epsilon\right\}.$$

Interesting that there is disagreement as to how the $\frac{1}{2}$ appears when picking the sum in the domain. In some sources, for example [24], which is a refereed publication, and therefore more reliable, one writes about midpoint of the segment. However, in other sources, such as [25], one mentions average over components of the vector in the domain. The concepts are clearly conflicting. Due to the fact that one is refereed and the other is not, we chose to believe and accept what is stated in [24]. The other point regards the existence of the 1 in the definition. Originally, as mentioned in [24], the choice of this 1 as parameter has to do with the function $\delta(\epsilon)$, which would then take $[0, 2]$ to $[0, 1]$. As changing this parameter would imply changing the target of the function, we must not change it.

Remark 2. If modulus of S -convexity is defined this way, one gets the usual one for convexity when $S = 1$, that is,

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in B_X, \|x - y\| \leq \epsilon\right\},$$

what is making of s_1 -convexity a proper extension of the concept in the case of the convexity modulus.

Remark 3. As it may be easily inferred from [24], the function $\delta\epsilon$ is nondecreasing and s_1 -convex.

B) For s_2 -convexity:

Definition 11. Let X be a normed space with $\dim X \geq 2$ and $B_X = \{x \in X : \|x\| \leq 1\}$ be the unit ball of X . The modulus of S -convexity $S\delta_X(\epsilon)$ of X , for $0 \leq \epsilon \leq 2$, is defined by

$$S\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2^s} : x, y \in B_X, \|x - y\| \leq \epsilon\right\}.$$

Remark 4. If modulus of S -convexity is defined this way, one gets the usual one for convexity when $S = 1$, that is,

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in B_X, \|x - y\| \leq \epsilon\right\},$$

what is making of s_2 -convexity also a proper extension of the concept in the case of the convexity modulus.

4.2 Modulus of Smoothness for S -convexity

A) For s_1 -convexity:

Definition 12. The modulus of smoothness $S\rho_X(\tau)$ of X , for $\tau > 0$, is defined by

$$S\rho_X(\tau) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2^{\frac{1}{s}}} - 1 : \|x\| = 1, \|y\| \leq \tau\right\}.$$

B) For s_2 -convexity:

Definition 13. The modulus of smoothness $S\rho_X(\tau)$ of X , for $\tau > 0$, is defined by

$$S\rho_X(\tau) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2^s} - 1 : \|x\| = 1, \|y\| \leq \tau\right\}$$

4.3 Dual

As per definition, the dual space is the space of the linear forms on V , denoted by V^* .

In this case, we may write:

- Modulus of convexity:

Definition 14. For $\epsilon \geq 0$,

$$S\tilde{\delta}(\epsilon) = \sup\left\{\frac{\tau\epsilon}{2} - S\rho_{X^*}(\tau) : \tau \geq 0\right\};$$

- Modulus of Smoothness/Continuity:

Definition 15. For the dual space of X , X^* , one has

$$S\rho_{X^*}(\tau) = \sup\left\{\frac{\tau\epsilon}{2} - S\delta_X(\epsilon) : 0 \leq \epsilon \leq 2\right\}$$

for every $\tau > 0$.

4.4 Uniform S -convexity

Definition 16. The space X is uniformly S -convex (u.s-c.) if $S\delta_X(\epsilon) > 0$ for every $\epsilon > 0$ and uniformly smooth (u.s-s.) if $\lim_{\tau \rightarrow 0} (\frac{S\rho_X(\tau)}{\tau}) = 0$.

4.5 Generalization remarks on moduli of S -convexity and S -smoothness of topological vector spaces

Definition 17. A u.s-c.(u.s-s.) space X has modulus of S -convexity (respectively, S -smoothness) of power type p if there exists $c, 0 < c < +\infty$ such that $S\delta_X(\epsilon) \geq c\epsilon^p$ (respectively, $S\rho_X(\tau) \leq c\tau^p$). The modulus of S -convexity of X is of power type p if and only if the modulus of S -smoothness of X^* is of power type q , where $\frac{1}{p} + \frac{1}{q} = 1$.

In order to work with optimization and local approximations to convexity, we introduce a few concepts to measure the distance between our S -convex spaces and the convex ones.

Based on the previous extension of results, already mentioned in this paper, for moduli, we gladly introduce:

4.6 Modulus of closeness to convexity

A) s_1 -convexity:

Definition 18. Let X be a normed space with $\dim X \geq 2$ and $B_X = \{x \in X : \|x\| \leq 1\}$ be the unit ball of X . The modulus of closeness to convexity $MCC\delta_X(\epsilon)$ is

$$MCC\delta_X(\epsilon) = \inf\left\{\frac{\|x+y\|}{2} - \frac{\|x+y\|}{2^{\frac{1}{s}}}\right\} : x, y \in B_X, \|x-y\| \leq \epsilon$$

B) s_2 -convexity:

Definition 19. Let X be a normed space with $\dim X \geq 2$ and $B_X = \{x \in X : \|x\| \leq 1\}$ be the unit ball of X . The modulus of closeness to convexity $MCC\delta_X(\epsilon)$ is

$$MCC\delta_X(\epsilon) = \inf\left\{\frac{\|x+y\|}{2^s} - \frac{\|x+y\|}{2}\right\} : x, y \in B_X, \|x-y\| \leq \epsilon$$

5 Extension of results from [14]

5.1 Results from [14]

Lemma 5.1. *Let (X_n) be a sequence of Banach spaces and $1 < p < +\infty$. For every ϵ , $0 \leq \epsilon \leq 2$, we put $\delta'(\epsilon) = \inf_n \inf_{(\frac{\epsilon}{2})^p \leq s \leq 1} s \delta_{X_n}(\frac{\epsilon}{s^{\frac{1}{p}}})$ and $\tilde{\delta}'(\epsilon) = \sup_{\tau \geq 0} \{\frac{\tau \epsilon}{2} - \rho'(\tau)\}$, where $\rho'(\tau) = \sup_n \sup_{u \leq 1} u^q \rho_{X_n^*}(\frac{\tau}{u})$, $\tau \geq 0$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then $\tilde{\delta}'(\epsilon) \geq (\gamma^{-1} - 1)\delta'(\gamma\epsilon)$, whenever $0 < \gamma < 1$ and $0 < \epsilon \leq 2$.*

5.2 Extension

Remark 5. We here swap letter s from [14] with letter m to make a distinction between the constant used in the lemma and our s for S -convexity.

Lemma 5.2. *Let (X_n) be a sequence of Banach spaces and $1 < p < +\infty$. For every ϵ , $0 \leq \epsilon \leq 2$, we put $S\delta'(\epsilon) = \inf_n \inf_{(\frac{\epsilon}{2})^p \leq m \leq 1} m S\delta_{X_n}(\frac{\epsilon}{m^{\frac{1}{p}}})$ and $S\tilde{\delta}'(\epsilon) = \sup_{\tau \geq 0} \{\frac{\tau \epsilon}{2} - S\rho'(\tau)\}$, where $S\rho'(\tau) = \sup_n \sup_{u \leq 1} u^q S\rho_{X_n^*}(\frac{\tau}{u})$, $\tau \geq 0$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then $S\tilde{\delta}'(\epsilon) \geq (\gamma^{-1} - 1)S\delta'(\gamma\epsilon)$, whenever $0 < \gamma < 1$ and $0 < \epsilon \leq 2$.*

Proof.

$$\begin{aligned}
\sup_{u \leq 1} u^q S\rho_{X_n^*}\left(\frac{\tau}{u}\right) &= \sup_{u \leq 1} \sup_{0 \leq \epsilon \leq 2} u^q \left\{ \frac{\tau}{2u} \epsilon - S\delta_{X_n}(\epsilon) \right\} \\
&= \sup_{0 \leq \epsilon \leq 2} \sup_{u \leq 1} \left\{ \frac{\tau}{2} u^{q-1} \epsilon - u^q S\delta_{X_n}(\epsilon) \right\} \\
&= \sup_{0 \leq \epsilon \leq 2} \sup_{\left(\frac{\epsilon}{2}\right)^{\frac{1}{q-1}} \leq u \leq 1} \left\{ \frac{1}{2} \tau \epsilon - u^q S\delta_{X_n}\left(\frac{\epsilon}{u^{q-1}}\right) \right\} \\
&= \sup_{0 \leq \epsilon \leq 2} \sup_{\left(\frac{\epsilon}{2}\right)^p \leq m \leq 1} \left\{ \frac{\tau \epsilon}{2} - m S\delta_{X_n}\left(\frac{\epsilon}{m^{\frac{1}{p}}}\right) \right\} \\
&= \sup_{0 \leq \epsilon \leq 2} \left\{ \frac{\tau \epsilon}{2} - \inf_{\left(\frac{\epsilon}{2}\right)^p \leq m \leq 1} m S\delta_{X_n}\left(\frac{\epsilon}{m^{\frac{1}{p}}}\right) \right\}
\end{aligned}$$

From the above we obtain

$$S\rho'(\tau) = \sup_n \sup_{0 \leq \epsilon \leq 2} \left\{ \frac{\tau \epsilon}{2} - \inf_{\left(\frac{\epsilon}{2}\right)^p \leq m \leq 1} m S\delta_{X_n}\left(\frac{\epsilon}{m^{\frac{1}{p}}}\right) \right\} = \sup_{0 \leq \epsilon \leq 2} \left\{ \frac{\tau \epsilon}{2} - S\delta'(\epsilon) \right\}$$

So $S\tilde{\delta}'(\epsilon) \leq S\delta'(\epsilon)$ for every $0 < \epsilon \leq 2$. Since $\frac{S\delta(\epsilon)}{\epsilon}$ is not decreasing (check [14]) we obtain that $\frac{S\delta'(\epsilon)}{\epsilon}$ is nondecreasing. With all this, $S\tilde{\delta}'$ is the maximal convex function minorizing $S\delta'$. And, as mentioned in [14], with Lemma 2 from [16], we obtain the conclusion. \square

Proposition 5.3. *Let $1 < p < +\infty$. There exists $K > 0$, which depends on p , such that for every sequence of Banach spaces (X_n) , if $X = [\sum_{n=1}^{\infty} X_n]_p$ then we have*

$$S\delta_X(\epsilon) \geq 2^{-p} K^{1-p} \epsilon^p \inf_{\frac{\epsilon}{(2K)} \leq t \leq 2} \frac{S\delta(t)}{t^p}$$

where $S\delta(t) = \inf_n S\delta_{X_n}(t)$

Proof. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of Banach spaces. If $\tau \geq 0$ and $0 \leq \epsilon \leq 2$ we put $S\rho'(\tau) = \sup_n \sup_{u \leq 1} u^q S\rho_{X_n^*}(\frac{\tau}{u})$, $S\delta'(\epsilon) = \inf_n \inf_{(\frac{\epsilon}{2})^p \leq m \leq 1} m S\delta_{X_n}(\frac{\epsilon}{m^{\frac{1}{p}}})$, and $S\tilde{\delta}'(\epsilon) = \sup_{\tau \geq 0} \{\frac{\tau\epsilon}{2} - S\rho'(\tau)\}$, where $\frac{1}{p} + \frac{1}{q} = 1$. From an adaptation of proposition 19 of [16] there exists $K > 0$, which depends only on p , such that $S\rho_{X^*}(\epsilon) \leq K S\rho'(\tau)$. So, from our last lemma, we obtain

$$\begin{aligned} S\delta_X(\epsilon) &\geq S\tilde{\delta}'(\epsilon) \geq K \sup_{\tau \geq 0} \left\{ \frac{\tau\epsilon}{2} K^{-1} - S\rho'(\tau) \right\} = K S\tilde{\delta}'\left(\frac{\epsilon}{K}\right) \geq K S\delta'\left(\frac{\epsilon}{2K}\right) \\ &= K \inf_n \inf_{(\frac{\epsilon}{4K})^p \leq m \leq 1} m S\delta_{X_n}\left(\frac{\epsilon}{2K m^{\frac{1}{p}}}\right) \\ &= K \inf_n \inf_{(\frac{\epsilon}{4K})^p \leq m \leq 1} \frac{S\delta_{X_n}\left(\frac{\epsilon}{2K m^{\frac{1}{p}}}\right)}{\frac{1}{m}} \\ &= K \inf_n \inf_{\frac{\epsilon}{(2K)} \leq t \leq 2} \left(\frac{\epsilon}{2K}\right)^p \frac{S\delta_{X_n}(t)}{t^p} \\ &= 2^{-p} K^{1-p} \epsilon^p \inf_{\frac{\epsilon}{(2K)} \leq t \leq 2} \frac{S\delta(t)}{t^p} \end{aligned}$$

\square

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