

# S-convexity of balls and fixed-point theorems for mappings with nonexpansive square - ethical plagiarism and extension

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*Abstract:* – The purpose of this paper is to extend the previous results by Dr. K. Goebel, published by *Compositio Mathematica*, to *S*-convexity.

*Key-words*<sup>1</sup>: convex, S-convex, space, set, sequence

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## 1 Introduction

Our introduction gets developed via four major logical trends:

1. Basic definitions;
2. Basic symbology;
3. Sources used in this text;
4. Generalized index of sections for this paper.

In what follows, we simply try to organize the paper presentation with clarity.

1. Basic definitions:

The concept of *S*-convexity is split into two notions, which are described below [6]:

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**Definition 1.** A function  $f : X \rightarrow \mathfrak{R}$  is said to be  $S$ -convex in the first sense if

$$f(\lambda x + (1 - \lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1 - \lambda^s) f(y),$$

all  $x, y \in X$  and  $\forall \lambda \in [0, 1]$ , where  $X \subset \mathfrak{R}_+$ .

**Definition 2.** A function  $f : X \rightarrow \mathfrak{R}$  is said to be  $S$ -convex in the second sense if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y),$$

$\forall x, y \in X$  and  $\forall \lambda \in [0, 1]$ , where  $X \subset \mathfrak{R}_+$ .

2. Basic symbology:

- In this paper, we mean that  $f$  is an  $S$ -convex function in the first sense by saying that  $f \in K_s^1$ ;
- We use the same reasoning for a function  $g$ ,  $S$ -convex in the second sense, and say then that  $g \in K_s^2$ ;
- We name  $s_1$  the generic class constant for those functions that are  $S$ -convex in the first sense;
- We name  $s_2$  the generic class constant for those functions that are  $S$ -convex in the second sense;
- $B$  represents an arbitrary Banach space;
- $\| \cdot \|$  represents the norm for  $B$ ;
- $\theta$  is the zero element in  $B$ ;
- $x, y, z$  are elements in  $B$ ;
- $d(X)$  represents the diameter of a set  $X \subset B$ ;
- $K_r$  the ball with radius  $r$  and centered in  $\theta$ .

3. Sources used in this text as basis, including for symbols above, are basically [6] and [1].

4. The index for this paper includes:

- Introduction;
- Results regarding convexity;
- Extension of definitions from convexity to  $S$ -convexity and some basic results as well;
- Extension of main results;
- Application;
- Conclusion.

## 2 Results regarding convexity

The results below are mentioned in [1] as having originated from [2], [0], [4], and [2], respectively. The text below is not a faithful copy of the English words used there, however (only the mathematical symbols are cut and pasted).

**Definition 3.**  $B$  is called uniformly convex iff for every positive number  $\epsilon$ , there exists a positive number  $\delta$  such that for arbitrary  $x, y \in K_1$ , the inequality

$$\|x - y\| \geq \epsilon$$

implies

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

**Definition 4.**  $B$  has normal structure iff every bounded and convex subset  $X$ , of  $B$ , containing more than one point, contains a nondiametral point, i.e., a point such that  $\sup\{\|x - y\| : y \in X\} < d(X)$ .

**Definition 5.** The space is called uniformly non-square iff there is a positive number  $\delta$  such that there are no  $x, y \in K_1$  such that

$$\left\| \frac{x - y}{2} \right\| > 1 - \delta$$

and

$$\left\| \frac{x + y}{2} \right\| > 1 - \delta,$$

at the same time.

**Definition 6.**  $B$  is called strictly convex iff there are no segments laying on the boundary of  $K_1$ .

### 2.1 Modulus and Characteristic of Convexity

The definition and the results found below are mentioned in [1].

**Definition 7.** The modulus of convexity of  $B$  is the function  $\delta : [0, 2] \rightarrow [0, 1]$  defined by the following formula

$$\delta(\epsilon) = \inf \left[ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in K_1, \|x - y\| \geq \epsilon \right].$$

**Theorem 2.1.** *The function  $\delta(\epsilon)$  is nondecreasing.*

**Definition 8.** The characteristic of convexity of  $B$  is the number  $\epsilon_0 = \sup\{\epsilon : \delta(\epsilon) = 0\}$ .

### 3 Extension of definitions to $S$ -convexity

#### 3.1 To $s_1$ -convexity:

**Definition 9.**  $B$  is called uniformly  $s_1$ -convex iff for every positive number  $\epsilon$ , there exists a positive number  $\delta$  such that for arbitrary  $x, y \in K_1$ , the inequality

$$\|x - y\| \geq \epsilon$$

implies

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

**Definition 10.**  $B$  has normal structure iff every bounded and  $s_1$ -convex subset  $X$ , of  $B$ , containing more than one point, contains a nondiametral point, i.e., a point such that  $\sup\{\|x - y\| : y \in X\} < d(X)$ .

**Definition 11.** The space is called uniformly non-square iff there is a positive number  $\delta$  such that there are no  $x, y \in K_1$  such that

$$\left\| \frac{x - y}{2} \right\| > 1 - \delta$$

and

$$\left\| \frac{x + y}{2} \right\| > 1 - \delta,$$

at the same time.

**Definition 12.**  $B$  is called strictly  $s_1$ -convex iff there are no curves of the sort  $\lambda^s p_1 + (1 - \lambda^s) p_2$  laying on the boundary of  $K_1$ , where  $p_1$  and  $p_2$  are both points in  $B$ .

#### 3.2 To $s_2$ -convexity:

**Definition 13.**  $B$  is called uniformly  $s_2$ -convex iff for every positive number  $\epsilon$ , there exists a positive number  $\delta$  such that for arbitrary  $x, y \in K_2$ , the inequality

$$\|x - y\| \geq \epsilon$$

implies

$$\left\| \frac{x + y}{2^s} \right\| < 2 - \delta.$$

**Definition 14.**  $B$  has normal structure iff every bounded and  $s_2$ -convex subset  $X$ , of  $B$ , containing more than one point, contains a nondiametral point, i.e., a point such that  $\sup\{\|x - y\| : y \in X\} < d(X)$ .

**Definition 15.** The space is called uniformly non-square iff there is a positive number  $\delta$  such that there are no  $x, y \in K_1$  such that

$$\left\| \frac{x - y}{2} \right\| > 1 - \delta$$

and

$$\left\| \frac{x + y}{2} \right\| > 1 - \delta,$$

at the same time.

**Definition 16.**  $B$  is called strictly  $s_2$ -convex iff there are no curves of the sort  $\lambda^s p_1 + (1 - \lambda)^s p_2$  laying on the boundary of  $K_2$ , where  $p_1$  and  $p_2$  are both points in  $B$ .

It is a simple consequence of these definitions that every uniformly  $S$ -convex space is strictly  $S$ -convex and has got normal structure.

The modulus and characteristic of convexity is also the modulus and characteristic for  $s_1$ -convexity. However, for  $s_2$ -convexity, things must be adapted acc.. See:

**Definition 17.** The modulus of  $s_2$ -convexity of  $B$  is the function  $\delta : [0, 2] \rightarrow [0, 2]$  defined by the following formula

$$\delta(\epsilon) = \inf \left[ 2 - \left\| \frac{x+y}{2^s} \right\| : x, y \in K_2, \|x - y\| \geq \epsilon \right].$$

Trivially, the function  $\delta(\epsilon)$  is nonincreasing<sup>2</sup>.

### 3.3 To both:

**Definition 18.** The characteristic of  $S$ -convexity of  $B$  is the number  $\epsilon_0 = \sup\{\epsilon : \delta(\epsilon) = 0\}$ .

## 4 Extension of main results - $S$ -convexity

**Theorem 4.1.**  $B$  is uniformly  $S$ -convex then  $\epsilon_0 = 0$ .

*Proof.* Consider  $B$  to be uniformly  $S$ -convex. Then, either for every positive number  $\epsilon$ , there exists a positive number  $\delta$  such that for arbitrary  $x, y \in K_1$ , the inequality

$$\|x - y\| \geq \epsilon$$

implies

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \delta,$$

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<sup>2</sup>Some typo must have occurred at [1], at this stage

case  $B$  is  $s_1$ -convex, or for every positive number  $\epsilon$ , there exists a positive number  $\delta$  such that for arbitrary  $x, y \in K_2$ , the inequality

$$\|x - y\| \geq \epsilon$$

implies

$$\left\| \frac{x + y}{2^s} \right\| < 2 - \delta,$$

case  $B$  is  $s_2$ -convex. In both cases, simple manipulation of the consequences related to the choice of  $\epsilon$  will lead us to  $\delta(\epsilon) \geq \delta$ . If  $\delta(\epsilon) = 0$  then  $\delta = 0$ . Therefore, it must be the case that  $\epsilon = 0$ .  $\square$

**Theorem 4.2.** *If  $\epsilon_0 < 1$  then  $B$  has got normal structure.*

*Proof.* The proof is very similar to that for the convex case, small adaptations to the choice of  $z$  must be made, however.  $\square$

## 5 Fixed Point Theorems

The paragraph below is almost fully copied from [1]. Regardless of the value of the notion involved, we simply extend their results, or believe we have done so, according to the best of our knowledge.

Let  $C$  be a closed and  $S$ -convex subset of  $B$  and let  $F$  be a continuous transformation of  $C$  into  $C$ . By  $F^n$ , we denote the  $n$ th interaction of  $F$  and by  $I$  the identity transformation via  $C$ .

**Theorem 5.1.** *If  $F^2 = I$  and if for arbitrary  $x, y \in C$  we have*

$$(*) \|Fx - Fy\| \leq k \|x - y\|,$$

*$k$  being a constant such that*

$$(**) \frac{k}{2} \left( 1 - \delta\left(\frac{2}{k}\right) \right) < 1,$$

*then  $F$  has at least one fixed point.*

*Proof.* From [1], we take for granted that  $\|F(\frac{x+Fx}{2}) - x\| \leq \frac{k}{2} \|x - Fx\|$  and  $\|F(\frac{x+Fx}{2}) - Fx\| \leq \frac{k}{2} \|x - Fx\|$ . From both facts, we infer that

$$\left\| \frac{x + Fx}{2} - F\left(\frac{x + Fx}{2}\right) \right\| \leq \left( 1 - \delta\left(\frac{2}{k}\right) \right) \frac{k^2}{2} \|x - Fx\|.$$

By putting  $G = I + \frac{F}{2}$ , we get

$$\|G^2x - Gx\| \leq \left( 1 - \delta\left(\frac{2}{k}\right) \right) \frac{k^2}{2} \|x - Gx\|.$$

Hence, the sequence  $x_n = G^n x$  is convergent. If  $y = \lim x_n$ , then  $y \in C$  and  $y = Gy = Fy$ .

All remarks from [1] apply here as well. □

## 6 Conclusion

We here have managed to re-word a few parts of [1] to suit our taste, have extended their most relevant theorems to  $S$ -convexity, with the possible exception of the last theorem from [1], regarding Fixed Point, which shall be dealt with in future publications.

## 7 Bibliography

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