Starants III - Application of the concepts exposed in [4] to [3]

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Abstract

In this paper, we further progress in our intentions to control disease spread in an effective manner by means of simple mathematical tools which are then supposed to be used for computational purposes in our project of mechanizing the control of disease spread and provide governmental bodies with righteous plus highly useful information. In this paper, we still do not make the applications of our Starant graphs clear, we mainly worry about re-writing [4] so that its theory applies to Starant graphs. Further work will achieve our desired results for our newly created graphs, as mentioned in [3].
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1 Introduction

Deterministic small-world communication networks were introduced by Comellas, Ozon, and Peters in [1]. They are supposed to have strong local clustering (nodes have many neighbors in common), small diameter (largest of the shortest distances between nodes must be small), and would be located between Regular Lattices, which are highly clustered, large worlds, where the diameter, or characteristic path length, grows linearly with the number of nodes, and Random Networks, which are poorly clustered, small worlds, where the diameter grows logarithmically with the number of nodes. We shall name them 'medium worlds'.
Circulant graphs are considered part of the deterministic small-world communication networks, once they have strong local clustering, but large average distance between pairs of nodes. They are included in the class of structured networks.
In this paper, we want to contribute to Comellas et al. findings, by establishing a mathematical way of writing about our own findings mentioned in [3].

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1.1 Notation & some definitions

1. $C_{n,\delta}$ - circulant graph of n nodes and δ (degree) links per node such that each node i is adjacent to nodes $(i \pm 1)$, $(i \pm 2)$, ..., $(i \pm \frac{\delta}{2}(mod\ \ n))$. This graph has got diameter $D = \lceil \frac{n}{\delta} \rceil$ whenever $\delta \neq 2$ and $D = \lfloor \frac{n}{2} \rfloor$ otherwise.

2. Star graph - rooted tree containing n nodes with a central node (root) of degree $(n-1)$.

3. Complete graph on n nodes - graph where every node has got degree $(n-1)$.

4. $S^C_{n,\delta}$ - string of n circulant graphs connected by means of K_2 exactly δ times for each circulant graph added (taking away the first and the last graph on the string which will use K_2 exactly $\frac{\delta}{2}$ times to make the connection): each vertex 'i' is connected to $\left(\frac{\delta}{2} - \lambda \pm i\right), 1 \leq \lambda \leq \frac{\delta}{2}$.

5. $C^C_{n,\delta}$ - circle of circulant graphs connected by means of K_2 exactly δ times for each graph added: each vertex 'i' is connected to $\left(\frac{\delta}{2} - \lambda \pm i\right), 1 \leq \lambda \leq \frac{\delta}{2}$.

6. S^C_n - string of n circulant graphs connected by means of K_2 as many times as we like for each circulant graph added.

7. C^C_n - circle of n circulant graphs connected by means of K_2 as many times as we like for each circulant graph added.

8. $SC_{n,\delta}$ - 'Starant' graph, that is, a circulant graph with degree δ containning a star inside of it whose vertices coincide with the vertices of the circulant graph, a total of n vertices.

9. $S^S_{n,\delta}$ - string of n star-graphs connected by means of K_2 exactly δ times for each star-graph added (taking away the first and the last graph on the string which will use K_2 exactly $\frac{\delta}{2}$ times to make the connection): each vertex 'i' is connected to $\left(\frac{\delta}{2} - \lambda \pm i\right), 1 \leq \lambda \leq \frac{\delta}{2}$.

10. $S^{C}_{n,\delta}$ - string of n circulant graphs connected by means of K_2 exactly δ times for each circulant graph added (taking away the first and the last graph on the string, which will use K_2 exactly $\frac{\delta}{2}$ times to make the connection): each vertex 'i' is connected to $\left(\frac{\delta}{2} - \lambda \pm i\right), 1 \leq \lambda \leq \frac{\delta}{2}$.

11. $C^{SC}_{n,\delta}$ - circle of n starant graphs connected by means of K_2 exactly δ times for each starant graph added: each vertex 'i' is connected to $\left(\frac{\delta}{2} - \lambda \pm i\right), 1 \leq \lambda \leq \frac{\delta}{2}$.

12. $C^{S}_{n,\delta}$ - circle of stars connected by means of K_2 exactly δ times for each star-graph added: each vertex 'i' will be connected to $\left(\frac{\delta}{2} - \lambda \pm i\right), 1 \leq \lambda \leq \frac{\delta}{2}$.

13. S^{S}_{n} - string of n star-graphs connected by means of K_2 as many times as we like for each star-graph added.

14. C^{S}_{n} - string of n star-graphs connected by means of K_2 as many times as we like for each star-graph added.

15. $\Gamma_{\Delta}(D)$ - cycle prefix digraph as mentioned in [4]. It is basically a digraph on an alphabet of $\Delta+1$ symbols $\{1, 2, ..., \Delta+1\}$ as follows: Each vertex $x_1x_2...x_D$ is sequence of distinct symbols from the alphabet. The adjacencies are given by

$$x_1x_2...x_D -> \begin{cases} x_2x_3x_4...x_Dx_{D+1} \\ x_2x_3x_4...x_Dx_1, \\ x_1x_2...x_{k-1}x_{k+1}...x_Dx_k \end{cases}$$

$$2 \leq k \leq D-1$$

$$x_{D+1} \neq x_1, x_2, ..., x_D$$

In [3] we argue that a person's circle of acquaintances might be mathematically represented by means of a starant graph, which is basically a circulant graph with a star inside of it, making its diameter as short as 2. We now want to consider a situation where we label each vertex of a starant with a number and treat the connections as cycles. No matter how many people a person knows one will always get the same diameter in a starant, 2. One

must be very attentive, though, for the fact that a person's circle of acquaintances might be an 'incomplete' starant, that is, it is still a star, that is for sure, but the vertices of the star might not have edges between them, that is, a person might know n people who actually do not know each other.
In this sense, the theory we presented before, in [3], is still not complete. But it is already a good start.
It is very logical to understand that if I have m people in my circle of acquaintances, all of them are going to be part of my starant.
Suppose that we have a labeled starant,$SC_{5,1}$, with the vertices from the circulant labeled with numbers from one to five and the root of the star labeled with the number 6. For this graph, it is true that the following cycles exist:

$$(12)(23)(34)(45)(15)(64)(65)(61)(62)(63).$$

Suppose now that we have another starant, also $SC_{5,1}$, which we connect with the previous one, let's rename it to $SC2_{5,1}$. We now label its vertices from 7 to 12, 12 being the root of the star. For $SC2_{5,1}$ we then get:

$$(78)(89)(9\quad 10)(10\quad 11)(7\quad 11)(12\quad 9)(12\quad 8)(12\quad 7)(12\quad 10)(12\quad 11)$$

Suppose now that $SC_{5,1}$ connects to $SC2_{5,1}$ via K_2, originating a new cycle: $(10\quad 1)$. Therefore, the longest of the shortest paths between them (diameter) is now the diameter of $SC_{5,1}$, 2, plus the diameter of $SC2_{5,1}$, 2, plus the edge connecting them, that is 5. One must bear in mind that this actually means that there is no intersection between the two circle of acquaintances, and we might represent this by means of the following symbology:

$$SC_{5,1} \bigcap SC2_{5,1} = \varnothing.$$

We know that one of the ways of decreasing this diameter is decreasing one of the starants diameter, that is, for instance, making their degree 1, instead of 2. If we do this to one of them, we then get diameter 4 to our set of two starants. If we do this to both of them, we get diameter 3. This proves that the connection of a starant to another takes something between 3 and 5 steps, that is, the total diameter of an 'Australian square', let's call it $D_{A_{n,\delta}}$ may be expressed as

$$3 \leq D_{A_{n,\delta}} \leq 5.$$

All this is well explained in [3].
However, in this paper, we explain our theory in a different way by means of

the terminology contained in [4]. Therefore, this paper is about explaining our theory further in a more mathematical way.

2 Motivation

Our motivation for this work is the fact that we had a clear application for the problem for machine reasoning and coding, and the paper [4] seems to provide us with the right tools to put forth this application for the general public.
We actually think that, so far, we had not done anything else than using not so technical symbols from graph theory to describe our work. By means of [4], we can finally do that.

3 Application of [4] to [3]

We here apply some of the theory contained in [4] to [3].

- $\Gamma_\delta(n)$ - cycle[2] prefix digraph as mentioned in [4] but with some adaptations. It is basically a digraph on an alphabet of $\delta + 1$ symbols $\{1, 2, ..., \delta + 1\}$ as follows: Each vertex $x_1x_2...x_n$ is sequence of distinct symbols from the alphabet. We here think of a star of $\delta + 1$ vertices as a good graphical representation for our choices of symbols, the picture of a running circular head with typographical symbols on each small active end, or a typewriter head would be more than adequate for our image.
  On the top of that, we have another set of 'almost' star graphs, basically a connected string, as described by us in [3], with an isolated point in the middle, the picture being a start of drawing for our starant graphs. We take n symbols from our typewriter tape, randomly, to label its vertices. One has to understand, here, that this is a graphical organization, ideal for classrooms, to understand a cycle prefix digraph. However, it also helps us applying the abstract concept to practical problems and optimize their resolution. Our main justification for this step is precisely studying matters of real life in a more scientific way.

[2]Observe here that we adopt the symbol δ instead of Δ once it matches our previous notation for the applications we intend

The right picture is the one of several star graphs labeled after an initial random choice for the middle vertex, our x_o. From the label of x_o, a pattern of labeling follows for all other star vertices.
If we label ourselves, or the core of each star, $x_o = x_1x_2...x_n$, the adjacencies are then given by:

$$x_1x_2...x_n- > \begin{cases} x_2x_3x_4...x_nx_{n+1} \\ x_2x_3x_4...x_nx_1, \\ x_1x_2...x_{k-1}x_{k+1}...x_nx_k \end{cases}$$

$$2 \leq k \leq n-1$$

$$x_{n+1} \neq x_1, x_2, ..., x_n$$

just by regarding n as being our total number of vertices for our stars. So far, this is just a simple choice over the theory in [4].
However, considering our intentions of the circle of acquaintances, we must go further.
The use of the labeling system created this way is the real-world matching with society individuals. By means of simple coding we may be able to represent a whole population of a suburb or town. With that, all our calculations become computer possible. With that, we acquire tools to prevent disease spread in a very useful, meaningful, and logical way.
Given that the work is no doubtful very extensive, we restrain ourselves to give small steps in this first work as to first apply labeling to each individual circle of acquaintances. In later work, we intend to progress into the random connections amongst the vertices of each star representing people's circle of acquaintances. What follows is then an American square, as mentioned in [5], which should already be enough for medical purposes in the computational sense.
We then devise each family doctor with a computer screen before them, with a logical entry for each individual alive in his/her area, able to provide any government person with an idea on how to stop disease spread as quickly as possible. Once we are able to map each individual's circle of acquaintance with technology as well, we have all disease spread well controlled by the government or any interested organization.

- The order of a cycle prefix digraph $\Gamma_\delta(n)$, as mentioned in [4], is

$$(\delta+1)_n = \frac{(\delta+1)!}{(\delta+1-n)!}$$

has diameter n, and is δ-regular ($\delta \geq n$).

Definition 4. *We call an Australian square the static picture of two starant graphs with no intersection between their set of vertices connected to each other.*

Definition 5. *We call American square the set of two Australian squares with no intersection between their sets of vertices organized in a square.*

- If d_k is the number of vertices at distance k, $k \leq n$, from a given vertex then, as mentioned in [4], we find that in [5] it is shown that there are $(\delta+1)_k$ vertices in $\Gamma_\delta(n)$ within distance k from the vertex $12...n$. Therefore, the value of d_k may be determined from there and is also computed in a different way from [6]. The result follows:

Proposition 1. *For $\Gamma_\delta(n)$, $\delta \geq n$, the number of vertices d_k at distance k from a given vertex is*

$$d_k = (\delta+1)\delta(\delta-1)...(\delta-k+3)(\delta-k+1)$$

where $1 < k \leq n$,$d_o = 1$, and $d_1 = \delta$

- In [4], it is mentioned that a distance matrix might be defined by means of a simple recursive relation:

$$(A_k)_{ij} = \begin{cases} 1 & \text{if } d(i,j) = k, \\ 0 & \text{otherwise} \end{cases}$$

where A is the adjacency matrix of a digraph Γ of order m and diameter n. The matrix A_k is a $k-$distance matrix thus defined.

Remark 1. When $\Gamma = \Gamma_\delta(n)$, the distance matrices A_k are polynomials of degree k on the adjacency matrix.

Proposition 2. *Let $\Gamma_\delta(n)$ be the cycle prefix digraph of degree δ and diameter n, A its adjacency matrix, A_k the $k-$distance matrix, and $v_k(x) \in \Re_k[x]$ the polynomial defined by $v_o(x) = 1, v_1(x) = x, v_k(x) = (x+1)x(x-1)...(x-k+3)(x-k+1), k = 2, ..., n$. Then $A_k = v_k(A)$, $k = 0, ..., n$.*

Corollary 1. *Given the cycle prefix digraph $\Gamma_\delta(n)$ and its adjacency matrix A, we have*

$$(A+I)A(A-I)...(A-(n-2)I) = J$$

Remark 2. The polynomial obtained above is the Hoffman polynomial of $\Gamma_\delta(n)$.

As a nice recollection, [4] also mentions the following theorem from [8]:

Theorem 5.1. *Let Γ be a strongly connected digraph with order n and adjacency matrix A. There exists a polynomial $P(x)$ such that $P(A) = J$ if an only if Γ is Δ-regular. In this case, the unique polynomial P of least degree is known as the Hoffman polynomial $H(x) = \frac{nS(x)}{S(\Delta)}$, where $(x-\Delta)S(x)$ is the minimal polynomial of A.*

- The adjacency algebra A of $\Gamma_\delta(n)$ is the algebra generated by all the powers of the adjacency matrix A of the cycle prefix digraph of diameter n and degree δ. Each power of A is a linear combination of the $n+1$ linearly independent matrices $A_o, A_1, A_2, ..., A_n$. Since $I, A, A^2, ..., A^n$ are also linearly independent, both $\{I, A, A^2, ..., A^n\}$ and $\{A_o, A_1, A_2, ..., A_n\}$ are bases for A. This property, in the undirected case, fully characterizes a distance regular graph [7], but is not a distance regular digraph [4].
  From the equations $A_k = p_o^k I + p_1^k A + ... + p_n^k A^n, 0 \leq k \leq n$, and the proposition above, we obtain the matrix

$$C = \left\{ \begin{array}{lllll} 1 & & & & \\ p_o^1 & 1 & & & \\ . & . & & & \\ p_o^{n-1} & p_1^{n-1}... & 1 & & \\ p_o^n & p_1^n & ... & p_{n-1}^n & 1 \end{array} \right\}$$

which is the transition matrix from $\{A_o, A_1, A_2, ..., A_n\}$ to $\{I, A, A^2, ..., A^n\}$, and hence the inverse transition matrix is (q_i^j), where $A^k = \sum_{i=0}^n q_i^k A_i$.

Proposition 3. *The number of walks of length k between any two vertices at distance j of $\Gamma_\delta(n)$ is q_j^k, $j \leq k \leq n$.*

Remark 3. In knowing the Hoffman polynomial of a cycle prefix digraph, one is able to determine its spectrum.

- Spectrum of $\Gamma_\delta(n)$

  Theorem 5.2. *Let $\Gamma_\delta(n)$ be the cycle prefix digraph of degree δ and diameter n and A its adjacency matrix. Then*

  1. *A diagonalizes.*
  2. *The eigenvalues of A are*

  $$\lambda_o = \delta, \lambda_1 = n-2, \lambda_2 = n-3, ..., \lambda_{n-1} = 0$$

  and

  $$\lambda_n = -1$$

  The above may be proved by means of the proof stated in [4] and it is a consequence of the ability to obtain the minimal polynomial of A via the Hoffman polynomial, which is $(x-\delta)(x+1)x...(x-(n-3))(x-(n-2))$.

6 Small worlds theories (see [2], for instance) and another evidence using our results

In real life, it is quite possible that someone we know is part of someone else's circle of acquaintances. This situation would originate, for instance, an Australian square with intersection different from the empty set. This is a case where the vertices of the starants will overlap. With this, there is one less step to go from one starant to the other and it is possible to reduce the distance between vertices to 2 instead of three. That would make possible to have a distance of 4 in each American square. That could be translated into having a diameter of n for n people. In considering symmetry, that would lead us to $\frac{n}{2}$ and a diameter of 50 for a network of 100 people. But symmetry happens both horizontally and vertically, and that could lead us to a diameter of $\frac{n}{4}$, 25 in our example. And because of the diagonals, one can see how 6 could be reached as an approximation to the result. But one can also see that the constraints involved, in order to get a result 6 have to be substantial. Of course, as Duncan states, it is 6 in average. We can say that it is very likely that someone that we know is also part of someone else's circle of acquaintances and we would expect them to be part of a certain number of circles in average, what could then reinforce their result. But we think that our objects are interesting per se and could be studied an applied like that.

7 Conclusion

In this paper, we have worked with new mathematical objects using the concepts of cycles and Starants. We introduced Australian squares and American squares. Our work seems to be, in a sense, on the back of Duncan's et al. reasoning, but it is still worth mentioning in order to open way to the study of graphical objects per se.

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