Exploring the concept of S-convexity

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Abstract: – The purpose of this paper is to distinguish, as much as possible, the concept of s-convexity from the concept of convexity and the concept of s-convexity in the first sense from the concept of s-convexity in the second sense. In this respect, the present work further develops a previous study by Hudzik and Maligranda (1994, [1]).

Key-words [1]*:* convex, s-convex, function

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1 Introduction

Recently, Hudzik and Maligranda ([1]) studied some classes of functions, the classes of s-convex functions. Although they claim, in their abstract, to be providing several examples and to be clarifying the idea introduced by Orlicz further, their work leaves plenty of room to build over the concept. Besides, their reference to Orlicz seems to be a bit mistaken and they seem to have created a new concept of convexity without noticing it.

The old conclusions presented here are:

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1. theoretical definitions of convex/s-convex functions;
2. Orliczs symbols are good to represent the classes of s-convex functions;
3. a theorem which acts as a generator of s-convex functions.

The new conclusions arisen from this paper are:

1. a rephrasing of the theoretical definitions of s-convex functions to look more similar to the definition of convex function;
2. an identity between the class of 1-convex functions and the class of convex functions;
3. a conjecture about the the looks of a $s-$convex function;
4. some theorems on functions that are s-convex in both senses;
5. a few other side results that might suit future work or are, at least, useful to clarify similarities and differences between functions that are s-convex in the first sense and the ones which are s-convex in the second sense.

The paper further tries to explain why the first s-convexity sense was abandoned by the literature in the field.
The paper is organized as follows: First, in section 2, we present the usual definition of both convex and s-convex functions. In section 3, we criticize the present presentation of definitions of s-convex functions. In section 4, we introduce Orliczs ways of referring to s-convex functions with views to have a more mathematical jargon to deal with them. In section 5, we re-write the definition of s-convex functions based on our new symbology. In section 6, we prove the equivalence between restrictions of convex functions and s-convex functions. In section 7, we present some consequences of the definition of s-convex functions. In section 8, we present some results on inequalities for s-convex functions. Section 9 brings reasons to why one must abandon the definition of s-convexity in the first sense and, possibly, presents a research question. Section 10 brings our conjecture whilst section 11 presents our conclusions.

2 The usual definition of convexity and s-convexity

The concept of convexity that is mostly cited in the bibliography is (as an example, [4]):

Definition 1. The function $(f : X \subset \Re - > \Re_f)^2$ is called convex if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

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[2] here, f means closure of $\Re$

holds $\forall \lambda \in [0,1]$, $\forall x, y \in X$ such that the right-hand side is well defined. It is called strictly convex if the above inequality strictly holds $\forall \lambda \in]0,1[$ and for all pairs of distinct points $x, y \in X$ with $f(x) < \infty$ and $f(y) < \infty$.

In some sources, such as [2], convexity is defined only in geometrical terms as being the property of a function whose graph bears tangents only under it. In their words,

Citation 1. f is called convex if the graph lies below the chord between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, and every linear function L, we have

$$sup_J(f - L) = sup_{\partial J}(f - L)$$

One calls f concave if $-f$ is convex.

The concept of s-convexity, on the other hand, is split into two notions which are described below with the basic condition that $0 < s \leq 1$. ([1])

Definition 2. A function $f : [0, \infty) - > \Re$ is said to be s-convex in the first sense if $f(ax+by) \leq a^s f(x) + b^s f(y)$, $\forall x, y \in [0, \infty)$ and $\forall a, b \geq 0$ with $a^s + b^s = 1$.

Definition 3. A function $f : [0, \infty) - > \Re$ is said to be s-convex in the second sense if $f(ax + by) \leq a^s f(x) + b^s f(y)$, $\forall x, y \in [0, \infty)$ and $\forall a, b \geq 0$ with $a + b = 1$.

3 What are the criticisms to the present definition of s-convexity?

- It seems that there is lack of objectivity in the present definition of s-convexity for there are some redundant things;

- It takes us a long time, the way the definition is written now, to work out the true difference between convex and s-convex functions;

- So far, we did not find references, in the bibliography, to the geometry of an s-convex function, what, once more, makes it less clear to understand the difference between an s-convex and a convex function whilst there are clear references to the geometry of the convex functions.

4 Proposed Symbology ([3])

- In this paper, we mean that f is an s-convex function in the first sense by saying that $f \in K_s^1$;

- We use the same reasoning for a function g, s-convex in the second sense and say then that $g \in K_s^2$;
- We name s_1 the generic class constant for those functions that are s-convex in the first sense;
- We name s_2 the generic class constant for those functions that are s-convex in the second sense.

5 The first few new results

5.1 Re-writting the definition of s-convex function

It is trivial to prove that $a, b \in [0, 1]$ is a consequence of the present definition of s-convexity.

Lemma 5.1. *If* $f \in K_s^1$ *or* $f \in K_s^2$ *then*

$$f(au + bv) \leq a^s f(u) + b^s f(v)$$

with $a, b \in [0, 1]$, *exclusively.*

Proof. We present the proof for K_s^1 only, since the proof for K_s^2 is analogous.
For K_s^1: We first prove that it is not the case that $a > 1$ and $b > 1$. Supposing that it is the case that $a > 1$ and $b > 1$, that implies having

$$a = 1 + \epsilon$$

$$b = 1 + \delta$$

$$a^s + b^s = 1, 0 < s \leq 1$$

Therefore,

$$(1 + \epsilon)^{\frac{1}{n}} + (1 + \delta)^{\frac{1}{n}} = 1, 1 \leq n < +\infty$$

As $x^{\frac{1}{n}}$ is a decreasing function of n, for $x > 1$, and, as $n- > +\infty$, the above result is not verified, being $a^s + b^s > 1$, $\urcorner(a > 1 \wedge b > 1)$.
Secondly, we prove that it is not the case that $a > 1$ and $b < 1$, or vice-versa, just by re-analyzing the previous case again. Therefore, $\urcorner(a < 1 \wedge b > 1) \wedge \urcorner(a > 1 \wedge b < 1)$.
Thirdly, we conclude that it must be the case that $(a \leq 1 \wedge b \leq 1)$. But since the definition of s-convexity uses $a, b \geq 0$, we have that

$$a, b \subset [0, 1]$$

□

With this, we may re-write the definitions of s-convexity in each of the senses as being:

Definition 4. A function $f : X- > \Re$ is said to be s-convex in the first sense if $f(\lambda x + (1 - \lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1 - \lambda^s) f(y)$, $\forall x, y \in X$ and $\forall \lambda \in [0, 1]$ where $X \subset \Re_+$.

Definition 5. A function $f : X- > \Re$ is said to be s-convex in the second sense if $f(\lambda x + (1 - \lambda) y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)$, $\forall x, y \in X$ and $\forall \lambda \in [0, 1]$ where $X \subset \Re_+$.

6 The classes K_1^1, K_1^2, and convex coincide when the domains are restricted to $\Re_+$

Theorem 6.1. *The classes K_1^1, K_1^2, and convex are equivalent when the domain is restricted to $\Re_+$.*

Proof. Just a matter of applying the definitions. ☐

Natural implication: All $1-$convex functions are convex.

7 Some natural consequences of the definition of s-convex functions

Theorem 7.1.

$$f \in K_s^1 \implies f\left(\frac{u+v}{2^{\frac{1}{s}}}\right) \leq \frac{f(u) + f(v)}{2}$$

Proof. Simply consider the case where $a^s = b^s = \frac{1}{2}$. ☐

Theorem 7.2.

$$f \in K_s^2 \implies f\left(\frac{u+v}{2}\right) \leq \frac{f(u) + f(v)}{2^s}$$

Proof. Simply consider the case where $a = b = \frac{1}{2}$. ☐

Theorem 7.3. *For a function that is both s_1 and s_2-convex, there is a perfect bijection between the set of (a's,b's) used in s_1 and the set of (a's, b's) used in s_2.*

Proof. Each a may be written as an a_1^s and each b as a b_1^s and vice-versa. This happens because $a, b \in [0, 1]$, $s \in [0, 1]$(each $\frac{1}{s}$-root in $(0, 1)$ will give us a number in $(0, 1)$). ☐

Theorem 7.4. *If a function belongs to both* K_s^1 *and* K_s^2*, then*

$$f(a_1u+b_1v)\leq a_1^sf(u)+b_1^sf(v)\leq a_2^sf(u)+b_2^sf(v)$$

for some $\{a_1,b_1,a_2,b_2\}\subset[0,1]$ *and such that it occurs to each and all of them.*

Proof. It follows from the bijection proved before. For each a_2, b_2 such that $a_2+b_2=1$, it corresponds a_1, b_1 such that $a_1^s+b_1^s=1$ and $a_2\geq a_1$, $b_2\geq b_1$ since $\{a,b\}\subset[0,1]$. $\square$

Theorem 7.5. *If a function belongs to both* K_s^1 *and* K_s^2 *and its domain coincides with its counter-domain then the composition* $f(f)$ *is* s_1^2*-convex.*

Proof. $f(a_1u+(1-a_1^s)^{\frac{1}{s}}v)\leq a_1^sf(u)+(1-a_1^s)f(v) \implies f(a_1^sf(u)+(1-a_1^s)f(v))\leq(a_1^s)^sf(f(u))+(1-a_1^s)^sf(f(v))=a_2^sf(f(u))+b_2^sf(f(v))$ $\square$

8 Some results in inequalities for s-convex functions

Consider the following functions:

- $F:[0,1]->\Re$,

  $$F(t)=\frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)dxdy$$

  ([5])

- $p:\Re->[0,+\infty]$ such that

  $$p(\lambda x+(1-\lambda)y)\leq p(x)+p(y),\forall\lambda\in(0,1),x,y\subset\Re$$

  (p-function).

Theorem 8.1. *Consider the function* F*, as described above, where* $f:[a,b]->\Re$ *is* s_1-*convex. Then:*

1. F *is* s_1-*convex in* $[0,1]$*;*
2. $2(b-a)F(0)=2(b-a)F(1)=\frac{2}{b-a}\int_a^b f(x)dx$ *is an upper bound for* $F(t)$*;*
3. $F(t)$ *is symmetric over* $t=\frac{1}{2}$.

Proof. 1. Take $\lambda, \beta \subset [0,1]$, $\lambda^s + \beta^s = 1$, $0 < s \leq 1$, $x_1, y_1 \subset D$, f being s_1-convex.

$$F(\lambda x_1 + \beta y_1) = \frac{1}{(b-a)^2} \int_a^b \int_a^b f(\lambda x_1 x + \beta y_1 x + (1 - \lambda x_1 - \beta y_1)y) dxdy$$

$$\leq \frac{1}{(b-a)^2} \int_a^b \int_a^b (\lambda^s f(x_1 x - x_1 y + y) + \beta^s f(y_1 x + y - y_1 y) dxdy$$

$$= \lambda^s F(x_1) + \beta^s F(y_1)$$

what implies that F is s_1-convex in $[0,1]$.

2.

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

implies that

$$A = \int_a^b \int_a^b f(tx + (1-t)y) dxdy \leq \int_a^b \int_a^b t^s f(x) dxdy + \int_a^b \int_a^b (1-t)^s f(y) dxdy$$

$$= t^s \int_a^b f(x)(b-a)dx + (1-t)^s \int_a^b f(y)(b-a)dy = t^s(b-a) \int_a^b f(x)dx + (1-t)^s(b-a) \int_a^b f(y)dy$$

$$= (t^s + (1-t)^s)(b-a) \int_a^b f(x)dx \leq (1 + (1-t)^s)(b-a) \int_a^b f(x)dx$$

since $t^s \leq 1$. Because $(1-t)^s \leq 1$ as well, this implies $A \leq 2(b-a) \int_a^b f(x)dx$.

3. $F(t)$ is symmetric about $t = \frac{1}{2}$ because $F(t) = F(1-t)$.

□

Consider now the following functions:

- $F_{g_1} : [0,1] -> \Re$,

$$F_{g_1}(t) = \int_a^b \int_a^b f(tx + (1-t)y) g(x) g(y) dxdy$$

  where $f : [a,b] \subset \Re_+ -> \Re$ is s_1-convex.

- $F_{g_2}(t) : [0,1] -> \Re$,

$$F_{g_2} = \int_a^b \int_a^b f(tx + (1-t)y) g(x) g(y) dxdy$$

  where $f : [a,b] \subset \Re_+ -> \Re$ is s_2-convex

Theorem 8.2. *One can say that:*

1. F_{g_1} *and* F_{g_2} *are both symmetric about* $t = \frac{1}{2}$*;*
2. F_{g_2} *is* s_2*-convex in* $[0, 1]$*;*
3. *We have the upper bound*

$$2F_{g_2}(1) = 2\int_a^b \int_a^b f(x)g(x)g(y)dxdy$$

Proof. 1. If we replace t with $(1-t)$ in the definition of F_{g_1} and F_{g_2}, we get the same result, thus both F_{g_1} and F_{g_2} are symmetric about $t = \frac{1}{2}$.

2.

$$F_{g_2}(\lambda t_1 + (1-\lambda)t_2) = \int_a^b \int_a^b f((\lambda t_1 + (1-\lambda)t_2)x + (1 - (\lambda t_1 + (1-\lambda)t_2))y)g(x)g(y)dxdy$$

But

$$(\lambda t_1 + (1-\lambda)t_2)x + (1 - (\lambda t_1 + (1-\lambda)t_2))y$$
$$= \lambda t_1 x + xt_2 - \lambda t_2 x + y - \lambda t_1 y - t_2 y + \lambda t_2 y = \lambda(t_1 x + y - t_1 y) + (1-\lambda)(t_2 x - t_2 y + y)$$

But, since f is s_2-convex,

$$\int_a^b \int_a^b f(\lambda(t_1 x + y - t_1 y) + (1-\lambda)(t_2 x - t_2 y + y))g(x)g(y)dxdy$$
$$\leq \int_a^b \int_a^b (\lambda^s f(t_1 x + y - t_1 y) + (1-\lambda)^s f(t_2 x - t_2 y + y))g(x)g(y)dxdy$$
$$= \lambda^s \int_a^b \int_a^b f(t_1 x + y - t_1 y)g(x)g(y)dxdy + (1-\lambda)^s \int_a^b \int_a^b f(t_2 x - t_2 y + y)g(x)g(y)dxdy$$
$$= \lambda^s \int_a^b \int_a^b f(t_1 x + (1-t_1)y)g(x)g(y)dxdy + (1-\lambda)^s \int_a^b \int_a^b f(t_2 x + (1-t_2)y)g(x)g(y)dxdy$$
$$= \lambda^s F_{g_2}(t_1) + (1-\lambda)^s F_{g_2}(t_2)$$

what proves that F_{g_2} is s_2-convex.

3.

$$F_{g_2}(t) = \int_a^b \int_a^b f(tx + (1-t)y)g(x)g(y)dxdy$$

$$\leq t^s \int_a^b \int_a^b f(x)g(x)g(y)dxdy + (1-t)^s \int_a^b \int_a^b f(y)g(x)g(y)dxdy$$

$$= (t^s + (1-t)^s) \int_a^b \int_a^b f(x)g(x)g(y)dxdy \leq 2 \int_a^b \int_a^b f(x)g(x)g(y)dxdy = 2F_{g_2}(1)$$

□

Theorem 8.3. *Consider a sum S of s_2-convex functions,*

$$S = \sum_{m=1}^{n} A_m(t)$$

where

$$A_m(t) = \int_a^b \int_a^b f_m(tx + (1-t)y)dxdy$$

In this context, we have:

1. *$Sup(S) = 2\sum_{m=1}^{n} A_m(0) = 2\sum_{m=1}^{n} A_m(1)$;*
2. *S is symmetric about $t = \frac{1}{2}$;*
3. *S is a p-function;*
4. *S is s_2-convex.*

Proof. 1. For each m,

$$A_m \leq t^s \int_a^b \int_a^b f_m(x)dxdy + (1-t)^s \int_a^b \int_a^b f_m(y)dxdy$$

since f_m is s-convex. Thus

$$A_m \leq [t^s + (1-t)^s] \int_a^b \int_a^b f_m(x)dxdy \leq 2 \int_a^b \int_a^b f_m(x)dxdy = 2A_m(0) = 2A_m(1)$$

2. For each m, $A_m(t) = A_m(1-t)$. Thus, A_m is symmetric about $t = \frac{1}{2}$. Thus, $S(1-t) = S(t)$ and S also is.

3. Take $\lambda \in (0,1)$

$$A_m(\lambda x + (1-\lambda)y) = \int_a^b \int_a^b f_m((\lambda x + (1-\lambda)y)x + (1-(\lambda x + (1-\lambda)y))y)dxdy$$

$$= \int_a^b \int_a^b f_m(\lambda x^2 + yx - \lambda yx + y - \lambda xy - y^2 + \lambda y^2)dxdy \leq \lambda^s A_m(x) + (1-\lambda)^s A_m(y) \leq A_m(x) + A_m(y)$$

$$\implies S(\lambda x + (1-\lambda)y) = \sum_{m=1}^{n} A_m(\lambda x + (1-\lambda)y) \leq \sum_{m=1}^{n} A_m(x) + \sum_{m=1}^{n} A_m(y) = S(x) + S(y)$$

Therefore, S is a p-function.

4. For each m, it is true that

$$A_m(\lambda x + (1-\lambda)y) \leq \lambda^s A_m(x) + (1-\lambda)^s A_m(y)$$

This way, it is true that

$$S(\lambda x + (1-\lambda)y) = \sum_{m=1}^{n} A_m(\lambda x + (1-\lambda)y) \leq \lambda^s \sum_{m=1}^{n} A_m(x) + (1-\lambda)^s \sum_{m=1}^{n} A_m(y)$$
$$= \lambda^s S(x) + (1-\lambda)^s S(y)$$

that is, S is s_2-convex.

□

9 Reasons to why the first s-convexity sense exposed by Hudzik and Maligranda got abandoned in the literature

Not much was written in regards to the definition of s-convexity in the first sense (see, for instance, [6]). There must be a reason for that.
First of all, it is not difficult to notice that if one considers $a = \frac{1}{4}, b = \frac{1}{4}, u = 0.5, v = 1$, for example, one gets that $au + bv = 0.125 + 0.25 = 0.375$. Therefore, if $s = \frac{1}{2}$, $au + bv$ would lie outside of the interval $[u, v]$. With this, the first sense of s-convexity, as mentioned by Hudzik and Maligranda, becomes a concept that can only be compared with the concept of convexity if some further restrictions are imposed to it. (Here, what we are taking into consideration is that in a convex function, the image considered in the definition is always inside of the interval, allowing us to have a clear geometrical comparison between that convexity definition and the $s-$convexity)

10 A new conjecture

Taking into account the relationship between a^s and a, we may wonder whether the following is true or not:

Conjecture 1. f is called s_2-convex, $s \neq 1$, if the graph lies below the 'bent chord' between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, and a special function L with a convexly[3] designed[4] curve of curvature ψ, corresponding to the figure $s_2 \in [0,1)$, we have

$$G(s) \geq sup_J(f-L) \geq sup_{\partial J}(f-L)$$

Proof. What we can prove, so far, regarding this conjecture, is that all the points of the curve below or above which our $s-$convex function graph should lie are above the chord drawn between the two images of our function under consideration, but for the initial and the final point of it. It is actually easy to see that in $f(au+bv)$, in order to have just $f(u)$ our a must be 1 and that will mean having $b=0$. In this event, the limiting curve will present $f(u)$. At the other end, the opposite happens, $a=0$ and $b=1$, giving $f(v)$ as part of the limiting curve. On the other hand, considering the definition of convex functions, when there is a chord between the images of u and v, it is easy to see that if $a+b=1$ $a^s \geq a$ and $b^s \geq b$, that is, all points of our limiting curve are above the chord if $s \neq 1$, but for the initial and for the final point. In order for our claim to be true, we still have to prove that our limiting designed curve[5] has got a maximal point and is ideologically smooth[6].

What we are actually doing is transforming a straight line into something else by raising the coefficients of the convex combination to an exponent s between zero and one. It seems, however, that the interval $[0,1]$ will be reduced to something between an $x>0$ and 1, the larger the exponent, the greater this x will be.

One can easily see that we could have a chord because of the expression $n(a^{n-1}f(u)+f(a)^{n-1}f(v))$ and the recollection that we fall into the same case as the convex functions. However, since a^s and b^s are above a and b, once $s \neq 1$, it is either the case that this chord would have a distance from $f(u)$ and $f(v)$ or it would actually reveal itself to be a convexly designed curve. Or it could be something totally different from any standard sort of curve. The inference that it would also be a chord would be implied from the fact that it is continuous in the interval, hence, differentiable. However, supposing it is not continuous, might lead to the desired result. In taking into account that all the a's and b's are being increased by some amount, one can

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[3] remember that a convex function has a concave curve

[4] here, we introduce a new concept, we call convexly designed curve any complete or incomplete curve, broken or continuous, which, in having its points connected in a smooth manner, gives a convex curve

[5] we again force a new definition here: we call designed curve the curve which is faulty by a few points or complete, imposing a single condition, that one is able to draw a continuous smooth function out of it

[6] we again impose a new definition: we call a curve ideologically smooth if by means of joining its points we can achieve a continuous smooth curve

easily understand that some gaps will appear. Therefore, our aimed graphical idea would be a ‘designed’ curve, rather than a ‘curve’. In this case, we would have to prove that by being able to put its points together in a smooth way one could reach a convex curve. We could do this by means of proving that raising the a’s and b’s that form a chord to the same power between 0 and 1 we ☐

11 Conclusions

In this paper, we proved that s-convexity may be stated in a very similar way to convexity, as written below:

Definition 6. the function $(f : X- > \Re_f)$ [7] is called convex if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

holds $\forall \lambda \in [0, 1]$, $\forall x, y \in X$.

For $0 < s_1, s_2 \leq 1$,

Definition 7. A function $f : X- > \Re$ is said to be s_1-convex if the inequality

$$f(\lambda x + (1-\lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1-\lambda^s)f(y)$$

holds $\forall \lambda \in [0, 1]$, $\forall x, y \in X$ such that $X \subset \Re_+$.

Definition 8. A function $f : X- > \Re$ is said to be s_2-convex if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds $\forall \lambda \in [0, 1]$, $\forall x, y \in X$ such that $X \subset \Re_+$.

The own re-definition of s-convexity included our way of referring to s-convex functions by creating class-like symbology for them:

- K_s^1 for the class of s-convex functions in the first sense, some s;
- K_s^2 for the class of s-convex functions in the second sense, some s;
- K_0 for the class of convex functions;
- s_1 for the constant s, $0 < s \leq 1$, used in the first definition of s-convexity;

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[7] here, f means closure of $\Re$

- s_2 for the constant s, $0 < s \leq 1$, used in the second definition of s-convexity. thirdly, we pointed out that the class of 1-convex functions is just a restriction of the class of convex functions, that is, when $X = \Re_+$,

$$K_1^1 \equiv K_1^2 \equiv K_0$$

In fourth, we introduced the following side-theorems:

Theorem 11.1. *For a function that is both s_1 and s_2-convex, there is a perfect bijection between the set of (a's,b's) used in s_1 and the set of (a's, b's) used in s_2.*

Theorem 11.2. *If a function belongs to both K_s^1 and K_s^2, then*

$$f(a_1u + b_1v) \leq a_1^s f(u) + b_1^s f(v) \leq a_2^s f(u) + b_2^s f(v)$$

for some $\{a_1, b_1, a_2, b_2\} \subset [0, 1]$ obeying K_s^1 and K_s^2 rules, and such that it occurs to each and all of them.

Theorem 11.3. *If a function belongs to both K_s^1 and K_s^2 and its domain coincides with its counter-domain then the composition $f(f)$ is s_1^2-convex.*

In fifth we bring our conjecture as a prospective future work

Conjecture 2. f is called s_2-convex, $s \neq 1$, if the graph lies below the 'bent chord' between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, and every smoothly graphed function L with convex curve (remember that a convex function has a concave curve), we have

$$G(s) \geq sup_J(f - L) \geq sup_{\partial J}(f - L)$$

along with the issue as to what restrictions may be imposed upon the definition of s_1-convexity as to make the concept closer to both the concept of s_2-convexity and convexity.

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