Formula for the Area of a Rectangle

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Here we describe Legendre's approach to obtain the formula for the area of a rectangle using Cauchy's functional equation.

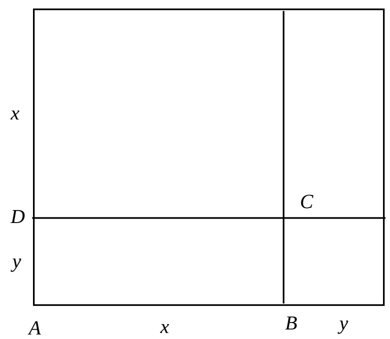
If it is assumed that the formula for the area of a square of side a is a^2, then the following consideration will give the fomula for the area for rectangle. Consider a square of side $x + y$ and partition the square into rectangles and squares as shown in the figure below. Then it is observed that

$$(x + y)^2 = x^2 + y^2 + 2 \times area\ of\ rectangle\ ABCD$$

and hence the area of the rectangle $ABCD = xy$.

The following approach based on functional equations will give formula for rectangle in general.

Functional equations are equations involving known and unknown functions. The following are two classical functional equations.



- Cauchy's functional equation

$$f(x+y) = f(x) + f(y) \tag{1}$$

- D'Alembert's functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y) \tag{2}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfies (1) for all reals, then it is easy to observe that for $m,\ n \in \mathbb{N}$, and $x \in \mathbb{R}$, (i) $f(0) = 0$, (ii) $f(x) = -f(-x)$, (iii) $f(nx) = nf(x)$, (iv) $f(\frac{m}{n}x) = \frac{m}{n}f(x)$. In short $f(rt) = rf(t),\ r \in \mathbb{Q},\ t \in \mathbb{R}$, and putting $t = 1$ we have

Proposition 1 *If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies equation (1) for all reals then $f(r) = cr,\ \forall\ r\ \in \mathbb{Q}$ for some c in $\mathbb{R}$.*

If f is assumed to be continuous then for $x \in \mathbb{R}$, $f(x) = \lim\limits_{n\to\infty} f(r_n) = \lim\limits_{n\to\infty} cr_n = cx$ where (r_n) is a sequence of rationals converging to x. It proves the following

Theorem 1 *(Cauchy, 1821): If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies eqn. (1) for all reals and is continuous, then $f(x) = cx,\ \forall\ x\ \in\ \mathbb{R}$ for some $c \in \mathbb{R}$.*

Later Darboux came up with a set of necessary conditions for existence of continuous solution for Cauchy's functional equation.

Theorem 2 *(Darboux, 1875,1880): Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying eqn. (1) for all reals. If f is*

(i) continuous at a point,

(ii) nonnegative (or nonpositive) for small positive numbers
or

(iii) bounded in an interval

then $f(x) = cx,\ \forall x \in \mathbb{R}$ for some $c\ \in\ \mathbb{R}$.

Part of the proof: We only prove the theorem for the case when (ii) is satisfied. Suppose $f(y) \geq 0$ for sufficiently small $y > 0$. Then

$$f(x+y) = f(x) + f(y) \geq f(x) \ \forall \ x \ \in \mathbb{R}$$

so that f is monotonically increasing. Now for $x \in \mathbb{R}$, let (r_n) and (R_n) be increasing and decreasing sequences respectively and converging to x. Then we have $r_n < x < R_n$ for each n. By Proposition 1, for some $c \in \mathbb{R}$

$$cr_n = f(r_n) \leq f(x) \leq f(R_n) = cR_n$$

from which it immediately follows that $f(x) = cx \ \ \forall \ \ x \in \mathbb{R}$. On the other hand, if $f(y) \leq 0$ for sufficiently small $y > 0$, then f is going to be a decreasing function and the very similar argument as above proves the theorem.

Remark : The above theorem holds for the function which is defined only on the set of all positive reals (or the set of all nonnegative reals).

Establishing the Formula : We now describe the method of Legendre (1791) using Cauchy's functional equation for obtaining the formula for the area of the rectangle. Let $f(x, y)$ be denote the area of the rectangle of sides x and y. By the very nature of area, it is clear that for x, y positive $f(x, y)$ is positive.

Further for x, y, x_1, x_2, y_1, y_2 positive,

$$f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y) \tag{3}$$

$$f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2) \tag{4}$$

Fix y in (3) and define $f_y(x) = f(x, y)$. Then we have

$$f_y(x_1 + x_2) = f_y(x_1) + f_y(x_2).$$

Thus f_y satisfies condition (ii) of Theorem 2 for positive values so $f_y(x) = k(y)x$, where the constant k naturally depends on the parameter y.

Again substituting this in (4), We have

$$k(y_1 + y_2) = k(y_1) + k(y_2)$$

By the positivity of $f(x, y)$ and Theorem (2) we have $k(y) = cy$ for some constant $c > 0$ so that $f(x, y) = cxy$.

Thus the area of the rectangle must be equal to cxy. The value of the constant c depends on the choice of the scale. If the scales is chosen so that the area of the square with sides of unit length is equal to 1, then $c = 1$. The volume of a cuboid can be derived in similar way.

References

[1] J.Aczél, *Lectures on functional equations and their applications*, Mathematics in science and engineering, Vol. 19, Academic press, 1966.

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