S-convexity IV

Extension of Mazur's lemma to the concept of $S-$convexity

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Abstract

In this article, we extend one more set of inequalities, from convexity to $S-$convexity.

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1 Introduction

We seem to have developed the precursor and so honorable work of Profs Hudzik and Maligranda to a palatable level of suitability for applications in diverse areas by making their theory more foundational in the Pure scope of the Science. In this further work, we want to provide more tools to the mathematicians, as well as other scientists involved in research, which will allow them to make use of such a beautiful novelty (1994): $S-$convexity.

We now extend convexity to $S-$convexity for a very well known inequalities development refereed by very respected names in the field, in terms of modern Science.

Our suggested application regards Pure Mathematics derivations only. We use the symbols defined in [1], along with the definitions stated in the next subsection:

- K_s^1 for the class of $S-$convex functions in the first sense, some S;

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- K_s^2 for the class of $S-$convex functions in the second sense, some S;
- K_0 for the class of convex functions;
- s_1 for the constant s_1, $0 < s_1 \leq 1$, used in the first definition of S-convexity;
- s_2 for the constant s_2, $0 < s_2 \leq 1$, used in the second definition of S-convexity.

  Remark 1. We have also pointed out that the class of 1-convex functions is just a restriction of the class of convex functions, that is, when $X = \Re_+$,

$$K_1^1 \equiv K_1^2 \equiv K_0.$$

1.1 Definitions

We use the definitions presented in [2]:

- *S-convex sets*
  - *s_1-convex sets*

    Definition 1. Let V be a vector space over $\Re$. A subset $X \subset V$ is called s_1-convex if every s_1-convex curve, defined by $\lambda^s x_1 + (1 - \lambda^s)x_2$, $\forall x_1, x_2 \in X$, intersects X in an interval, that is:

$$(\lambda^s x_1 + (1 - \lambda^s)x_2) \subset X$$

    when $0 < \lambda < 1$ and $x_1, x_2 \in X$.

  - *s_2-convex sets*

Definition 2. Let V be a vector space over $\Re$. A subset $X \subset V$ is called s_2-convex if every s_2-convex curve, defined by $\lambda^s x_1 + (1-\lambda)^s x_2$, $\forall x_1, x_2 \in X$, intersects X in an interval, that is:

$$(\lambda^s x_1 + (1-\lambda)^s x_2) \subset X$$

when $0 < \lambda < 1$ and $x_1, x_2 \in X$.

- *S-convex combinations*
  - *s_1-convex combinations*

Definition 3. An s_1-convex combination of a set of vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n}$ is a linear combination of those vectors in which the coefficients are all nonnegative, and the individual raise to a power 's' results in their sum being one, that is: $\lambda_1^s \overrightarrow{v_1} + \lambda_2^s \overrightarrow{v_2} + ... + \lambda_n^s \overrightarrow{v_n}$, with $\sum_{p=1}^{n} \lambda_p^s = 1$ and $\lambda_p \geq 0$, $\forall p/p \in N, 1 \leq p \leq n$, is an s_1-convex combination of vectors.

The set of all s_1-convex combinations of the vectors is their s_1-convex span, denoted by $SCS_1(\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n})$.

  - *s_2-convex combinations*

Definition 4. An s_2-convex combination of a set of vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n}$ is a 'bent' combination[2] of those vectors in which the coefficients are all nonnegative and hold sum one, that is: $\lambda_1^s \overrightarrow{v_1} + \lambda_2^s \overrightarrow{v_2} + ... + \lambda_n^s \overrightarrow{v_n}$, with $\sum_{p=1}^{n} \lambda_p = 1$ and $\lambda_p \geq 0$, $\forall p/p \in N, 1 \leq p \leq n$, is an s_2-convex combination of vectors.

The set of all s_2-convex combinations of the vectors is their s_2-convex span, denoted by $SCS_2(\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n})$.

[2] Simply a choice of name to distinguish between it and the L. C.

- s_1-*convex functions*

Definition 5. A function $f : X- > \Re$, $f \in C^1$, is said to be s_1-convex if the inequality

$$f(\lambda x + (1-\lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1-\lambda^s) f(y)$$

holds $\forall \lambda \in [0,1]$, $\forall x, y \in X$ such that $X \subset \Re_+$.

- s_2-*convex functions*

Definition 6. f is called s_2-convex, $s \neq 1$, if the graph lies below a 'bent chord' (L) between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, it is true that

$$sup_J(L-f) \geq sup_{\partial J}(L-f)$$

- s_2-*convex functions*

Definition 7. A function $f : X- > \Re$, $f \in C^1$, is said to be s_2-convex if the inequality

$$f(\lambda x + (1-\lambda) y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds $\forall \lambda \in [0,1]$, $\forall x, y \in X$ such that $X \subset \Re_+$.

2 Extension of Mazur's lemma to $S-$convexity

2.1 Mazur's lemma

This must be the best contribution to the scholarship in the subject provided by us, once the analytical definitions have been settled in [2].

Lemma 2.1. *Assume $u_n- > u$ weakly in a normed linear space. Then there exists, for any $\epsilon > 0$, an $S-$convex combination: $\sum_{k=1}^n \lambda_k^s u_k (\lambda_k \geq 0, \sum_{k=1}^n \lambda_k = 1)$ (for s_2-convex combination), and $\sum_{k=1}^n \lambda_k^s u_k (\lambda_k \geq 0, \sum_{k=1}^n \lambda_k^s = 1)$ (for s_1-convex combination), of $\{u_k : k = 1, 2, ...\}$ such that $||u - \sum_{k=1}^n \lambda_k^s u_k|| \leq \epsilon$ where $||v||$ is a norm of v in the space.*

Proof. The extension to s_1-convexity is trivial, once if a convex combination exists then there is λ_w^s satisfying the primary condition for the sum, its value being equivalent to the original one. For the second type of $S-$convexity, given that λ_k exists, replace that λ_k with $\lambda_w^{\frac{1}{s}}$. Once they hold the same value, everything is the same. □

3 Extension of the major inequality under consideration

We now base ourselves in [3], in order to get the new theorems, loins of a very simple and trivially proved extension from convexity to $S-$convexity.

3.1 Major inequality under consideration

Original inequality

Theorem 3.1. *Suppose that a sequence* $\{u_i\}_{i=1}^{+\infty}$ *weakly converges in* $(L_\mu^p(\Re^n))^m$ *to* u *as* $i->+\infty$*, where* $p \in [1, +\infty)$ *and* m *and* n *are two positive integers. Assume that all the values of* u *and* u_i $(i = 1, 2, 3, ...)$ *belong to an open convex set* K *in* $\Re^n$ *and that* $f(x)$ *is a nonnegative convex function from* K *to* $\Re$*. Then*

$$\lim_{i->+\infty} \int_\Omega f(u_i)d\mu \geq \int_\Omega f(u)d\mu$$

for any measurable set $\Omega \subseteqq \Re^n$*.*

3.2 Extension

- *Extension to* s_1-*convexity*

Theorem 3.2. *Suppose that a sequence* $\{u_i\}_{i=1}^{+\infty}$ *weakly converges in* $(L_\mu^p(\Re^n))^m$ *to* u *as* $i->+\infty$*, where* $p \in [1,+\infty)$ *and* m *and* n *are two positive integers. Assume that all the values of* u *and* u_i $(i = 1,2,3,...)$ *belong to an open* s_1-*convex set ([2])* K *in* $\Re^n$ *and that* $f_s(x)$ *is a nonnegative* s_1-*convex function from* K *to* $\Re$*. Then*

$$\lim_{i->+\infty} \int_\Omega f_s(u_i)d\mu \geq \int_\Omega f_s(u)d\mu$$

for any measurable set $\Omega \subseteqq \Re^n$*.*

Proof. Trivial. □

- *Extension to* s_2-*convexity*

  The same theorem may be adapted to fit S_2-convexity as well. See:

  Theorem 3.3. *Suppose that a sequence* $\{u_i\}_{i=1}^{+\infty}$ *weakly converges in* $(L_\mu^p(\Re^n))^m$ *to* u *as* $i->+\infty$*, where* $p \in [1,+\infty)$ *and* m *and* n *are two positive integers. Assume that all the values of* u *and* u_i $(i = 1,2,3,...)$ *belong to an open* s_2*-convex set* K *in* $\Re^n$ *and that* $f_s(x)$ *is a nonnegative* s_2-*convex function from* K *to* $\Re$*. Suppose, at the same time, that* $f_s(x)$ *was generated via a lifting of a few points from* $f(x)$*,* $f(x)$ *being convex. Then*

  $$\lim_{i->+\infty} \int_\Omega f_s(u_i)d\mu \geq \int_\Omega f_s(u)d\mu \geq \int_\Omega f(u)d\mu$$

  for any measurable set $\Omega \subseteqq \Re^n$*.*

4 Applications

Theorem 4.1. *Assume that a sequence* $\{u_i\}_{i=1}^{+\infty}$ *weakly* converges in* $(L_\mu^\infty(\Re^n))^m$ *to* u *as* $i->+\infty$*, where* m *and* n *are two positive integers. Assume that all the values of* u *and* u_i $(i = 1,2,3,...)$ *belong to an open* $S-$*convex set* K *of* $\Re^m$ *and that* $f(x)$ *is a nonnegative* $S-$*convex function from* K *to* $\Re$*. Then the inequality of our first theorem holds for any measurable set* $\Omega \subseteq \Re^n$*.*

Theorem 4.2. *Assume that a sequence* $\{u_i\}_{i=1}^{+\infty}$ *weakly* converges in* $(L_\mu^\infty(\Re^n))^m$ *to* u *as* $i->+\infty$*, where* m *and* n *are two positive integers. Assume that all the values of* u *and* u_i $(i = 1,2,3,...)$ *belong to a closed S-convex set* K *in* $\Re^m$ *and that* $f(x)$ *is a continuous* $S-$*convex function from* K *to* $\Re$*. Then the inequality of our first theorem holds for any bounded measurable set* $\Omega \subset \Re^n$*.*

5 References

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