S-convexity revisited (fuzzy)

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Abstract

In this revisional article, we criticize (strongly) the use made by Medar (et al), and those whose work they base themselves on, of the name 'convexity' and definitions which intend to relate to convex functions, or cones, or sets, but actually seem to contradict the original definitions. We then believe to have fixed the 'denominations', associated with Medar's (et al) work, to match the existing literature in the field which precedes his work (by long). We also expand his work scope by introducing s_1-convexity concepts to his group of definitions, which encompasses only convex and its proper extension, s_2-convex, so far.

Key-words: convex, $S-$convex,s_1-convex, s_2-convex, process, processes, function, $s-$convex, fuzzy.

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1 Introduction

The notion of convex processes seem to have been defined in 1967 by Rockafellar ([7], [8]). Convex processes were, back then, defined as being set-valued maps whose graphs were closed convex cones. Therefore, Rockafellar should be starting the terminology and definitions involved in the work of Medar (et al).

Medar et. al. [12] mention the benefits of this sort of theory for Optimization purposes (with reference to ([3], [4], [9], [10], [11])). Convex cones are defined as sets of points containing all linear combinations, with coefficients in $\Re_+$, of their points. With this, there is straight analogy with the concept of convex functions, once if the frontier is guaranteed to belong to the function, then the inequality will always be verified.

However, Medar (et al) seem to be stating something different whilst making use of the same denominations, what is making their work 'look' unsound. He seems to agree, however, with [13].

We here, then, intend to help Medar (et al) to reach coherence with the scholarship in the field.

Once the definition of convex fuzzy processes already besets revision, $S-$convex fuzzy processes are obviously mistakenly defined as well. Breckner has introduced $S-$convex functions as a generalization for convex functions in 1978, according to [12], what must mean Breckner has 'ownership' of its name. We here make use of our later work, based also on Breckner's result (as well as others'), work from 2001 ([5]).

In this work, we revise the fuzzy version of Breckner's definitions, proposed by Medar (et al), making use of our developments ([5]) of the work by Hudzik and Maligranda (1994, [2]).

2 Notations and Definitions

2.1 Notations

We use the symbology defined in [5]:

- K_s^1 for the class of $S-$convex functions in the first sense, some s;
- K_s^2 for the class of $S-$convex functions in the second sense, some s;
- K_0 for the class of convex functions;
- s_1 for the variable S, $0 < s_1 \leq 1$, used for the first type of S-convexity;
- s_2 for the variable S, $0 < s_2 \leq 1$, used for the second type of s-convexity.

  Remark 1. The class of 1-convex functions is simply a restriction of the class of convex functions, which is attained when $X = \Re_+$,

$$K_1^1 \equiv K_1^2 \equiv K_0.$$

2.2 Definitions

We use the definitions presented in [5]:

Definition 3. *A function $f : X -> \Re$ is said to be s_1-convex if the inequality*

$$f(\lambda x + (1-\lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1-\lambda^s) f(y)$$

holds $\forall \lambda \in [0,1]$; $\forall x, y \in X$; $X \subset \Re_+$.

Remark 2. If the complementary concept is verified, then f is said to be s_1-concave.

Definition 4. *A function $f : X- > \Re$ is called s_2-convex, $s \neq 1$, if the graph lies below a 'bent chord' (L) between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, it is true that*

$$sup_J(L-f) \geq sup_{\partial J}(L-f).$$

Definition 5. *A function $f : X- > \Re$ is said to be s_2-convex if the inequality*

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds $\forall \lambda \in [0,1]$; $\forall x, y \in X$; $X \subset \Re_+$.

Remark 3. If the complementary concept is verified, then f is said to be s_2-concave.

6 Preliminaries

6.1 Fuzzy Process

From [12], we learn that:

Definition 7. *Any application $F : \Re^m- > F(\Re^n)$ is called a fuzzy process.*

Definition 8. *A fuzzy process $F : \Re^m- > F(\Re^n)$ is called convex if it satisfies the following relation*

$$F\{(1-a)x_1 + ax_2\}(y) \geq sup_{y_1,y_2:(1-a)y_1+ay_2=y} \quad min\{F(x_1)(y_1), F(x_2)(y_2)\}$$

for all $x_1, x_2 \in \Re^m, a \in (0,1)$*, and* $y \in \Re^n$*.*

Remark 4. Problem with this definition is the conflict with the convex pre-existing definitions by the time Medar (et al) come up with this one, or Syau ([10]). If ever thinking of analogy with convex functions, Medar (et al) would be doing something similar to

$$F\{(1-a)x_1+ax_2\}(y) \leq (1-a)F(x_1)(y_1)+aF(x_2)(y_2) \leq F(x_1)(y_1)+F(x_2)(y_2),$$

for all $x_1, x_2 \in \Re^m$, $a \in [0,1]$, and $y \in F(\Re^m)$.
The choice of the last right bound, however, implies similarity with $p-$functions, instead. Therefore, it would make no sense to use the name 'convex' when there is better choice.
On the other hand, if one chooses to take the minimum between those measures, or sets, instead of the sum, it must be another concept, totally different from convexity.
If the support line was intended, once Medar (et al) mention support in his original work, admitting there was a typo, taking [1] as line of reasoning, then they mean the supporting line taking the minimum between the two measures as a basis. However, this is simply a consequence of the fact that the function is convex, not a definition, or an 'iff' theorem.
This way, the mention to the expression 'convex' is still not appropriate.
Alternatively, if Medar (et al) think of convex cones, it is then necessary that $F(\Re^m)$ (which might coincide with $F(\Re^n)$, not sure, this is not our field, we can only deal with what we know with certainty, even though the whole set of symbols used by Medar and collaborators is not making much sense) has got all its linear combinations, formed departing from positive coefficients,

contained in the own image of the function. This way, picked a $y_1 \in F(\Re^m)$ (or $F(\Re^n)$, if ever coinciding with $F(\Re^m)$), and a $y_2 \in F(\Re^m)$, there should be y, which is the response of F for at least one point of its domain. Supposing that this is what Medar (et al) mean, it is not only missing mentioning that both y_1 and y_2 belong to the image of F, but that the inequality is satisfied only if that y is found in the right side of it, independently of the looks of the element of the domain, so that it starts making sense. On the other hand, this is the only condition for a cone to be convex, so that the right side still looks a bit surreal.

Definition 9. *Let $s \in (0,1]$. A fuzzy process $F : C \subseteq \Re^m -> F(\Re^n)$ is said to be an $s-$convex fuzzy process on C if for all $a \in (0,1)$ and for all $x, y \in C$ it satisfies the condition*

$$(1-a)^s F(x) + a^s F(y) \subseteq F\{(1-a)x + ay\}$$

Remark 5. Once more, swapping the direction of the relation of inclusion would be advisable (at least). A bit confusing, as well, as to why they would now choose inclusion instead of linear comparison.

Definition 10. *Let $s \in (0,1]$. A fuzzy process $F : C \subseteq \Re^m -> F(\Re^n)$ is said to be an $s-$concave fuzzy process on C, if, for all $a \in (0,1)$ and for all $x, y \in \Re^m$ it satisfies the condition*

$$F\{(1-a)x + ay\} \subseteq (1-a)^s F(x) + a^s F(y)$$

Remark 6. Here, one notices that there is, once more, almost full analogy, or relationship, with the concept of s_2-concave functions, at the possible exception of the inclusion symbol, which is the opposite, in terms of direction, of what would be expected.

Theorem 10.1. *Let $F : C \subseteq \Re^m -> F(\Re^n)$ be a fuzzy process on C. Then, F is $s-$concave if and only if*

$$F((1-a)x_1 + ax_2)(y) \leq sup_{y_1,y_2:(1-a)^s y_1 + a^s y_2 = y} \quad min\{F(x_1)(y_1), F(x_2)(y_2)\}$$

for all $a \in (0,1)$ and for all $x, y \in C$.

Remark 7. All previous remarks apply here.

Theorem 10.2. *Let $F : C \subseteq \Re^m -> F(\Re^n)$ be a fuzzy process on C such that*

- $F(x+y) \subseteq F(x) + F(y)$;
- $F(tx) = t^s F(x)$.

  Then F is a $s-$concave fuzzy process on C.

Remark 8. That would clearly define s_2-convex fuzzy process on C, not $s-$concave, if names and similarities matter for anything in Mathematics, for in the above case:

$$F(tx + (1-t)y) \subseteq F(tx) + F((1-t)y) = t^s F(x) + (1-t)^s F(y).$$

11 Generalized conclusion

Both terminology and symbols used by Medar (et al), as well as his coherence in development, seems to be not good enough.

Basically, they seem to refer to intervals between x_1 and x_2, but make their 'a' belong to $(0,1)$, instead of $[0,1]$, for unclear reasons. On the other hand, they seem not to have decided between the relation of inclusion or the relation of comparison when defining their mathematical objects. One would suppose they should choose one of them only, and stick to that, especially because $S-$convexity is supposed to be an extension of the concept of convexity.

Another point is that they seem to also be mixing the notion of convex cones with the notion of convex functions whilst defining new concepts. They seem to create a very unlikely-to-exist tie between any linear combination of points in the image with mandatory linear combinations in the domain, if ever having the concept of convex cones in mind.

Last point refers to the possibility of Medar (et al) including s_1-convexity in their works as well, the same way they (seem to) apply s_2-convexity concepts, in analogy, to processes. In the latter case, an $S-$convex process, for instance, could as well read:

$$F\{(1-a^s)^{\frac{1}{s}}x + ay\} \leq (1-a^s)F(x) + a^sF(y).$$

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