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Unique continuation property for a higher order nonlinear Schrödinger equation

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Abstract

We prove that, if a sufficiently smooth solution u to the initial value problem associated with the equation

$$\partial_t u + i\alpha \partial_x^2 u + \beta \partial_x^3 u + i\gamma |u|^2 u + \delta |u|^2 \partial_x u + \epsilon u^2 \partial_x \bar{u} = 0, \quad x, t \in \mathbb{R},$$

is supported in a half line at two different instants of time then $u \equiv 0$. To prove this result we derive a new Carleman type estimate by extending the method introduced by Kenig et al. in [Ann. Inst. H. Poincaré Anal. Non Linéaire 19 (2002) 191–208].

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1. Introduction

Let us consider the following equation:

$$\partial_t u + i\alpha \partial_x^2 u + \beta \partial_x^3 u + i\gamma |u|^2 u + \delta |u|^2 \partial_x u + \epsilon u^2 \partial_x \bar{u} = 0, \quad x, t \in \mathbb{R}, \quad (1.1)$$

where $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$, $\gamma, \delta, \epsilon \in \mathbb{C}$ and $u = u(x, t)$ is a complex valued function.

This mixed model of Korteweg–de Vries (KdV) and Schrödinger type was proposed by Hasegawa and Kodama in [7] and [11] to describe the nonlinear propagation of pulses in optical fibers.

In this work we are interested to study the unique continuation property for Eq. (1.1). Unique continuation property says, if a solution u to certain evolution equation vanishes on some nonempty open subset \mathcal{O} of Ω then it vanishes in the horizontal component of \mathcal{O} , where Ω is the domain of the evolution operator under consideration.

As far as we know, the first work in this direction for a general class of dispersive equation is due to Saut and Scheurer [14]. They used Carleman type estimate to obtain unique continuation property for such class of equations. In particular, the KdV equation falls in the class considered in [14] and so the unique continuation property is valid for solution of the KdV equation [14, Theorem 4.2]. Later, B. Zhang [16] used inverse scattering transform and some results from Hardy function theory to prove that the solution to the KdV equation cannot be supported in the horizontal half lines at two different moments unless it vanishes identically. To get this result he imposed certain decay condition on the solution. This slightly stronger result recovers the result due to Saut and Scheurer [14]. Zhang [16] also proved the same result for the modified KdV equation using Miura's transform. Recently, Bourgain [2] introduced a different approach and proved that, if a solution u to a dispersive equation has compact support in a nontrivial time interval $I = [t_1, t_2]$ then u vanishes identically. Use of complex variables technique along with the Paley–Wiener theorem are the main ingredients in Bourgain's approach. In fact, the result obtained by Bourgain [2] is a direct corollary of Theorem 4.2 in [14] in the case of the KdV equation, but it has merit of being simple and more general method to be applicable in higher spatial dimensions too. Quite recently, Kenig et al. [9] introduced a new method and proved that, if a sufficiently smooth solution u to a generalized KdV equation is supported in a half line at two different instants of time then u vanishes identically. To get this result they derived a Carleman type estimate exploiting the structure of the generalized KdV equation and combined this with the result in [14] mentioned above.

The main result of this work is in the same spirit to that for the generalized KdV equation obtained in [9] and reads as follows.

Theorem 1.1. *Let $u \in C([t_1, t_2]; H^s) \cap C^1([t_1, t_2]; H^1)$, $s \geq 4$, be a solution of the equation (1.1) with $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}$, $\beta \neq 0$. If there exists $t_1 < t_2$ such that*

$$\text{supp } u(\cdot, t_j) \subset (-\infty, a), \quad j = 1, 2, \quad (1.2)$$

$$\text{or } (\text{supp } u(\cdot, t_j) \subset (b, \infty), \quad j = 1, 2). \quad (1.3)$$

Then $u(t) = 0$ for all $t \in [t_1, t_2]$.

To prove this theorem, we follow the scheme introduced in [9] by deriving a new Carleman type estimate (see (3.14) below) related to Eq. (1.1). When $\alpha = 0$ the linear part of (1.1) reduces to that of the KdV type equation and in this case one can utilize the estimate derived in [9]. So we suppose $\alpha \neq 0$ in our analysis. In this case also, it seems that one can combine the term containing α with the nonlinear terms and use the theory developed in [9] directly. But this idea does not work because the term containing α does not have a decay property like other nonlinear terms (see Remark 3.10 below). Therefore, it is necessary to obtain a new Carleman type estimate for Eq. (1.1). Here we do it, in some sense, by extending the estimate in [9] to the mixed equation (1.1) of type KdV and Schrödinger. The existence of the third order derivative in the linear part of (1.1) (i.e., $\beta \neq 0$) helps us to get the exponential decay of the solution to (1.1), otherwise the situation is much complex as can be seen in the work of Kenig et al. [8]. Decay property of the solution is fundamental to obtain Carleman type estimate. Once this new Carleman type estimate is established, the proof of Theorem 1.1 follows using the technique introduced in [9]. To conclude the proof of Theorem 1.1, instead of the result in [14] which is much stronger, we use the following result.

Theorem 1.2. *Let $u \in C([-T, T]; H^s)$, $s \geq 4$, be a solution of Eq. (1.1). If there exists a nontrivial time interval $I = [-T, T]$ such that*

$$\text{supp } u(t) \subseteq [-B, B], \quad \forall t \in I,$$

where $B > 0$ is a constant, then $u(t) = 0$ for all $t \in I$.

Remark 1.3. Note that, Eq. (1.1) belongs to the class considered in [14] with appropriate choice of the terms. So, Theorem 1.2 can be proved following the scheme introduced in [14]. However, to follow the scheme in [14] one has to use very strong estimates of Carleman type. Here we are going to give a simple proof by generalizing the method introduced in [2] which utilizes the analyticity of the terms obtained after applying the Fourier transform in Duhamel's formula. The Paley–Wiener theorem plays a central role in this approach. We believe that this result will be interesting for readers because the method employed in the proof can also be used to more general class of evolution equations in higher spatial dimensions.

The initial value problem (IVP) associated to (1.1) has also been studied by several authors in recent years. Taking $\alpha, \beta, \gamma, \delta$ and ϵ as real constants, Laurey [12] proved that the IVP associated to (1.1) for given data in $H^s(\mathbb{R})$ is locally well-posed when $s > 3/4$ and globally well-posed when $s \geq 1$. Later, using the techniques developed by Kenig et al. [10], Staffilani [15] improved the result in [12] by showing that the IVP associated to (1.1) is locally well-posed in $H^s(\mathbb{R})$, $s \geq 1/4$. Recently, using the method introduced by Bourgain [1] and further developed by Fonseca et al. [6], Carvajal and Linares [4] proved that the IVP associated to (1.1) is globally well-posed in $H^s(\mathbb{R})$, $s > 5/9$ (see also [3]). The IVP associated to (1.1) when α and β are functions of $t \in [-T_0, T_0]$ for some $T_0 > 0$ and $\beta(t) \neq 0$ for all $t \in [-T_0, T_0]$ has also been a matter of study (see, for instance, [12] and [5]).

The remainder of this article is organized as follows. In Section 2, we prove Theorem 1.2 and in Section 3, we prove our main result, the Theorem 1.1. Finally, in Appendix A, we state a known result which is utilized in this article.

Notation. Now we give some notations which will be used throughout this work. We use \mathcal{F} to denote Fourier transform and is defined by $(\mathcal{F}f)(\xi) = (2\pi)^{-1/2} \int \exp(-ix\xi) f(x) dx$. We use $C^{3,1}$ to denote the space of functions which are C^3 in space variable and C^1 in time variable and $C_0^{3,1}$ to denote the space of real valued functions in $C^{3,1}$ with compact support. We write $A \lesssim B$ if there exists a constant $c > 0$ such that $A \leq cB$. Also, we use ρ^{*3} to denote the convolution product $\rho * \rho * \rho$ and $\text{supp } f$ to denote support of f .

2. Solution with compact support in a time interval

This section is devoted to prove Theorem 1.2. We proceed by proving following results that will be useful in our analysis.

2.1. Preliminary results

For $\lambda \in \mathbb{R}$ and $u = u(x, t)$, a solution of Eq. (1.1), let us define

$$\rho(\lambda) = \sup_{|\lambda'| \geq |\lambda|} \sup_{t \in I} |\mathcal{F}(u(t))(\lambda')| = \sup_{|\lambda'| \geq |\lambda|} u^*(\lambda') \tag{2.1}$$

and

$$M = \sup_{t \in I} \|u(t)\|_{H^s}. \tag{2.2}$$

First, let us start with the following lemma.

Lemma 2.1. *Let $u \in C([-T, T]; H^s)$, $s \geq 4$, be a solution of Eq. (1.1) and B as in Theorem 1.2; then we have*

$$\rho(\lambda) \lesssim \frac{\sqrt{B} M}{1 + \lambda^4}. \tag{2.3}$$

Proof. Using Cauchy–Schwarz inequality we get

$$\sup_{t \in I} \int_{\mathbb{R}} |u(t)(x)| dx \lesssim \sqrt{B} M.$$

Hence for all $t \in I$,

$$|\mathcal{F}(u(t))(\lambda)| \lesssim \sqrt{B} M. \tag{2.4}$$

Since $u \in C([-T, T]; H^s)$, $s \geq 4$, using Cauchy–Schwarz inequality combined with (2.2) we obtain

$$|\lambda|^s |\mathcal{F}(u(t))(\lambda)| = |\mathcal{F}(D_x^s u(t))(\lambda)| \leq \int_{\mathbb{R}} |D_x^s u(t)(x)| dx \lesssim \sqrt{B} M. \tag{2.5}$$

Combining (2.4) and (2.5) we get the required result. \square

Remark 2.2. (1) If there exists t such that $u(t) \neq 0$ and $u(t)$ has compact support then $\rho(\lambda) > 0$ for all $\lambda \in \mathbb{R}$.

(2) If u is such that $\sup_{[-T, T]} \|u(t)\|_{H^n} \leq M_0$, for all $n \in \mathbb{N}$ and B as in Theorem 1.2, then

$$\rho(\lambda) \lesssim \frac{\sqrt{B} M_0}{1 + \lambda^n}. \quad (2.6)$$

This in turn implies that $u(t) = 0$ for all $t \in I$.

Next, using (2.3), we have the following lemma whose proof is similar to the proof of lemma in [2, p. 440].

Lemma 2.3. *Let $u = u(x, t)$ be a solution of Eq. (1.1) with the same hypothesis of Theorem 1.2. Suppose that there exists $t \in I$ such that $u(t) \neq 0$, then there are numbers $c > 0$ and $N_0 > 0$ such that, for any $N > N_0$, there are arbitrarily large values of λ (i.e., for all $\lambda' > 0$ there exists λ with $|\lambda| > \lambda'$), such that*

$$\rho(\lambda) > c\rho^{*3}(\lambda) \quad (2.7)$$

and

$$\rho(\lambda) > e^{-|\lambda|/N}. \quad (2.8)$$

Remark 2.4. If $\lambda > 0$ satisfies (2.7) and (2.8) then so does $-\lambda$.

In sequel we use the following result whose proof follows from Lemma 2.3.

Corollary 2.5. *Suppose the hypotheses in Lemma 2.3 are valid, then there exist arbitrarily large λ and N and $t_\lambda := t_1 \in I$, such that*

$$\rho(\lambda) = |\mathcal{F}(u(t_1))(\lambda)| \gtrsim \rho^{*3}(\lambda) \quad \text{and} \quad \rho(\lambda) > e^{-|\lambda|/N}.$$

In what follows we need some derivative estimates for entire functions. Using harmonic measure and A.A. Markov inequality for polynomials (i.e., $\|P'\|_{L^\infty[0,1]} \leq 2n^2 \|P\|_{L^\infty[0,1]}$, where n is the degree of the polynomial P , see [13, p. 40]), Bourgain [2] proved the following lemma.

Lemma 2.6. *Let $\Phi(z)$ be an entire function bounded and integrable on the real line such that $|\Phi(\lambda + i\theta)| \lesssim e^{|\theta|B}$, $\lambda, \theta \in \mathbb{R}$, $B > 0$. Let $g(x) = x(1 + |\log x|)$, $x > 0$, and suppose*

$$\tilde{\rho}(\lambda) = \sup_{|\lambda'| \geq \lambda > 0} |\Phi(\lambda')|.$$

Then,

$$\sup_{|\lambda'| \geq \lambda > 0} |\Phi'(\lambda')| \lesssim Bg(\tilde{\rho}(\lambda)). \quad (2.9)$$

Proof. See Lemma 2.2 in [2]. \square

Corollary 2.7. Let $v(\theta, \lambda) := \sup_{|\lambda'| \geq \lambda > 0} |\Phi(\lambda' + i\theta)|$. Under the hypothesis of Lemma 2.6 if $\theta \in \mathbb{R}$ is such that $|\theta| \lesssim 1/B(1 + |\log \tilde{\rho}(\lambda)|)$; then

$$v(\theta, \lambda) \leq 2\tilde{\rho}(\lambda). \tag{2.10}$$

Proof. See Corollary 2.9 in [2]. \square

Corollary 2.8. Let $t \in I$, $\Phi(z) = \mathcal{F}(u(t))(z)$, θ as in Corollary 2.7 and ρ as in (2.1), then for $|\theta'| \leq |\theta|$ fixed, we have

$$|\Phi'(\lambda - \lambda' + i\theta')| \lesssim B[\rho(\lambda) + \rho(\lambda - \lambda')] [1 + |\log \rho(\lambda)|]. \tag{2.11}$$

Proof. Let us consider $\Theta(z) = \Phi(z + i\theta')$, $z = x + iy$, $x, y \in \mathbb{R}$. By (2.10), Θ is bounded and integrable on the real line. Using (2.9) for Θ , we have

$$\begin{aligned} |\Phi'(\lambda - \lambda' + i\theta')| &\leq \sup_{|\lambda| \geq \bar{\lambda} > 0} |\Phi'(\lambda + i\theta')| = \sup_{|\lambda| \geq \bar{\lambda} > 0} |\Theta'(\lambda)| \\ &\lesssim Bw(\bar{\lambda})(1 + |\log w(\bar{\lambda})|) \lesssim B\rho(\bar{\lambda})(1 + |\log \rho(\bar{\lambda})|), \end{aligned} \tag{2.12}$$

where $\bar{\lambda} = \min\{|\lambda|, |\lambda - \lambda'|\}$, $w(\bar{\lambda}) = \sup_{|\lambda| \geq \bar{\lambda} > 0} |\Phi(\lambda + i\theta)|$ and in the third inequality the estimate (2.10) has been used.

It is clear that for large λ , $|\log \rho(\lambda)| = -\log \rho(\lambda)$ and

$$\rho(\bar{\lambda}) \leq \rho(\lambda) + \rho(\lambda - \lambda'). \tag{2.13}$$

Also, if $\rho(\bar{\lambda}) < 1$, then as $\bar{\lambda} \leq |\lambda|$ we have $-\log \rho(\bar{\lambda}) \leq -\log \rho(\lambda)$. If $\rho(\bar{\lambda}) \geq 1$, using (2.3) and taking λ large we get $\rho(\bar{\lambda})\rho(\lambda) \lesssim \rho(\lambda)/(1 + \bar{\lambda}^4) < 1$ and therefore, $\log \rho(\bar{\lambda}) \leq -\log \rho(\lambda)$. Hence in any case

$$\log |\rho(\bar{\lambda})| \leq |\log \rho(\lambda)|. \tag{2.14}$$

Now combining (2.12), (2.13) and (2.14) we get the required estimate. \square

Lemma 2.9. If $\rho(\lambda)$ is a function defined in (2.1) then there exists $a_0 > 0$ such that

$$g(\lambda) = \int_{\mathbb{R}} \int_{|\xi| \geq |\lambda|} |\xi| \rho(\lambda - \xi - \eta) \rho(\eta) d\xi d\eta \leq |\lambda| a_0 + a_0.$$

Proof. Since g is an even function we can suppose $\lambda > 0$. By (2.3) we have

$$\begin{aligned} g(\lambda) &\lesssim \int_{\mathbb{R}} \rho(\eta) \int_{|\xi| \geq |\lambda|} \frac{|\xi|}{1 + (\lambda - \xi - \eta)^4} d\xi d\eta \\ &= \int_{\mathbb{R}} \rho(\eta) \left[\int_{\lambda}^{\infty} \frac{|\xi|}{1 + (\lambda - \xi - \eta)^4} d\xi + \int_{-\infty}^{-\lambda} \frac{|\xi|}{1 + (\lambda - \xi - \eta)^4} d\xi \right] d\eta. \end{aligned} \tag{2.15}$$

Let us use $\xi - \lambda = x$ in the first integral to get

$$\int_0^{\infty} \frac{x + \lambda}{1 + (\eta + x)^4} dx = \lambda \int_0^{\infty} \frac{1}{1 + (\eta + x)^4} dx + \int_0^{\infty} \frac{x}{1 + (\eta + x)^4} dx$$

$$\leq \lambda b_1 + |\eta| b_1 + b_0.$$

Also using $\xi - \lambda = x$ in the second integral we get

$$\int_{-\infty}^{-2\lambda} \frac{-x - \lambda}{1 + (\eta + x)^4} dx = -\lambda \int_{-\infty}^{-2\lambda} \frac{1}{1 + (\eta + x)^4} dx - \int_{-\infty}^{-2\lambda} \frac{x}{1 + (\eta + x)^4} dx$$

$$\leq |\eta| b_1 + b_0.$$

Therefore,

$$g(\lambda) \lesssim \int_{\mathbb{R}} \rho(\eta) (\lambda b_1 + 2|\eta| b_1 + 2b_0) d\eta \leq \lambda a_0 + a_0.$$

This completes the proof of the lemma. \square

In what follows we consider λ, t_1 as in the Corollary 2.5, $t_2 \in I$ such that

$$|\Delta t| = |t_2 - t_1| = T, \quad (2.16)$$

and θ small such that $\operatorname{sgn} \theta = -\operatorname{sgn}(\beta \Delta t)$. After choosing sign for θ let us choose sign for λ such that $\operatorname{sgn} \lambda = -\operatorname{sgn}(\alpha \theta \Delta t)$. Therefore, $\beta \theta \Delta t < 0$ and $\alpha \theta \lambda \Delta t \leq 0$.

2.2. Reduction of the problem using Fourier transform

Using Duhamel's formula we have for $t_1, t_2 \in I$,

$$u(t_2)(x) = U(t_2 - t_1)u(t_1)(x) - \int_{t_1}^{t_2} U(t_2 - \tau) F(u)(\tau)(x) d\tau, \quad (2.17)$$

where $U(t)u = e^{-tP(D)}u$, $P(D)u = i\alpha \partial_x^2 u + \beta \partial_x^3 u$ and $F(u) = i\gamma |u|^2 u + \delta |u|^2 \partial_x u + \epsilon u^2 \partial_x \bar{u}$.

Taking Fourier transform \mathcal{F} in the space variable in (2.17) we get

$$\mathcal{F}(u(t_2))(\lambda) = e^{i(\beta \lambda^3 + \alpha \lambda^2)(t_2 - t_1)} \mathcal{F}(u(t_1))(\lambda)$$

$$- \int_{t_1}^{t_2} e^{i(\beta \lambda^3 + \alpha \lambda^2)(t_2 - \tau)} \mathcal{F}(F(u)(\tau))(\lambda) d\tau.$$

Since $u(t)$ has compact support, by Paley–Wiener theorem $\mathcal{F}(u(t))$ and $\mathcal{F}(F(u)(t))$ have analytic continuation in \mathbb{C} and we get

$$\mathcal{F}(u(t_2))(\lambda + i\theta) = e^{i(\beta(\lambda + i\theta)^3 + \alpha(\lambda + i\theta)^2)(t_2 - t_1)} \mathcal{F}(u(t_1))(\lambda + i\theta)$$

$$- \int_{t_1}^{t_2} e^{i(\beta(\lambda + i\theta)^3 + \alpha(\lambda + i\theta)^2)(t_2 - \tau)} \mathcal{F}(F(u)(\tau))(\lambda + i\theta) d\tau. \quad (2.18)$$

Taking absolute value in (2.18) we get for $\Delta t = t_2 - t_1$,

$$\begin{aligned}
 e^{|\theta|B} &\gtrsim |\mathcal{F}(u(t_2))(\lambda + i\theta)| \\
 &\geq e^{-3\Delta t\lambda^2\beta\theta - 2\alpha\lambda\Delta t\theta + \theta^3\beta\Delta t} \left[|\mathcal{F}(u(t_1))(\lambda + i\theta)| \right. \\
 &\quad \left. - \left| \int_0^{\Delta t} e^{-i(\beta(\lambda+i\theta)^3 + \alpha(\lambda+i\theta)^2)\tau} \mathcal{F}(F(u)(\tau + t_1))(\lambda + i\theta) d\tau \right| \right]. \tag{2.19}
 \end{aligned}$$

Let

$$J_0 = \int_0^{\Delta t} e^{(-i(\beta(\lambda+i\theta)^3 + \alpha(\lambda+i\theta)^2)\tau)} \mathcal{F}(F(u)(\tau + t_1))(\lambda + i\theta) d\tau.$$

Let us denote $(\Delta t)^+ := \Delta t$ if $\Delta t > 0$ and $(\Delta t)^- := -\Delta t$ if $\Delta t < 0$, then we obtain

$$|J_0| \leq \int_0^{(\Delta t)^\pm} e^{(-3\lambda^2|\beta\theta|\tau - 2\tau|\alpha\lambda\theta| \mp \theta^3\beta\tau)} |\mathcal{F}(F(u)(t_1 \pm \tau))(\lambda + i\theta)| d\tau.$$

In what follows we consider $(\Delta t)^+ = \Delta t > 0$, the case $(\Delta t)^- = -\Delta t > 0$ is similar. From (2.19) we get

$$\begin{aligned}
 e^{|\theta|B} e^{(3\Delta t\lambda^2\beta\theta + 2\alpha\lambda\Delta t\theta - \theta^3\beta\Delta t)} &\gtrsim |\mathcal{F}(u(t_1))(\lambda + i\theta)| \\
 &\quad - \int_0^{|\Delta t|} e^{-3\lambda^2|\beta\theta|\tau - 2\tau|\alpha\lambda\theta| - \theta^3\beta\tau} |\mathcal{F}(F(u)(\tau + t_1))(\lambda + i\theta)| d\tau. \tag{2.20}
 \end{aligned}$$

For arbitrarily small values of θ , (2.20) becomes,

$$\begin{aligned}
 e^{-3T\lambda^2|\beta\theta|} &\gtrsim |\mathcal{F}(u(t_1))(\lambda + i\theta)| \\
 &\quad - \int_0^T e^{-\tau(3\lambda^2|\beta\theta| + 2|\alpha\lambda\theta|)} |\mathcal{F}(F(u)(\tau + t_1))(\lambda + i\theta)| d\tau \\
 &\geq I_1 - I_2 - I_3, \tag{2.21}
 \end{aligned}$$

where

$$I_1 = |\mathcal{F}(u(t_1))(\lambda)| - \int_0^T e^{-\tau(3\lambda^2|\beta\theta| + 2|\alpha\lambda\theta|)} |\mathcal{F}(F(u)(\tau + t_1))(\lambda)| d\tau, \tag{2.22}$$

$$I_2 = |\mathcal{F}(u(t_1))(\lambda + i\theta) - \mathcal{F}(u(t_1))(\lambda)|, \tag{2.23}$$

$$I_3 = \int_0^T e^{-\tau(3\lambda^2|\beta\theta|+2|\alpha\lambda\theta|)} |\mathcal{F}(F(u)(\tau+t_1))(\lambda+i\theta) - \mathcal{F}(F(u)(\tau+t_1))(\lambda)| d\tau. \quad (2.24)$$

2.3. Proof of Theorem 1.2

This section is devoted to carry out the proof of Theorem 1.2. The principal point in the proof is the use of the estimate (2.21) by finding lower bound for I_1 and upper bounds for I_2 and I_3 .

Proof of Theorem 1.2. We prove it by contradiction. Suppose

$$u(t) \neq 0 \quad \text{for some } t \in I. \quad (2.25)$$

As mentioned above, to arrive at contradiction we use Lemma 2.3 to get appropriate estimates for I_1 , I_2 and I_3 and apply those in (2.21) for large λ and small θ .

Using Corollary 2.8 with $\lambda' = 0$ and $|\theta| \lesssim 1/B(1 + |\log \rho(\lambda)|)$ in (2.23) we get

$$I_2 < c_0 \rho(\lambda), \quad (2.26)$$

where $c_0 < 1/3$ is a constant.

Now we proceed to estimate I_1 and I_3 . Let $F_1(u) = i\gamma|u|^2u$, $F_2(u) = \delta|u|^2\partial_x u$ and $F_3(u) = \epsilon u^2 \partial_x \bar{u}$ so that

$$\mathcal{F}(F_1(u)) = i\gamma \mathcal{F}(u) * \mathcal{F}(u) * \mathcal{F}(\bar{u}), \quad (2.27)$$

$$\mathcal{F}(F_2(u)) = i\delta \mathcal{F}(u) * (\xi \mathcal{F}(u)(\xi)) * \mathcal{F}(\bar{u}), \quad (2.28)$$

$$\mathcal{F}(F_3(u)) = i\epsilon \mathcal{F}(u) * \mathcal{F}(u) * (\xi \mathcal{F}(\bar{u})(\xi)). \quad (2.29)$$

Estimate for I_1 . For simplicity let us introduce a notation $\nu := 3\lambda^2|\beta\theta| + 2|\alpha\lambda\theta|$. Using Corollary 2.5 we get from (2.22),

$$I_1 > \rho(\lambda) - I_1^1 - I_1^2, \quad (2.30)$$

where

$$I_1^1 := \int_0^T e^{-\tau\nu} |\mathcal{F}(F_1(u))| d\tau \quad \text{and} \quad I_1^2 := \int_0^T e^{-\tau\nu} |\mathcal{F}(F_2(u)) + \mathcal{F}(F_3(u))| d\tau.$$

Using definition of $\rho(\lambda)$, (2.27) and Lemma 2.3 we get

$$\begin{aligned} I_1^1 &\leq |\gamma| \rho^{*3} \int_0^T e^{-\tau(3\lambda^2|\beta\theta|+2|\alpha\lambda\theta|)} d\tau = \frac{|\gamma| \rho^{*3} (1 - e^{-T(3\lambda^2|\beta\theta|+2|\alpha\lambda\theta|)})}{3\lambda^2|\beta\theta| + 2|\alpha\lambda\theta|} \\ &\leq \frac{|\gamma| \rho^{*3}}{2|\alpha\lambda\theta|} \leq \frac{|\gamma| \rho(\lambda)}{2c|\alpha\lambda\theta|}. \end{aligned} \quad (2.31)$$

Now, using Lemma 2.9, we get

$$\begin{aligned}
 I_1^2 &\leq \int_0^T e^{-\tau\nu} \rho * \rho * (|\xi|\rho(\xi))(\lambda) |\delta + \epsilon| d\tau \\
 &= |\delta + \epsilon| \rho * \rho * (|\xi|\rho(\xi))(\lambda) \int_0^T e^{-\tau\nu} d\tau.
 \end{aligned}
 \tag{2.32}$$

On the other hand, we have

$$\rho * \rho * (|x|\rho(x))(\lambda) \leq |\lambda|\rho^{*3}(\lambda) + \rho(\lambda) \int_{|\xi|\geq|\lambda|} |\xi|\rho(\lambda - \xi - \eta)\rho(\eta) d\xi d\eta.$$

Therefore (2.32) yields

$$\begin{aligned}
 I_1^2 &\leq |\delta + \epsilon| (|\lambda|\rho^{*3} + a_0\rho(\lambda)(1 + |\lambda|)) \int_0^T e^{-\tau\nu} d\tau \\
 &\leq |\delta + \epsilon| \left(\frac{|\lambda|\rho(\lambda)(c^{-1} + a_0)}{3\lambda^2|\theta\beta|} + \frac{\rho(\lambda)a_0}{2|\alpha\lambda\theta|} \right) = k_0 \frac{\rho(\lambda)}{|\lambda\theta|}.
 \end{aligned}
 \tag{2.33}$$

Hence from (2.30), (2.31) and (2.33) we obtain the following estimate for I_1 :

$$I_1 \geq \rho(\lambda) - \frac{\rho(\lambda)}{|\lambda\theta|} \left(k_0 + \frac{|\gamma|}{2c|\alpha|} \right) > 3c_0\rho(\lambda),
 \tag{2.34}$$

where in the last inequality we have chosen $|\theta| \geq c_1/|\lambda|$.

Remark 2.10. (1) To estimate I_1^2 in the case $\delta = 2\epsilon$ and nonlinearity $i\gamma|u|^2u + \epsilon(|u|^2u)_x$ we can use the following identity:

$$\begin{aligned}
 &\mathcal{F}(u) * (x\mathcal{F}(u)(x)) * \overline{\mathcal{F}^{-1}(u)}(z) + \frac{1}{2}\mathcal{F}(u) * \mathcal{F}(u) * (x\overline{\mathcal{F}^{-1}(u)}(x))(z) \\
 &= \frac{z}{2}\mathcal{F}(u) * \mathcal{F}(u) * \overline{\mathcal{F}^{-1}(u)}(z).
 \end{aligned}
 \tag{2.35}$$

(2) In the case when the solution is real and nonlinearity is of the form $u^{p_0}u_x$, $p_0 \in \mathbb{N}$, we can use

$$((xf(x)) * \overbrace{f * \dots * f}^n)(z) = \frac{z}{n+1} \overbrace{(f * \dots * f)}^{n+1}(z).$$

(3) The general case follows by using Lemma 2.9.

Estimate for I_3 . Let $L_1 := |\mathcal{F}(F_1)(\lambda + i\theta) - \mathcal{F}(F_1)(\lambda)|$, $L_2 := |\mathcal{F}(F_2)(\lambda + i\theta) - \mathcal{F}(F_2)(\lambda)|$ and $L_3 := |\mathcal{F}(F_3)(\lambda + i\theta) - \mathcal{F}(F_3)(\lambda)|$, so that I_3 can be written as

$$I_3 \leq \int_0^T (L_1 + L_2 + L_3)e^{-\tau\nu} d\tau.$$

Now, using Lemma 2.3, Corollary 2.8 and taking $|\theta| \lesssim 1/B(1 + |\log \rho(\lambda)|)$, we get

$$\begin{aligned}
 L_1 &= |\gamma| \left| \mathcal{F}(u) * \mathcal{F}(u) * \overline{\mathcal{F}^{-1}(u)}(\lambda + i\theta) - \mathcal{F}(u) * \mathcal{F}(u) * \overline{\mathcal{F}^{-1}(u)}(\lambda) \right| \\
 &= |\gamma| \left| \int_{\mathbb{R}^2} \mathcal{F}(u)(\lambda + i\theta - \xi - \eta) \mathcal{F}(u)(\eta) \overline{\mathcal{F}^{-1}(u)}(\xi) d\eta d\xi \right. \\
 &\quad \left. - \int_{\mathbb{R}^2} \mathcal{F}(u)(\lambda - \xi - \eta) \mathcal{F}(u)(\eta) \overline{\mathcal{F}^{-1}(u)}(\xi) d\eta d\xi \right| \\
 &\leq |\gamma| |\theta| \int_{\mathbb{R}^2} \sup_{|\theta'| \leq |\theta|} |(\mathcal{F}(u(t_1 + \tau)))'(\lambda + i\theta' - \xi - \eta)| \rho(\eta) \rho(\xi) d\xi d\eta \\
 &\leq |\gamma| \int_{\mathbb{R}^2} (\rho(\lambda) + \rho(\lambda - \xi - \eta)) \rho(\eta) \rho(\xi) d\xi d\eta \\
 &\leq |\gamma| (\rho(\lambda) k_1^2 + \rho^{*3}) \leq \rho(\lambda) (k_1^2 |\gamma| + c^{-1} |\gamma|), \tag{2.36}
 \end{aligned}$$

where $k_1 = \int_{\mathbb{R}} B_1 / (1 + \lambda^4) d\lambda$.

As in L_1 , using Lemmas 2.3 and 2.9, we obtain

$$L_2 \leq |\delta| \rho(\lambda) (k_2 + a_0) + |\delta| \rho(\lambda) |\lambda| (c^{-1} + a_0)$$

and

$$L_3 \leq |\epsilon| \rho(\lambda) (k_2 + a_0) + |\epsilon| \rho(\lambda) |\lambda| (c^{-1} + a_0).$$

Therefore,

$$L_2 + L_3 \lesssim (|\epsilon| + |\delta|) (\rho(\lambda) + \rho(\lambda) |\lambda|). \tag{2.37}$$

Now from (2.36) and (2.37) we get

$$L_1 + L_2 + L_3 \lesssim \rho(\lambda) + \rho(\lambda) |\lambda|.$$

Hence, we obtain the following estimate for I_3 :

$$I_3 \leq \int_0^T (L_1 + L_2 + L_3) e^{-3\tau(\lambda^2|\beta\theta| + 2|\alpha\lambda\theta|)} d\tau \leq \frac{\rho(\lambda)}{|\lambda\theta|} c_2 < c_0 \rho(\lambda), \tag{2.38}$$

where we have chosen θ such that $|\theta| > c_2 / (c_0 |\lambda|)$.

The use of (2.26), (2.34) and (2.38) in the estimate (2.21) yields

$$e^{-3T\lambda^2|\beta\theta|} \gtrsim 3c_0 \rho(\lambda) - c_0 \rho(\lambda) - c_0 \rho(\lambda) > c_0 e^{-|\lambda|/N}. \tag{2.39}$$

Since $|\lambda\theta| > c_3$, the estimate (2.39) gives

$$m_0 e^{-m_1 |\lambda|} \geq e^{-|\lambda|/N},$$

with $m_0 > 0, m_1 > 0$ constants. But this inequality is false for large values of N and λ if we choose $N > 2/m_1$ and $|\lambda| > 2|\log m_0|/m_1$. This concludes the proof of the theorem. \square

Remark 2.11. Note that if $c_3 = \max\{c_1, c_2/c_0\}$, $c > 0$ is a constant and λ is sufficiently large then it is easy to get the following relation:

$$\frac{c_3}{|\lambda|} < \frac{c}{B(1 + |\log \rho(\lambda)|)}.$$

So, we can choose θ such that $|\theta|$ lies between these two values.

3. Solution with compact support at two different time

As mentioned in the introduction, this section is devoted to prove Theorem 1.1. Let us start with the following remark of technical character.

Remark 3.1. We can suppose $\beta > 0$. In fact, for $\alpha \neq 0$ we can suppose $\beta = |\alpha|/3$.

If $\beta < 0$ we define $w(x, t) = u(x, -t)$ then w is a solution to Eq. (1.1) satisfying (1.2) or (1.3) and the coefficient of the third derivative is positive.

If $\beta > 0$ and $\alpha \neq 0$ we define $w(x, t) = u(\tilde{a}^{-1}x, t)$ with $\tilde{a} = |\alpha|/3\beta$, then w is a solution of the equation

$$w_t + i\alpha\tilde{a}^2 w_{xx} + \beta\tilde{a}^3 w_{xxx} + i\gamma|w|^2 w + \delta\tilde{a}|w|^2 w_x + \epsilon\tilde{a}w^2 \bar{w}_x = 0.$$

Now $|\alpha|\tilde{a}^2/3 = \beta\tilde{a}^3 = |\alpha|^3/(27\beta^2)$, moreover it is easy to see that if u satisfies hypothesis (1.2) or (1.3), then w also satisfies the same hypothesis.

3.1. Exponential decay of solution

In sequel we need the following results.

Lemma 3.2. Let $u = u(x, t)$ be a solution of Eq. (1.1) such that

$$\sup_{t \in [-T, T]} \|u(\cdot, t)\|_{H^1(\mathbb{R})} = c_0 < \infty.$$

If for $\eta > 0$, $e^{\eta x} u_0 \in L^2(\mathbb{R})$ then $e^{\eta x} u \in C([-T, T]; L^2(\mathbb{R}))$.

Proof. By Remark 3.1 we can suppose that $\beta > 0$ and $|\alpha|/3 \leq \beta$. Let us consider, $\varphi_n \in C^\infty(\mathbb{R})$, $0 \leq \varphi_n \leq e^{\eta x}$, $\varphi_n(x) = e^{\eta x}$, $x \leq n$; $\varphi_n(x) = e^{2n\eta}$, $x > 10n$; $0 \leq \varphi'_n(x) \leq \eta\varphi_n(x)$ and $|\varphi_n^{(j)}(x)| \leq \eta^j \varphi_n(x)$, $j = 2, 3$.

Multiplying (1.1) by $\varphi_n \bar{u}$ and taking real part we get after integration,

$$\begin{aligned} \frac{d}{dt} \int |u|^2 \varphi_n &= 2\alpha \operatorname{Im} \int \partial_x^2 u \bar{u} \varphi_n - 2\beta \operatorname{Re} \int \partial_x^3 u \bar{u} \varphi_n \\ &\quad - 2(\delta + \epsilon) \operatorname{Re} \int \partial_x u \bar{u} |u|^2 \varphi_n. \end{aligned} \tag{3.1}$$

Integrating by parts we get

$$\operatorname{Re} \int \partial_x u \bar{u} |u|^2 \varphi_n = -\frac{1}{4} \int |u|^4 \varphi_n', \quad (3.2)$$

$$2 \operatorname{Re} \int \partial_x^3 u \bar{u} \varphi_n = - \int |u|^2 \varphi_n^{(3)} + 3 \int |\partial_x u|^2 \varphi_n', \quad (3.3)$$

$$\operatorname{Im} \int \partial_x^2 u \bar{u} \varphi_n = \operatorname{Im} \int u \overline{\partial_x u} \varphi_n'. \quad (3.4)$$

Since $\varphi_n' \geq 0$ and $\operatorname{Im} \int u \overline{\partial_x u} \varphi_n' \leq \frac{1}{2} (\int |u|^2 \varphi_n' + \int |\partial_x u|^2 \varphi_n')$, we get from (3.1)–(3.4),

$$\begin{aligned} \frac{d}{dt} \int |u|^2 \varphi_n &\leq |\alpha| \int |u|^2 \varphi_n' + |\alpha| \int |u_x|^2 \varphi_n' + \beta \int |u|^2 \varphi_n^{(3)} \\ &\quad - 3\beta \int |\partial_x u|^2 \varphi_n' + \frac{(\delta + \epsilon)}{2} \int |u|^4 \varphi_n' \\ &= (|\alpha| - 3\beta) \int |\partial_x u|^2 \varphi_n' + |\alpha| \eta \int |u|^2 \varphi_n + \beta \eta^3 \int |u|^2 \varphi_n \\ &\quad + \frac{(\delta + \epsilon)}{2} \eta \int |u|^4 \varphi_n \\ &\leq (|\alpha| \eta + \beta \eta^3) \int |u|^2 \varphi_n + \frac{(\delta + \epsilon)}{2} \eta \|u\|_\infty^2 \int |u|^2 \varphi_n. \end{aligned} \quad (3.5)$$

Integrating and applying Gronwall's inequality we get

$$\int |u|^2 \varphi_n \leq \int |u_0|^2 \varphi_n e^{\mathcal{C}},$$

where $\mathcal{C} = |\alpha| \eta + \beta \eta^3 + c_0^2 \eta (\delta + \epsilon) / 2$ and in the limit $u e^{\eta x} \in C([-T, T]; L^2(\mathbb{R}))$. \square

In general we have the following lemma whose proof follows by induction.

Lemma 3.3. *Let $n \in \mathbb{Z}$, $n \geq 1$ and $u = u(x, t)$ be a solution of Eq. (1.1) such that*

$$\sup_{t \in [-T, T]} \|u(\cdot, t)\|_{H^n(\mathbb{R})} < \infty.$$

If for $\eta > 0$, $e^{\eta x} u_0, \dots, e^{\eta x} \partial_x^{n-1} u_0 \in L^2(\mathbb{R})$ then

$$\sup_{t \in [-T, T]} \|e^{\eta x} u(\cdot, t)\|_{H^{n-1}(\mathbb{R})} < \infty.$$

3.2. Carleman type estimates

Let us prove some multiplier estimates which will be used in sequel.

Proposition 3.4. *If $\phi(\xi) = a_3 \xi^3 + a_2 \xi^2 + a_1 \xi + a_0$, $a_3 \neq 0$ and $f \in C_0^{3,1}(\mathbb{R}^2)$ then*

$$\left\| \int_{\mathbb{R}} e^{i(x,t) \cdot (\xi, \phi(\xi))} \mathcal{F}(f)(\xi, \phi(\xi)) d\xi \right\|_{L^8(\mathbb{R}^2)} \lesssim |a_3|^{-1/4} \|f\|_{L^{8/7}(\mathbb{R}^2)}. \quad (3.6)$$

Proof. Let us define

$$V(t)v_0(x) := \int_{\mathbb{R}} e^{i(x\xi + t\phi(\xi))} \mathcal{F}(v_0)(\xi) d\xi = \mathcal{F}^{-1}(e^{it\phi(\cdot)} \mathcal{F}(v_0)(\cdot))(x). \tag{3.7}$$

Then we have

$$\int_{\mathbb{R}} V(t-t')f(x, t') dt' = \int_{\mathbb{R}} e^{i(x,t)\cdot(\xi, \phi(\xi))} \mathcal{F}(f)(\xi, \phi(\xi)) d\xi.$$

Therefore, (3.6) is equivalent to

$$\left\| \int_{\mathbb{R}} V(t-t')f(x, t') dt' \right\|_{L^8(\mathbb{R}^2)} \lesssim |a_3|^{-1/4} \|f\|_{L^{8/7}(\mathbb{R}^2)}. \tag{3.8}$$

Using Theorem A.1 in Appendix A for $q = 8$ we get

$$\|V(t)v_0\|_{L^8(\mathbb{R})} \lesssim |a_3t|^{-1/4} \|v_0\|_{L^{8/7}(\mathbb{R})}. \tag{3.9}$$

Now, using the Minkowski inequality, (3.9) and fractional integration, we get

$$\begin{aligned} \left\| \int_{\mathbb{R}} V(t-t')f(x, t') dt' \right\|_{L^8(\mathbb{R}^2)} &\leq \left\| \int_{\mathbb{R}} \|V(t-t')f(x, t')\|_{L^8_x(\mathbb{R})} dt' \right\|_{L^8_t(\mathbb{R})} \\ &\lesssim |a_3|^{-1/4} \left\| \int_{\mathbb{R}} \frac{\|f(\cdot, t')\|_{L^{8/7}_x(\mathbb{R})}}{|t-t'|^{1/4}} dt' \right\|_{L^8_t(\mathbb{R})} \\ &\lesssim |a_3|^{-1/4} \|f\|_{L^{8/7}(\mathbb{R}^2)}, \end{aligned} \tag{3.10}$$

which concludes the proof. \square

Proposition 3.5. For all $h \in C_0^{3,1}(\mathbb{R}^2)$ and $l \in \mathbb{R}, l \neq 0$, we have

$$\|h\|_{L^8(\mathbb{R}^2)} \lesssim |\beta|^{-1/4} \|(\partial_t + \beta\partial_x^3 + i\alpha\partial_x^2 + \zeta\partial_x + i\theta + l)h\|_{L^{8/7}(\mathbb{R}^2)}. \tag{3.11}$$

Proof. Let

$$Sh(x, t) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(x,t)\cdot(\xi, \tau)} \frac{1}{(\tau - \phi(\xi)) - il} \mathcal{F}(h)(\xi, \tau) d\xi d\tau,$$

where $\phi(\xi) = \beta\xi^3 + \alpha\xi^2 - \zeta\xi - \theta$.

So, (3.11) follows if we prove

$$\|Sh(x, t)\|_{L^8(\mathbb{R}^2)} \lesssim |\beta|^{-1/4} \|h\|_{L^{8/7}(\mathbb{R}^2)}. \tag{3.12}$$

Now we have

$$Sh(x, t) = \int_{\mathbb{R}} b(s)V(s)h(\cdot, t-s)(x) ds, \tag{3.13}$$

where $V(s)$ is the unitary group defined in (3.7) with $a_3 = \beta$, $a_2 = \alpha$, $a_1 = -\zeta$, $a_0 = -\theta$ and

$$b(s) = \int_{\mathbb{R}} \frac{e^{is\tau}}{\tau - il} d\tau = \begin{cases} 2i\pi e^{-ls} \chi_{\{s \leq 0\}}(s) & \text{if } l < 0, \\ 2i\pi e^{-ls} \chi_{\{s \geq 0\}}(s) & \text{if } l > 0. \end{cases}$$

As $|b(s)| \leq 2\pi$ for all $l, s \in \mathbb{R}$, using (3.9) and fractional integration we can obtain the required result. \square

Remark 3.6. By the result of the previous section, if $f \in C_0^{3,1}(\mathbb{R}^2)$ is such that $(\partial_t + i\alpha\partial_x^2 + \beta\partial_x^3)f = 0$ then $f = 0$.

Now, we prove a Carleman type estimate which is the main ingredient in our argument to prove Theorem 1.1.

Lemma 3.7. If $f \in C_0^{3,1}(\mathbb{R}^2)$ then

$$\|e^{\lambda x} f\|_{L^8(\mathbb{R}^2)} \leq c(\beta) \|e^{\lambda x} (\partial_t + i\alpha\partial_x^2 + \beta\partial_x^3)f\|_{L^{8/7}(\mathbb{R}^2)}, \quad (3.14)$$

for all $\alpha, \lambda \in \mathbb{R}$ with $c = c(\beta)$ independent of λ, α .

Proof. It is enough to prove (3.14) for $\lambda = \pm 1$. For, if

$$\|e^x f\|_{L^8(\mathbb{R}^2)} \leq c \|e^x (\partial_t + i\alpha\partial_x^2 + \beta\partial_x^3)f\|_{L^{8/7}(\mathbb{R}^2)}, \quad (3.15)$$

and $\lambda > 0$, then making change of variables $(y, s) = (x/\lambda, t/\lambda^3)$, we get

$$\|e^x f_\lambda\|_{L^8(\mathbb{R}^2)} = \lambda^{1/2} \|e^{\lambda y} f\|_{L^8(\mathbb{R}^2)} \quad (3.16)$$

and

$$\begin{aligned} & \|e^x (\partial_t + i\alpha\partial_x^2 + \beta\partial_x^3)f_\lambda\|_{L^{8/7}(\mathbb{R}^2)} \\ &= \lambda^{1/2} \|e^{\lambda y} (\partial_s + i\alpha\lambda\partial_y^2 + \beta\partial_y^3)f\|_{L^{8/7}(\mathbb{R}^2)}. \end{aligned} \quad (3.17)$$

Since $c = c(\beta)$ is independent of α and λ , using (3.16) and (3.17) in (3.15) we get (3.14). The case $\lambda < 0$ is similar and the case $\lambda = 0$ follows from the case $\lambda \neq 0$ by taking the limit as $\lambda \rightarrow 0$.

Observe that, if we prove

$$\begin{aligned} \|g\|_{L^8(\mathbb{R}^2)} &\leq c(\beta) \|(\partial_t + \beta\partial_x^3 - 3\beta\partial_x^2 + 3\beta\partial_x - \beta + i\alpha + i\alpha\partial_x^2 - 2i\alpha\partial_x)g\|_{L^{8/7}(\mathbb{R}^2)} \\ &= c(\beta) \|(\partial_t + \beta(\partial_x - 1)^3 + i\alpha(\partial_x - 1)^2)g\|_{L^{8/7}(\mathbb{R}^2)}, \end{aligned} \quad (3.18)$$

for all $g \in C_0^{3,1}(\mathbb{R}^2)$, then (3.15) follows by taking $g(x, t) = e^x f(x, t)$. Hence our interest here is to prove (3.18). For this we use Propositions 3.4 and 3.5.

Let us define

$$Lh = (\partial_t + \beta\partial_x^3 - 3\beta\partial_x^2 + 3\beta\partial_x - \beta + i\alpha\partial_x^2 - 2i\alpha\partial_x + i\alpha)h.$$

Then

$$\mathcal{F}(Lh)(\xi, \tau) = [i(\tau - \phi(\xi)) + \zeta(\xi)]\mathcal{F}(h)(\xi, \tau) = \varphi(\xi, \tau)\mathcal{F}(h)(\xi, \tau),$$

where $\phi(\xi) = \beta\xi^3 + \alpha\xi^2 - 3\beta\xi - \alpha$, $\zeta(\xi) = 3\beta\xi^2 + 2\alpha\xi - \beta$.

As $\beta \neq 0$, we have $\varphi(\xi, \tau) = 0$ at the pair of points

$$P_{\pm} = (\xi_0^{\pm}, \tau_0^{\pm}) = \left(-\frac{\alpha}{3\beta} \pm \sqrt{\left(\frac{\alpha}{3\beta}\right)^2 + \frac{1}{3}}, \phi(\xi_0^{\pm}) \right).$$

For $\alpha \neq 0$, by Remark 3.1, we can consider $\beta > 0$ and $|\alpha|/3\beta = 1$. Therefore we have

$$\xi_0^{\pm} = \begin{cases} \pm 1/\sqrt{3} & \text{if } \alpha = 0, \\ -1 \pm \sqrt{4/3} & \text{if } \alpha > 0, \\ 1 \pm \sqrt{4/3} & \text{if } \alpha < 0. \end{cases}$$

In this way we get

$$\xi_0^+ > 0, \quad \xi_0^- < 0, \quad |\xi_0^{\pm}| \sim 1. \tag{3.19}$$

Since h has compact support, we prove (3.18) by supposing $h \in \mathcal{S}(\mathbb{R}^2)$, with $\mathcal{F}(h)(P_{\pm}) = 0$. Therefore (3.18) is equivalent to the following multiplier estimate:

$$\|Sh(x, t)\|_{L^8(\mathbb{R}^2)} \leq c(\beta)\|h\|_{L^{8/7}(\mathbb{R}^2)}, \tag{3.20}$$

where

$$Sh(x, t) = \iint_{\mathbb{R} \times \mathbb{R}} e^{i(x,t) \cdot (\xi, \tau)} \frac{1}{i(\tau - \phi(\xi)) + \zeta(\xi)} \mathcal{F}(h)(\xi, \tau) d\xi d\tau. \tag{3.21}$$

We prove (3.20) in the case $\text{supp } \mathcal{F}(h) \subset \{(\xi, \tau): \xi \geq 0\}$. The case $\text{supp } \mathcal{F}(h) \subset \{(\xi, \tau): \xi < 0\}$ is similar.

From Littlewood–Paley theory it is enough to show (3.20) for each $M_k h$ (see Step 5 in [9]), where

$$\mathcal{F}(M_k h)(\xi, \tau) = \chi_{[1/2, 1]}(|\xi - \xi_0^+|/2^{-k})\mathcal{F}(h)(\xi, \tau), \quad k \in \mathbb{Z}.$$

So, we suppose that

$$\text{supp } \mathcal{F}(h) \subseteq \{(\xi, \tau): \xi \geq 0, 2^{-k-1} \leq |\xi - \xi_0^+| < 2^{-k}\}.$$

We consider the cases $k \leq 0$ and $k > 0$ separately. First, let us consider the case $k \leq 0$.

If $\xi \in \text{supp } \mathcal{F}(h)$, then from (3.19) we get

$$\zeta(\xi) \sim |\xi - \xi_0^+| |\xi - \xi_0^-| \sim 2^{-k} 2^{-k}.$$

In fact, as $|\xi| \lesssim 2^{-k}$, $|\xi_0^-| \lesssim 2^{-k}$, we have $|\xi - \xi_0^-| \lesssim 2^{-k}$. On the other hand, if $c_0 = \xi_0^+ - \xi_0^-$, then we have $1 < c_0 < 4$ and for all $k \in \mathbb{Z}$,

$$|\xi - \xi_0^-| \geq \max\{c_0 - 2^k, 2^{-k-1} - c_0\}. \tag{3.22}$$

Hence, $|\xi - \xi_0^-| \gtrsim 2^{-k}$ if $k \leq 0$. Using Proposition 3.5 it is enough to prove the inequality for the multiplier

$$\begin{aligned} & \frac{1}{i(\tau - \phi(\xi)) + \zeta(\xi)} - \frac{1}{i(\tau - \phi(\xi)) + 2^{-2k}} \\ &= \frac{(2^{-2k} - \zeta(\xi))}{[i(\tau - \phi(\xi)) + \zeta(\xi)][i(\tau - \phi(\xi)) + 2^{-2k}]} := m_1(\xi, \tau). \end{aligned}$$

If $\mathcal{F}(S_k h)(x, t) := m_1(x, t)\mathcal{F}(M_k h)(x, t)$, we have

$$S_k h(x, t) = \int_{\mathbb{R}} e^{it\tau} \int_{\mathbb{R}} e^{i(x,t) \cdot (\xi, \phi(\xi))} \mathcal{F}(h_\tau)(\xi, \phi(\xi)) d\xi d\tau,$$

where

$$\mathcal{F}(h_\tau)(\xi, \lambda) = \frac{(2^{-2k} - \zeta(\xi))}{[i\tau + \zeta(\xi)][i\tau + 2^{-2k}]} \mathcal{F}(M_k h)(\xi, \tau + \lambda).$$

Using Proposition 3.4 and the fact that h_τ is a multiplier in $L^p(\mathbb{R})$, $1 < p < \infty$, with norm

$$\|h_\tau\|_p \lesssim \frac{2^{-2k} |\beta|^{-1/4}}{\tau^2 + 2^{-4k}}, \quad (3.23)$$

we get

$$\|S_k h\|_{L^8(\mathbb{R}^2)} \leq \int_{\mathbb{R}} \|h_\tau\|_{L^{8/7}(\mathbb{R}^2)} d\tau \leq c(\beta) \|M_k h\|_{L^{8/7}(\mathbb{R}^2)}.$$

Now, we consider the case $k > 0$. From (3.19) and (3.22) we have, for $\xi \in \text{supp } \mathcal{F}(h)$,

$$\zeta(\xi) \sim |\xi - \xi_0^+| |\xi - \xi_0^-| \sim 2^{-k}.$$

The rest of the proof is similar to the case $k \leq 0$, with 2^{-k} in place of 2^{-2k} . \square

As a consequence of Lemma 3.7 it is not difficult to prove the following results using Lemma 2.5 in [9].

Lemma 3.8. *Let $g \in C^{3,1}(\mathbb{R} \times [t_1, t_2])$. If for all $\eta > 0$,*

$$\sum_{j \leq 2} |\partial_x^j g(x, t)| \leq c_\eta e^{-\eta|x|}, \quad t \in [t_1, t_2],$$

and $g(x, t_1) = g(x, t_2) = 0$ for all $x \in \mathbb{R}$, then

$$\|e^{\lambda x} g\|_{L^8(\mathbb{R} \times [t_1, t_2])} \leq c(\beta) \|e^{\lambda x} (\partial_t + i\alpha \partial_x^2 + \beta \partial_x^3) g\|_{L^{8/7}(\mathbb{R} \times [t_1, t_2])}, \quad (3.24)$$

for all $\lambda \in \mathbb{R}$, $c = c(\beta)$ independent of λ and α .

Lemma 3.9. *Let $u \in C([t_1, t_2]; H^4(\mathbb{R})) \cap C^1([t_1, t_2]; H^1(\mathbb{R}))$ be a solution of Eq. (1.1) such that*

$$\text{supp } u(x, t_1) \subseteq (-\infty, a].$$

Then for all $\eta > 0$,

$$\sum_{j \leq 2} |\partial_x^j u(x, t)| \leq c_{a,\eta} e^{-\eta x}, \quad t \in [t_1, t_2], \quad x \in \mathbb{R}.$$

3.3. Proof of the main result

Proof of Theorem 1.1. If $u \in C([t_1, t_2]; H^s(\mathbb{R})) \cap C^1([t_1, t_2]; H^1(\mathbb{R}))$, $s \geq 4$, is a solution to Eq. (1.1) then we have

$$(\partial_t + i\alpha\partial_x^2 + \beta\partial_x^3)u_R(x, t) = -\mu_R V u + F_R, \tag{3.25}$$

where $u_R(x, t) = \mu_R(x)u(x, t)$, $\mu_R = \mu(x/R)$, $\mu \in C^\infty(\mathbb{R})$, is nondecreasing for $x > 0$, given by

$$\mu(x) = \begin{cases} 1 & \text{if } x \geq 2, \\ 0 & \text{if } x < 1, \end{cases} \tag{3.26}$$

$$F_R = i\alpha(\mu_R''u + 2\mu_R'\partial_x u) + \beta(\mu_R'''u + 3\mu_R''\partial_x u + 3\mu_R'\partial_x^2 u) \tag{3.27}$$

and

$$V = i\gamma|u|^2 + \delta\bar{u}\partial_x u + \epsilon u\partial_x \bar{u} \in L^p(\mathbb{R} \times [t_1, t_2]), \quad p \geq 1. \tag{3.28}$$

Since $\sup_{t \in [t_1, t_2]} \|u(\cdot, t)\|_{H^4(\mathbb{R})} < \infty$, (3.28) is justified by using Sobolev embedding theorem ($\|u\|_{L^q(\mathbb{R})} \leq c_q \|u\|_{H^{1/2}(\mathbb{R})}$, $q > 2$).

From Lemma 3.9 we get

$$\sum_{j=0}^2 |\partial_x^j(u_R)(x, t)| \leq C(R, \mu, a, \eta)e^{-\eta x}, \quad t \in [t_1, t_2], x \in \mathbb{R}.$$

Moreover by (1.2), $u_R(x, t_1) = u_R(x, t_2) = 0$ for large R . Therefore, by Lemma 3.8, Hölder’s inequality and (3.28), we get

$$\begin{aligned} & \|e^{\lambda x} u_R\|_{L^8(\mathbb{R} \times [t_1, t_2])} \\ & \leq c \|e^{\lambda x} u_R\|_{L^8(\mathbb{R} \times [t_1, t_2])} \|V\|_{L^{4/3}(\{|x| \geq R\} \times [t_1, t_2])} + c \|e^{\lambda x} F_R\|_{L^{8/7}(\mathbb{R} \times [t_1, t_2])} \\ & \leq \frac{1}{2} \|e^{\lambda x} u_R\|_{L^8(\mathbb{R} \times [t_1, t_2])} + c \|e^{\lambda x} F_R\|_{L^{8/7}(\mathbb{R} \times [t_1, t_2])}. \end{aligned} \tag{3.29}$$

Since all the terms in F_R have compact support in $[R, 2R]$, from (3.29) we easily obtain

$$\left(\int_{t_1}^{t_2} \int_{x > 2R} e^{8\lambda(x-2R)} |u(x, t)|^8 dx dt \right)^{1/8} \leq C(R) < \infty, \tag{3.30}$$

which is contradiction for $\lambda > 0$ arbitrarily large unless $\text{supp } u(t) \subseteq (-\infty, 2R]$ for $t \in [t_1, t_2]$.

On the other hand, if we consider $\mu \in C^\infty(\mathbb{R})$, nonincreasing for $x < 0$, defined by

$$\mu(x) = \begin{cases} 1 & \text{if } x \leq -2, \\ 0 & \text{if } x > -1, \end{cases}$$

and use the above argument, it easy to get

$$\left(\int_{t_1}^{t_2} \int_{x < -2R} e^{8\lambda(x+2R)} |u(x, t)|^8 dx dt \right)^{1/8} \leq C(R) < \infty,$$

which is again a contradiction for $\lambda < 0$, arbitrarily large unless $\text{supp } u(t) \subseteq [-2R, \infty)$, for $t \in [t_1, t_2]$.

Hence, $\text{supp } u(t) \subseteq [-2R, 2R]$, for all $t \in [t_1, t_2]$, and using Theorem 1.2 we conclude that $u(t) = 0$, for all $t \in [t_1, t_2]$. \square

Remark 3.10. The proof of Theorem 1.1 does not follow in a straightforward manner from the theory developed in [9] although we can write (3.25) in the form

$$\{\partial_t + \beta \partial_x^3\} u_R(x, t) = -\mu_R V u + F_R,$$

where V as in (3.28) and F_R in (3.27) replaced with

$$F_R = -\mu_R (i\alpha \partial_x^2 u) + \beta (\mu_R''' u + 3\mu_R'' \partial_x u + 3\mu_R' \partial_x^2 u).$$

As mentioned in the introduction, we cannot do with this modification because the term $-\mu_R (i\alpha \partial_x^2 u)$ in F_R does not behave like other terms. This is due to the lack of compact support and decay property for this term. As seen in the proof of Theorem 1.1 above, the existence of compact support for the terms in F_R is essential in the argument we employed.

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Appendix A

Theorem A.1. Let $u_0 \in L^{q'}(\mathbb{R})$, $\phi(\xi) = a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3$, $a_j \in \mathbb{R}$, $a_3 \neq 0$ and define

$$Lu_0(x) = \int_{\mathbb{R}} e^{i\phi(\xi) + ix\xi} \mathcal{F}(u_0)(\xi) d\xi = u_0 * I^\phi(x),$$

where

$$I^\phi(x) = \int_{\mathbb{R}} e^{i\phi(\xi) + ix\xi} d\xi.$$

Then for $0 \leq \theta \leq 1$, $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{q} = \frac{(1-\theta)}{2}$,

$$\|Lu_0\|_{L^q(\mathbb{R})} \leq C |a_3|^{-\theta/3} \|u_0\|_{L^{q'}(\mathbb{R})}, \tag{A.1}$$

where C is a constant independent of a_j , $j = 0, \dots, 3$.

Proof. See in Carvajal and Linares [5]. \square

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