Starants

M.R.Pinheiro[1]

Abstract

Progressing further on the work of Comellas et al. into deterministic small-world networks, and intending a future new application of their graphs, we introduce a new form of graph: Starants.
keywords: Small-world, communication networks, networks, combinatorial problems.
AMS: 05C12.

1 Introduction

Deterministic small-world communication networks were introduced by Comellas, Ozon, and Peters in [1]. They are supposed to have strong local clustering (nodes have many neighbors in common), small diameter (largest of the shortest distances between nodes must be small), and would be located between Regular Lattices, which are highly clustered, large worlds, where the diameter, or characteristic path length, grows linearly with the number of nodes, and Random Networks, which are poorly clustered, small worlds, where the diameter grows logarithmically with the number of nodes. We shall name them 'medium worlds'.
Circulant graphs are considered part of the deterministic small-world communication networks, once they have strong local clustering, but large average distance between pairs of nodes. They are included in the class of structured networks.
In this paper, we want to contribute to Comellas et al. findings, which focused on replacing probabilistic models with deterministic small-world networks and non-random interconnection patterns, by adding some constraints to their variables plus proposing a new form of graph: Starants. Small diameters not only make calculations easier but are consistent with the concept of small-world networks besides being more relevant than average distance in the study of communication networks most of the time.
Following Comellas et al., our model will be more adequate for situations where the number of neighbors of a node must be fixed because of technical

[1]UNESA, RJ-BR. mrpprofessional@yahoo.com.

considerations.
Contrary to Comellas et al. work, though, we would like to consider any possible number of nodes for our graphs.
From Comellas et al. [1], we use:

Lemma 2. *Let S be a segment of $C_{n,\delta}$; δ even, with length $(k-1)\delta+1 < l_s \leq k\delta+1$, $k \geq 1$. The maximum distance between any node of S and one of the end nodes of S is k.*

Theorem 2.1. *Given $C_{n,\delta}$; δ even, and $D < D_{C_{n,\delta}}$, the number of hubs required to construct a new graph G with diameter at most $D_G \leq D$ from $C_{n,\delta}$ by using a graph H of diameter D_H to interconnect the hubs is*

$$\begin{cases} h_e = \lceil \frac{2n}{\delta(D-D_H)+2} \rceil & or \\ h_o = \lceil \frac{2(n-\delta)}{\delta(D-D_H-1)+2} \rceil \end{cases}$$

depending on $(D-D_H)$ being even (h_e) or odd (h_o).

Theorem 2.2. *Given $C_{n,\delta}$; δ even, let H denote a graph with h nodes and diameter D_H. There is a graph G with n nodes and h hubs (using graph H to interconnect the hubs) which has diameter $D_G \leq 2k+D_H$, where $k = \left\lceil \frac{\lceil \frac{n}{h} \rceil - 1}{\delta} \right\rceil$. If the condition $n - [(k-1)\delta+1](h-1) \leq k\delta+1$ is also satisfied, then the diameter is $D_G \leq 2k-1+D_H$.*

The above results rely on lemma 1 and Comellas et al. constraints are enough for them. We also use the result below [1] regarding clustering:

Proposition 1. *The clustering parameter of $C_{n,\delta}$ is*

$$C_{C_{n,\delta}} = \frac{3(\delta-2)}{4(\delta-1)}$$

This result relies on the fact that nodes i and $(i+j)$ have $\delta-(j+1)$ common neighbors, $1 \leq j \leq \frac{\delta}{2}$.

2.1 Notation & some definitions

1. $C_{n,\delta}$ - circulant graph of n nodes and δ (degree) links per node such that each node i is adjacent to nodes $(i \pm 1)$, $(i \pm 2)$, ..., $(i \pm \frac{\delta}{2} (mod\ \ n))$. This graph has got diameter $D = \lceil \frac{n}{\delta} \rceil$ whenever $\delta \neq 2$ and $D = \lfloor \frac{n}{2} \rfloor$ otherwise.

2. Star graph - rooted tree containing n nodes with a central node (root) of degree $(n-1)$.

3. Complete graph on n nodes - graph where every node has got degree $(n-1)$.

4. $S^C_{n,\delta}$ - string of n circulant graphs connected by means of K_2 exactly δ times for each circulant graph added (taking away the first and the last graph on the string which will use K_2 exactly $\frac{\delta}{2}$ times to make the connection): each vertex 'i' is connected to $\left(\frac{\delta}{2} - \lambda \pm i\right), 1 \leq \lambda \leq \frac{\delta}{2}$.

5. $C^C_{n,\delta}$ - circle of circulant graphs connected by means of K_2 exactly δ times for each graph added: each vertex 'i' is connected to $\left(\frac{\delta}{2} - \lambda \pm i\right), 1 \leq \lambda \leq \frac{\delta}{2}$.

6. S^C_n - string of n circulant graphs connected by means of K_2 as many times as we like for each circulant graph added.

7. C^C_n - circle of n circulant graphs connected by means of K_2 as many times as we like for each circulant graph added.

8. $SC_{n,\delta}$ - 'Starant' graph, that is, a circulant graph with degree δ containning a star inside of it whose vertices coincide with the vertices of the circulant graph, a total of n vertices.

9. $S^S_{n,\delta}$ - string of n star-graphs connected by means of K_2 exactly δ times for each star-graph added (taking away the first and the last graph on the string which will use K_2 exactly $\frac{\delta}{2}$ times to make the connection): each vertex 'i' is connected to $\left(\frac{\delta}{2} - \lambda \pm i\right), 1 \leq \lambda \leq \frac{\delta}{2}$.

10. $S^C_{n,\delta}$ - string of n circulant graphs connected by means of K_2 exactly δ times for each circulant graph added (taking away the first and the last graph on the string, which will use K_2 exactly $\frac{\delta}{2}$ times to make the connection): each vertex 'i' is connected to $\left(\frac{\delta}{2} - \lambda \pm i\right), 1 \leq \lambda \leq \frac{\delta}{2}$.

11. $C^{SC}_{n,\delta}$ - circle of n starant graphs connected by means of K_2 exactly δ times for each starant graph added: each vertex 'i' is connected to $\left(\frac{\delta}{2} - \lambda \pm i\right), 1 \leq \lambda \leq \frac{\delta}{2}$.

12. $C^{S}_{n,\delta}$ - circle of stars connected by means of K_2 exactly δ times for each star-graph added: each vertex 'i' will be connected to $\left(\frac{\delta}{2} - \lambda \pm i\right), 1 \leq \lambda \leq \frac{\delta}{2}$.

13. S^{S}_{n} - string of n star-graphs connected by means of K_2 as many times as we like for each star-graph added.

14. C^{S}_{n} - string of n star-graphs connected by means of K_2 as many times as we like for each star-graph added.

Section 2 introduces constraints on Comellas et al. work. Section 3 is about networks formed by circulant graphs. Section 4 discusses the choice of hubs in order to decrease the diameter of a network to a size D_G. Section 5 deals with clustering and regularity. Section 6 brings networks of star graphs. Section 7 is about reduction of diameter whilst, in section 8, we introduce the Starant graphs. Section 9 deals with clustering in respect to the newly created types of networks. The work in the paper ends with a summary of its contents and some prediction of future work.
In the next sections we make use of the following result:

Theorem 2.3. *The longest path that there may exist between two nodes of a circulant graph of degree* 2 *measures* $\lfloor \frac{n}{2} \rfloor$.

Proof. We use lemma 2 from [1] and the fact that when $\delta = 2$ the distance between nodes is maximum. See: $l_s = min(|i-j|, n-|i-j|)$ according to lemma 2 from [1]. This way, considering nodes i and j, and the vertices of our circulant graph to be labeled with the elements of the group $mod(n)$, plus assuming that $j > i$, we have:

$$\begin{cases} n-(j-i) \leq (j-i) \iff \frac{n}{2} \leq (j-i) \\ (j-i) < n-(j-i) \iff \frac{n}{2} > (j-i) \end{cases}$$

Therefore

$$\begin{cases} (j-i) \geq \frac{n}{2} \iff l_s = n-(j-i) \\ (j-i) < \frac{n}{2} \iff l_s = (j-i) \end{cases}$$

But

$$(j-i) \geq \frac{n}{2} \implies n-(j-i) \leq \frac{n}{2} \implies l_s \leq \frac{n}{2}$$

and

$$(j-i) < \frac{n}{2} \implies l_s < \frac{n}{2}$$

Therefore, $l_s = \frac{n}{2}$ is an upper bound for l_s. If n is even, take $k = \frac{n}{2} = \lfloor \frac{n}{2} \rfloor$ in lemma 2 from [1]; otherwise, take $k = \frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor$. In any case, one can say that $k = \lfloor \frac{n}{2} \rfloor$. □

3 Limiting δ

In [1], Comellas et al. assume that nodes i and $(i+j)$ have $\delta - (j+1), 1 \leq j \leq \frac{\delta}{2}$ common neighbors. Hence, when i and j are $\frac{\delta}{2}$ apart, they will have $(\frac{\delta}{2} - 1)$ neighbors.
One of the counter-examples to the above theory is $C_{9,6}$: any vertices n and $(n+3)$ should have 2 common neighbors but, in fact, they have 3 common neighbors.

Lemma 4. *If δ is the maximum degree that a vertex may have in the circulant graph C_n (C being allowed to be either regular or random), and n is the number of vertices in $C_n, \delta < \frac{2n}{3}$.*

Proof. If i and j have common neighbors to the right of i, for instance, there should not be common neighbors to its left. Analogously, when i and j have common neighbors to the left of i, there should not be common neighbors to its right, and the proof of one case is analogous to the other. We shall, therefore, consider just the first case in our proof. According to Comellas et al., $0 < (j-i) \leq \frac{\delta}{2}$. The situation to be excluded, taking into account our first supposition regarding neighborhood is, therefore, $(i-u)(mod \;\; n) \equiv (j+v)(mod \;\; n)$

where

$$0 < u, v \leq (j-i)(mod \;\; n)$$

what implies that

$$0 < (j-i+u+v)(mod \;\; n) \leq \frac{\delta}{2} + 2(j-i)(mod \;\; n)$$

what implies that

$$0 < (j-i+u+v)(mod \;\; n) \leq \frac{3\delta}{2}$$

finally implying that we should exclude the cases where

$$0 < kn \leq \frac{3\delta}{2}$$

what brings us to our constraint once the lowest value for k is 1. □

5 Networks formed by one set of circulant graphs

String of Circulant graphs

We connect them in a sequence: 'm' with '$(m+1)$', but we stop connecting them when n is reached. We make use of K_2 just once for each graph added, departing from the most extreme vertex in each graph in relation to the last chosen hub.

Theorem 5.1. *In this case, the new diameter is*

$$D_{S^C_{n,2}} = \sum_{m=1}^{n} \left\lceil \frac{n_m}{\delta_m} \right\rceil + (n-1)$$

Proof. The result is trivial, one just has to consider adding an external edge to the graph and recall that the longest of the shortest paths of a circulant graph is $\lceil \frac{n}{\delta} \rceil$. □

Circle of Circulant graphs

We connect the circulant graphs in a sequence, as before, but we add an extra connection, made with k_2, to join graph 'n' to graph '1'.

Theorem 5.2. *The diameter of a circle of circulant graphs is*

$$D_{C^C_{n,2}} = \left\lfloor \frac{\sum_{m=1}^{n} \lceil \frac{n_m}{\delta_m} \rceil + n}{2} \right\rfloor$$

Proof. Trivial. □

6 Diameter reduction

All proofs on hubs and reduction of diameter of [1] are based on the use of the results concerning segments:

Definition 1. - Segment is a graph induced by two consecutive hubs, considering all vertices and edges between them. Hubs for us, so far, were the chosen vertices in each star/circulant to be connected by means of k_2. From now on, hubs will be vertices chosen amongst the hubs used to connect graphs in a circular way to be inter-linked in a direct way using a graph of known diameter. Therefore, a segment will still be defined as a graph induced by two consecutive hubs.

Lemma 7. *If S is a segment of $C^C_{n,\delta}$; δ even; $2(k-1) < l_s \leq 2k+1$; $k \geq 1$ will mean that the maximum distance between any node of S and one of its final nodes (hub) is k.*

Proof. We know that the diameter of $C^C_{n,2}$ is

$$D_{C^C_{n,2}} = \left\lfloor \frac{\sum_{m=1}^{n} \lceil \frac{n_m}{\delta_m} \rceil + n}{2} \right\rfloor$$

and this is the maximum diameter that one can have for $C^C_{n,\delta}$, since the connection between any two circulants is minimum in this case. Therefore, since

$$\left\lfloor \frac{n'}{2} \right\rfloor = \left\lfloor \frac{\sum_{m=1}^{n} \lceil \frac{n_m}{\delta_m} \rceil + n}{2} \right\rfloor$$

$$= \begin{cases} p \text{ if } n' = 2p + R = l_s, 0 < R < 2 \\ p \text{ if } n' = 2p = l_s \end{cases}$$

$n' \leq 2k+1$ implies $k = p$, that is, k is the maximum distance between a node and an endpoint, as required. □

Theorem 7.1. *Take $D < D_{C^C_{n,2}}$. The number of hubs required to build a new graph G with diameter $D_G \leq D$ from $C^C_{n,2}$ using a graph of diameter D_H to interconnect the hubs is*

$$\begin{cases} h_e = \left\lceil \frac{\sum_{m=1}^{n} \lceil \frac{n_m}{\delta_m} \rceil + n}{D - D_H + 1} \right\rceil \text{ or} \\ h_o = \left\lceil \frac{\sum_{m=1}^{n} \lceil \frac{n_m}{\delta_m} \rceil + n - 2}{D - D_H} \right\rceil \end{cases}$$

depending on whether $(D - D_H)$ is even (h_e) or odd (h_o).

Proof. We have to split the proof into cases:

- CASE 1 : If $(D - D_H)$ is even, choose 'h' circulant graphs of $C^C_{n,2}$ to be hubs such that each segment S has got length $2(k-1)+1 < l_s \leq (2k+1)$ for some $k \leq 1$. Thus, the maximum distance from any node of C to a hub is k (lemma 1), provided that k is minimum, and the maximum distance between any two hubs of G is D_H. Let's worry about h now. Let $h = \left\lceil \frac{\sum_{m=1}^{n} \lceil \frac{n_m}{\delta_m} \rceil + n}{2k+1} \right\rceil$. All segments have length $(2k + 1)$ at most. This way, the diameter $D_G \leq 2k + D_H = D \implies k = \frac{D - D_H}{2} \implies h = \left\lceil \frac{\sum_{m=1}^{n} \lceil \frac{n_m}{\delta_m} \rceil + n}{D - D_H + 1} \right\rceil$.

- CASE 2 : If $(D - D_H)$ is odd, do

$$h = \left\lceil \frac{\sum_{m=1}^{n} \lceil \frac{n_m}{\delta_m} \rceil + n - 2}{2k + 1} \right\rceil$$

(this way, we will cover the whole graph minus $\frac{2}{2k+1}$). Thus, $(h - 1)$ segments will have length $(2k+1)$ and the remaining segment will have length $\sum_{m=1}^{n} \lceil \frac{n_m}{\delta_m} \rceil + n - (h - 1)(2k + 1)$. But if $h = \left\lceil \frac{\sum_{m=1}^{n} \lceil \frac{n_m}{\delta_m} \rceil + n - 2}{2k+1} \right\rceil$

then

$$h \geq \frac{\sum_{m=1}^{n} \lceil \frac{n_m}{\delta_m} \rceil + n - 2}{2k + 1}$$

or,

$$h(2k+1) \geq \sum_{m=1}^{n} \left\lceil \frac{n_m}{\delta_m} \right\rceil + n - 2$$

thus

$$h(2k+1) - (2k+1) \geq \sum_{m=1}^{n} \left\lceil \tfrac{n_m}{\delta_m} \right\rceil + n - 2 - (2k+1)$$

or

$$(h-1)(2k+1) \geq \sum_{m=1}^{n} \left\lceil \frac{n_m}{\delta_m} \right\rceil + n - 3 - 2k$$

what implies that

$$\sum_{m=1}^{n} \left\lceil \frac{n_m}{\delta_m} \right\rceil + n - (h-1)(2k+1) \geq \sum_{m=1}^{n} \left\lceil \frac{n_m}{\delta_m} \right\rceil + n - \sum_{m=1}^{n} \left\lceil \frac{n_m}{\delta_m} \right\rceil - n + 3 + 2k$$

that is, the length of the remaining segment is less than or equal to $3 + 2k = 2(k+1) + 1$. Well, $D \geq D_G \implies D = D_H + 2l_s \geq D_G$ according to our choice for $D - D_H$ even, but this will not work here. Now we need to have an odd number instead of $2l_s$. Since subtracting one may return 0, we add one unit to the result. This is why our last segment had to be less than $(k+1)$. Therefore, $(2k+1)$ is the maximum distance from point to hub, and

$$D_G \leq (2k+1) + D_H = D \implies k = \frac{D - D_H - 1}{2}$$

$$\implies \left\lceil \frac{\sum_{m=1}^{n} \lceil \frac{n_m}{\delta_m} \rceil + n - 2}{D - D_H} \right\rceil$$

.

□

Theorem 7.2. *Consider $C^C_{n,2}$, H being a graph with 'n' nodes ($h \leq n$) and diameter D_H. Consequently, there is a graph G with 'n' nodes and 'h' hubs (using H to connect hubs) which has diameter $D_G \leq 2k + D_H$ where*

$$k = \left\lceil \frac{\left\lceil \frac{\sum_{m=1}^{n} \lceil \frac{n_m}{\delta_m} \rceil + n}{h} \right\rceil - 1}{2} \right\rceil.$$

If the condition $\sum_{m=1}^{n} \lceil \frac{n_m}{\delta_m} \rceil + n - (2k-1)(h-1) \leq (2k+1)$ is also satisfied, the diameter is $D_G \leq 2k - 1 + D_H$.

Proof. Since there are h hubs, one can divide $C^C_{n,2}$ into segments of length $\left\lceil \frac{\sum_{m=1}^{n} \lceil \frac{n_m}{\delta_m} \rceil + n}{h} \right\rceil$ at most. Let $k \in Z$ such that

$$2k - 1 < \left\lceil \frac{\sum_{m=1}^{n} \lceil \frac{n_m}{\delta_m} \rceil + n}{h} \right\rceil \leq 2k + 1$$

and $k = \left\lceil \frac{\lceil \frac{\sum_{m=1}^{n} \lceil \frac{n_m}{\delta_m} \rceil + n}{h} \rceil - 1}{2} \right\rceil$. Build a graph G using H to interconnect hubs of $C^C_{n,2}$. By lemma 1, the distance between any node of G and a hub is k at most and the distance between any two hubs of G is D_H at most. This way, $D_G \leq 2k + D_H$. If

$$\sum_{m=1}^{n} \lceil \frac{n_m}{\delta_m} \rceil + n - (2k-1)(h-1) \leq 2k + 1,$$

we can choose the hubs such that $(h-1)$ of the segments have maximum length $(2k-1)$ and the remaining segment has maximum length $2k+1$. by lemma 1, the distance to a hub is $(k-1)$ at most for every node in $(h-1)$ of the segments and k at most for all nodes of the remaining segment. This gives $D_G \leq D_H + 2k - 1$. □

8 Clustering and regularity

Theorem 8.1. *The clustering parameter of $C^C_{n,\delta}$ is*

$$C_{C^C_{n,\delta}} = \frac{\frac{3}{2}\left(\frac{\sum_{m=1}^{n} \delta_m (\frac{\delta_m}{2} - 1)}{\sum_{m=1}^{n} \delta_m (\delta_m - 1)} + \frac{\frac{\delta}{2} - 1}{\delta - 1} \right)}{n+1}$$

where δ_m stands for degree of graph m from the circle and δ stands for the degree of the circle itself.

Proof. Since the number of triangles containing node 'i' is still $\frac{3}{4}\delta(\frac{\delta}{2}-1)$ for each circulant internally, but also for each hub, we just apply the concept from [1] to calculate our C_v, C_v being the quotient of the number of edges between a vertex neighbors and the possible number of edges. This way, for each circulant,

$$C_v^C = \frac{\frac{3}{4}\delta(\frac{\delta}{2}-1)}{\frac{\delta(\delta-1)}{2}} = \frac{3}{2}\left(\frac{\frac{\delta}{2}-1}{\delta-1}\right)$$

and for the circle of circulants, we consider the average of C_v over all vertices, that is:

$$C_{C^C_{n,\delta}} = \frac{\frac{\sum_{m=1}^{n}\frac{3}{4}\delta_m(\frac{\delta_m}{2}-1)}{\sum_{m=1}^{n}\frac{\delta_m}{2}(\delta_m-1)} + \frac{3}{2}\left(\frac{\frac{\delta}{2}-1}{\delta-1}\right)}{n+1}$$

that is

$$C_{C^C_{n,\delta}} = \frac{\frac{3}{2}\left(\frac{\sum_{m=1}^{n}\delta_m(\frac{\delta_m}{2}-1)}{\sum_{m=1}^{n}\delta_m(\delta_m-1)} + \frac{\frac{\delta}{2}-1}{\delta-1}\right)}{n+1}$$

□

9 Networks formed by one set of star graphs

String of stars

We connect them in a sequence, making use of K_2 once for each graph added (but the last one) in a regular way, joining the furthest vertices from one graph to the furthest ones in the other.

Theorem 9.1. *In this case, the new diameter is*

$$D_{S^S_{n,2}} = 2n + (n-1) = 3n - 1$$

Proof. Trivial. □

Circle of Stars

We build a string of stars and connect the last star-graph to the first one.

Theorem 9.2. *In this case, our new diameter is*

$$D_{C^S_{n,2}} = \left\lfloor \frac{3n}{2} \right\rfloor$$

Proof. Trivial. ☐

10 Reduction of diameter

It is very important to recover the concept of segment for sets of stars and circulants.

Definition 2. We called hubs the chosen vertices in each star/circulant in terms of connecting it to K_2. We now re-define hubs to be vertices chosen to be inter-linked in a direct way using a graph of known diameter. This way, a segment is still a graph induced by two consecutive hubs.

Lemma 11. *If S is a segment of $C^S_{n,\delta}$; δ even; $2(k-1) < l_s \leq 2k+1; k \geq 1$ means that the maximum distance between any node of S and one of its final nodes (hub) is k.*

Proof. The proof is the same as for the circle of circulants. ☐

Theorem 11.1. *Take $D < D_{C^S_{n,2}}$. The number of hubs required to build a new graph G with diameter $D_G \leq D$ from $C^S_{n,2}$ using a graph of diameter D_H to interconnect the hubs is*

$$\begin{cases} h_e = \lceil \frac{3n}{D-D_H+1} \rceil \text{ if } (D-D_H) \text{ is even} \\ h_o = \lceil \frac{3n-2}{D-D_H} \rceil \quad \text{if } (D-D_H) \text{ is odd} \end{cases}$$

Proof. We again have to split the proof in cases:

1. CASE 1: If $(D - D_H)$ is even, choose 'h' nodes of $C^S_{n,2}$, each node being a circulant (and we can always choose them such that the length of S is as required) to become hubs such that each segment S has got length $2(k+1)+1 < l_s < (2k+1)$ for some $k \leq 1$. Thus, the maximum distance from any node G to a hub is k (lemma 1), provided that k is minimum, and the maximum distance between any two hubs of G is D_H. Let's determine the value of h now. Let $h = \lceil \frac{3n}{2k+1} \rceil$. All segments have length $(2k+1)$ at most. This way, the diameter $D_G \leq 2k + D_H = D \implies k = \frac{D-D_H}{2} \implies h = \lceil \frac{3n}{D-D_H+1} \rceil$.

2. CASE 2: If $(D - D_H)$ is odd, do $h = \lceil \frac{3n-2}{2k+1} \rceil$ (this way, we will cover the whole graph minus $\frac{2}{2k+1}$). Thus, $(h-1)$ segments will have length $(2k+1)$ and the remaining segment will have length $3n-(h-1)(2k+1)$. That is,

$$D_G \leq (2k+1) + D_H = D$$

$$k = \frac{D - D_H - 1}{2} \implies h = \left\lceil \frac{3n-2}{D - D_H} \right\rceil$$

□

Theorem 11.2. *Consider $C^S_{n,2}$, H being a graph with 'n' nodes and 'h' hubs (using H to connect hubs) which has diameter $D_G \leq 2k + D_H$ where $k = \lceil \frac{\lceil \frac{3n}{h} \rceil - 1}{2} \rceil$. If the condition*

$$3n - (2k-1)(h-1) \leq 2k+1$$

is also satisfied, the diameter is $D_G \leq 2k - 1 + D_H$.

Proof. The proof follows similar reasoning to the one applied in the previous theorem. □

12 Starant Graphs

Definition 3. A Starant graph is a circulant graph with n vertices and degree δ with a star of n vertices inserted in it where both the vertices from the star graph and the vertices from the circulant graph coincide.

Theorem 12.1. *The diameter of a starant graph with degree up to $(n-2)$ is equal to 2.*

Proof. There is a single vertex to which a vertex x is not directly connected to. Let's call it y. It is trivial to infer that there will be two steps until one can reach vertex y. □

Lemma 13. *The only way of decreasing the diameter of a starant graph is having $\delta = n - 1$.*

Proof. The worst spaced vertices in a circulant graph are spaced by $\lfloor \frac{n}{2} \rfloor$. There is only one way of making one step from vertex m to vertex $(m + \lceil \frac{n}{2} \rceil)$. This way is having $\frac{\delta}{2} = \lfloor \frac{n}{2} \rfloor$. But this will imply that $\delta \geq n$, what can only mean that $\delta = n$ since δ cannot overcome n. □

Networks formed by one set of starant graphs

String of Starants

Theorem 13.1. *In this case, the new diameter is* $(3n-1)$.

Proof. If each starant graph has got diameter equal to 2 and we make $(n-1)$ insertions of K_2, the result will follow naturally. □

Circle of Starant graphs

Theorem 13.2. *In this case, the new diameter is* $D_{C_n^{SC}} = \lfloor \frac{3n}{2} \rfloor$.

Proof. Trivial. □

14 Clustering

The notion of clustering is a little bit intricate in our case since it involves each circulant/star-graph, plus the clustering of their ring. The same combinatorial point of view from [1] can be adopted here, though.

15 Circle of Starant graphs

Lemma 16. *Each Starant graph has clustering parameter equal to*

$$\frac{3}{4}\left(\frac{\delta - \frac{2}{3}}{\delta + 1}\right)$$

Proof. Since we now have a circulant graph plus a star-graph inserted in it, each vertex i will have one more neighbor and we just have to add one to the vertices in common that vertex i and vertex $(i+j)$ will have. This way, the sum of all triangles to which vertex i belongs will give us

$$\sum_{j=1}^{\frac{\delta}{2}} (\delta - j) = \frac{3}{4}\delta\left(\frac{\delta}{2} - \frac{1}{3}\right)$$

The number of possible connections between neighbors of vertex i is now $\frac{(\delta+1)\delta}{2}$ and the ratio between both of them will give us the clustering, as defined by Comellas et al. in [1], that is

$$C_{SC_{n,\delta}} = \frac{\sum_{j=1}^{\frac{\delta}{2}}(\delta - j)}{\frac{(\delta+1)\delta}{2}} = \frac{\frac{3}{4}\delta\left(\frac{\delta}{2}-\frac{1}{3}\right)}{\frac{(\delta+1)\delta}{2}} = \frac{3}{4}\left(\frac{\delta-\frac{2}{3}}{\delta+1}\right)$$

□

Theorem 16.1. *The clustering parameter of a circle of starant graphs is*

$$C_{C^{SC}_{n,\delta}} = \frac{\frac{3}{2}(\sum_{m=1}^{n}\frac{\frac{\delta_m}{2}-\frac{1}{3}}{\delta_m+1} + \frac{\frac{\delta}{2}-1}{\delta-1})}{n+1}$$

Proof. Considering that each vertex i, from a starant graph, has $\frac{3}{4}\delta(\frac{\delta}{2}-\frac{1}{3})$ triangles to which it belongs, and each hub from the circle will obey the rule for circulant graphs, that is, will have $\frac{3}{4}\delta(\frac{\delta}{2}-1)$ triangles to which it belongs, and that each starant has got $\frac{(\delta+1)\delta}{2}$ possible edges amongst its neighbors, whilst our circle will have $\frac{\delta(\delta-1)}{2}$ possible edges amongst its neighbors, just like the usual circulant graphs, the result for C_v, as explained earlier, follows naturally. See: for the circle itself, we have

$$C_v^C = \frac{\frac{3}{4}\delta(\frac{\delta}{2}-1)}{\frac{\delta(\delta-1)}{2}} = \frac{3}{2}\left(\frac{\frac{\delta}{2}-1}{\delta-1}\right)$$

whilst, for each starant graph, we have:

$$C_v^{SC} = \frac{\frac{3}{2}(\frac{\delta}{2}-\frac{1}{3})}{\delta+1}$$

The only information we find about the subject in [2] is that C_v is the fraction of allowable edges that actually exist and C is the average of C_v over all v. This is not enough since we are now dealing with a set of starants in a circulant arrangement. Since we have to consider all starants in our counting, it seems that taking the average to be our C_v makes sense. This way, for the circle of n starant graphs, we have:

$$C_{C^{SC}_{n,\delta}} = \frac{\sum_{m=1}^{n} C_{v_m}^{SC} + C_v^C}{n+1}$$

that is

$$C_{C^{SC}_{n,\delta}} = \frac{\frac{3}{2}(\sum_{m=1}^{n} \frac{\frac{\delta_m}{2} - \frac{1}{3}}{\delta_m + 1} + \frac{\frac{\delta}{2} - 1}{\delta - 1})}{n+1}$$

□

17 Conclusion

In this paper, we have investigated all possible string and ring situations with starants. Future work will deal with regularity. We also intend to discuss the applicability of starant graphs to represent people's circle of acquaintances, plus develop a medium-world theory to account for this, which will include work on the effects of subtracting edges from a starant graph and having non-regular ones.

References

[1] F. Comellas, J.G.Peters, & J. Ozon. *Deterministic small-world communication networks*. Information Processing Letters 76(83-90), 2000.

[2] S.H. Strogatz & D.J. Watts. *Collective dynamics of 'small-world' networks*. Nature(393), 1998.