S-convexity VII

(Playing up with Lazhar and his inequalities)

Author: M.R. Pinheiro[1]

Abstract

In this further little article, we simply extend Lazhar's work on inequalities for convex functions to those a little bit beyond: S-convex functions.

Key-words: $S-$convexity, convex, $S-$convex, function, inequality, extension, bounds, improvement, refinement.

AMS:26A51

1 Introduction

We seem to have developed the precursor and so honorable work of Profs Hudzik and Maligranda to a palatable level of suitability for applications in diverse areas by making their theory more foundational in the Pure scope of the Science. In this further work, we wish to extend Lazhar's work to S-convexity functions.

V. I. Prof. Lazhar has made use, as seen on [3], of the sources [1], [2], [5]. We obviously simply trust Prof. Lazhar's citations refereed by the so respectful editorial board of JIPAM.

Little by little, the use of S-convexity is proven. By our extensions of results and foundational works, we have developed many tools that may be used in Optimization when dealing with functions that almost look like convex functions but are not. By splitting the domain of the function into intervals, one may make the whole function passive of work in Optimization with little effort.

In the next section, the set of symbols used is explained in detail. **Section** 3 will bring the results exposed by Lazhar in his precursor work. **Section** 4 brings our new theorems, stated as they would be after the extension to S-convexity, followed by proofs of each one.

We use the symbols defined in [4]:

- K_s^1 for the class of $S-$convex functions in the first sense, some S;
- K_s^2 for the class of $S-$convex functions in the second sense, some S;
- K_0 for the class of convex functions;

---

[1] I.R., Porto Alegre, Brazil. (mrpprofessional@yahoo.com). Questions, comments, or corrections to this document may be directed to the e-mail address.

- s_1 for the constant S, $0 < S \leq 1$, used in the first definition of S-convexity;
- s_2 for the constant S, $0 < S \leq 1$, used in the second definition of S-convexity. Thirdly, we pointed out that the class of 1-convex functions is just a restriction of the class of convex functions, that is, when $X = \Re_+$,

$$K_1^1 \equiv K_1^2 \equiv K_0.$$

1.1 Definitions

We use the definitions presented in [1]:

Definition 1. A function $f : X- > \Re$, f continuous (see [1] for argumentation), is said to be s_1-convex if the inequality

$$f(\lambda x + (1-\lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1-\lambda^s) f(y)$$

holds $\forall \lambda \in [0,1]$, $\forall x, y \in X$ such that $X \subset \Re_+$.

Definition 2. f is called s_2-convex, $s \neq 1$, if the graph lies below a 'bent chord' (L) between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, it is true that

$$sup_J(L-f) \geq sup_{\partial J}(L-f)$$

Definition 3. A function $f : X- > \Re \in C^1$ is said to be s_2-convex if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds $\forall \lambda \in [0,1]$, $\forall x, y \in X$ such that $X \subset \Re_+$.

2 Lazhar's precursor theorems

Theorem 2.1. *If f is a convex function and $x_1, x_2, ..., x_n$ lie in its domain, then*

$$\sum_{i=1}^{n} f(x_i) - f(\frac{x_1 + ... + x_n}{n})$$
$$\geq \frac{n-1}{n}\left[f\left(\frac{x_1+x_2}{2}\right) + ... + f\left(\frac{x_{n-1}+x_n}{2}\right) + f\left(\frac{x_n+x_1}{2}\right)\right].$$

Theorem 2.2. *If f is a convex function and $a_1, ..., a_n$ lie in its domain, then*

$$(n-1)[f(b_2 + ... + f(b_n)] \leq n[f(a_1 + ... + f(a_n) - f(a)],$$

where $a = \frac{a_1+...+a_n}{n}$ and $b_i = \frac{na-a_i}{n-1}$, $i = 1,...,n$.

3 Our theorems: extensions of Lazhar's work to S-convex functions

As a conclusion for this one more precursor paper, we mention our own results, all based on Lazhar's previous developments.

Theorem 3.1. *If f is an S_1-convex function and $x_1, x_2, ..., x_n$ lie in its domain, then*

$$\sum_{i=1}^{n} f(x_i) - f\left(\frac{x_1+...+x_n}{n^{\frac{1}{s}}}\right)$$

$$\geq \frac{n-1}{n}\left[f\left(\frac{x_1+x_2}{2^{\frac{1}{s}}}\right)+...+f\left(\frac{x_{n-1}+x_n}{2^{\frac{1}{s}}}\right)+f\left(\frac{x_n+x_1}{2^{\frac{1}{s}}}\right)\right].$$

Proof. Using the condition of S_1-convexity, with $t=\frac{1}{2^{\frac{1}{s}}}$, we obtain:

$$f\left(\frac{x_1+x_2}{2^{\frac{1}{s}}}\right)+...+f\left(\frac{x_{n-1}+x_n}{2^{\frac{1}{s}}}\right)+f\left(\frac{x_n+x_1}{2^{\frac{1}{s}}}\right)$$

$$\leq f(x_1)+f(x_2)+...+f(x_n).$$

However,

$$\sum_{i=1}^{n} f(x_i) = \frac{n}{n-1}\sum_{i=1}^{n} f(x_i) - \frac{1}{n-1}\sum_{i=1}^{n} f(x_i),$$

$$\sum_{i=1}^{n} f(x_i) = \frac{n}{n-1}\left[\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n}\frac{1}{n}f(x_i)\right].$$

Replacing $\sum_{i=1}^{n} f(x_i)$ with its equivalent expression, as above, one gets:

$$f\left(\frac{x_1+x_2}{2}\right)+...+f\left(\frac{x_{n-1}+x_n}{2}\right)+f\left(\frac{x_n+x_1}{2}\right)$$

$$\leq \frac{n}{n-1}\left[\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n}\frac{1}{n}f(x_i)\right].$$

With the subsequent application of the condition of S_2-convexity, one gets:

$$f\left(\frac{x_1+x_2}{2}\right)+...+f\left(\frac{x_{n-1}+x_n}{2}\right)+f\left(\frac{x_n+x_1}{2}\right)$$

$$\leq \frac{n}{n-1}\left[\sum_{i=1}^{n} f(x_i) - f\left(\frac{\sum_{i=1}^{n} x_i}{n^{\frac{1}{s}}}\right)\right].$$

□

Theorem 3.2. *If f is an S_2-convex function and $x_1, x_2, ..., x_n$ lie in its domain, then*

$$\sum_{i=1}^{n} f(x_i) - f(\frac{x_1 + ... + x_n}{n})$$

$$\geq \frac{n^s - 1}{n^s}\left[f\left(\frac{x_1 + x_2}{2}\right) + ... + f\left(\frac{x_{n-1} + x_n}{2}\right) + f\left(\frac{x_n + x_1}{2}\right)\right].$$

Proof. Using the condition of S_2-convexity, with $t = \frac{1}{2}$, we obtain:

$$f\left(\frac{x_1 + x_2}{2}\right) + ... + f\left(\frac{x_{n-1} + x_n}{2}\right) + f\left(\frac{x_n + x_1}{2}\right)$$

$$\leq f(x_1) + f(x_2) + ... + f(x_n).$$

However,

$$\sum_{i=1}^{n} f(x_i) = \frac{n^s}{n^s - 1}\sum_{i=1}^{n} f(x_i) - \frac{1}{n^s - 1}\sum_{i=1}^{n} f(x_i),$$

$$\sum_{i=1}^{n} f(x_i) = \frac{n^s}{n^s - 1}\left[\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n}\frac{1}{n^s} f(x_i)\right].$$

Replacing $\sum_{i=1}^{n} f(x_i)$ with its equivalent expression, as above, one gets:

$$f\left(\frac{x_1 + x_2}{2}\right) + ... + f\left(\frac{x_{n-1} + x_n}{2}\right) + f\left(\frac{x_n + x_1}{2}\right)$$

$$\leq \frac{n^s}{n^s - 1}\left[\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n}\frac{1}{n^s} f(x_i)\right].$$

With the subsequent application of the condition of S_2-convexity, one gets:

$$f\left(\frac{x_1 + x_2}{2}\right) + ... + f\left(\frac{x_{n-1} + x_n}{2}\right) + f\left(\frac{x_n + x_1}{2}\right)$$

$$\leq \frac{n^s}{n^s - 1}\left[\sum_{i=1}^{n} f(x_i) - f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)\right].$$

□

Remark 1. Considering the the extended theorem for K_s^1 for $n = 3$, we get:

$$\sum_{i=1}^{3} f(x_i) - f\left(\frac{x_1 + x_2 + x_3}{3^{\frac{1}{s}}}\right)$$

$$\geq \frac{2}{3}\left[f\left(\frac{x_1 + x_2}{2^{\frac{1}{s}}}\right) + f\left(\frac{x_2 + x_3}{2^{\frac{1}{s}}}\right) + f\left(\frac{x_3 + x_1}{2^{\frac{1}{s}}}\right)\right].$$

Remark 2. Considering the the extended theorem for K_s^2 for $n = 3$, we get:

$$f(x_1) + f(x_2) + f(x_3) - f\left(\frac{x_1 + x_2 + x_3}{3}\right) \geq \frac{3^s - 1}{3^s}\left[f\left(\frac{x_1 + x_2}{2}\right) + f\left(\frac{x_2 + x_3}{2}\right) + f\left(\frac{x_3 + x_1}{2}\right)\right].$$

Theorem 3.3. *If f is an S_1-convex function and $a_1, ..., a_n$ lie in its domain, then*

$$(n-1)[f(b_2) + ... + f(b_n)] \leq n[f(a_1) + ... + f(a_n) - f(a)],$$

$$\textit{where } a = \tfrac{a_1 + ... + a_n}{n^{\frac{1}{s}}} \textit{ and } b_i = \tfrac{n^{\frac{1}{s}} a - a_i}{(n-1)^{\frac{1}{s}}},\ i = 1, ..., n.$$

Proof. The same proof used by Lazhar applies here, once more. However, the last parcel in the negative sum which finalizes the proof gets its denominator replaced with $n^{\frac{1}{s}}$.

□

Theorem 3.4. *If f is an S_2-convex function and $a_1, ..., a_n$ lie in its domain, then*

$$(n^s - 1)[f(b_2) + ... + f(b_n)] \leq n^s[f(a_1) + ... + f(a_n) - f(a)],$$

$$\textit{where } a = \tfrac{a_1 + ... + a_n}{n} \textit{ and } b_i = \tfrac{na - a_i}{(n-1)},\ i = 1, ..., n.$$

Proof. The same proof used by Lazhar applies here, once more. However, as seen before in this same paper, the intermediate steps in the proof will bear n^s where it previously read n.

□

References:

[1] D. S. Mitrinović, J. E. Peča̋ric, A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

[2] Kiran Kedlaya, *$A < B$ (A is less than B), based on notes for the Math Olympiad Program (MOP) Version 1.0*, 1999.

[3] Lazhar Bougoffa, *New inequalities about convex functions*, JIPAM, V. 7, I. 4, Art. 148, 2006.

[4] M. R. Pinheiro, *Exploring the concept of $S-$convexity*, Aequationes Mathematicae, 2006.

[5] T. Popoviciu, *Sur certaines inégalitées qui caractérisent les functions convexes*, An. Sti. Univ. Al. I. Cuza Iasi. I-a, Mat. (N. S), 1965.