

**CALCULUS 4**

**FOURIER SERIES**

**Definition to know**

A Fourier series is an expansion of a periodic function  $f(x)$  in terms of an infinite sum of sines and cosines. The Fourier series make use of the orthogonality relationships of the sine and cosine functions.

The functions  $\text{Sin}(m\pi x/L)$  and  $\text{Cos}(m\pi x/L)$ ,  $m = 1, 2, \dots$  form a mutually orthogonal set of functions on the interval  $-L \leq x \leq L$  satisfying the following relations:

$$\int_{-L}^L \text{Cos} \frac{m\pi x}{L} \text{Cos} \frac{n\pi x}{L} dx = \begin{cases} 0, m \neq n \\ L, m = n \end{cases}$$

$$\int_{-L}^L \text{Cos} \frac{m\pi x}{L} \text{Sin} \frac{n\pi x}{L} dx = 0, \text{ all } m, n$$

$$\int_{-L}^L \text{Sin} \frac{m\pi x}{L} \text{Sin} \frac{n\pi x}{L} dx = \begin{cases} 0, m \neq n \\ L, m = n \end{cases}$$

The Fourier series of a function  $f(x)$  is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \text{Cos} \frac{n\pi x}{L} + b_n \text{Sin} \frac{n\pi x}{L} \right)$$

where  $a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \text{Cos} \frac{n\pi x}{L} dx, n = 0, 1, 2, \dots$$

and  $b_n = \frac{1}{L} \int_{-L}^L f(x) \text{Sin} \frac{n\pi x}{L} dx, n = 1, 2, 3, \dots$

**Orthogonal Function**

The standard inner product  $(u, v)$  of two real-valued functions  $u$  and  $v$  on the interval  $\alpha \leq x \leq \beta$  is defined

$$\text{by } (u, v) = \int_{\alpha}^{\beta} u(x)v(x) dx$$

The functions  $u$  and  $v$  are said to be orthogonal on  $\alpha \leq x \leq \beta$  if their inner product is zero that is

$$\int_{\alpha}^{\beta} u(x)v(x) dx = 0$$

**The Fourier Convergence Theorem**

Suppose that  $f$  and  $f'$  are piecewise continuous on the interval  $-L \leq x < L$  so that it is periodic with period  $2L$ . Then  $f$  has a Fourier series.

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left( a_m \text{Cos} \frac{m\pi x}{L} + b_m \text{Sin} \frac{m\pi x}{L} \right)$$

whose coefficients are:

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \text{Cos} \frac{(n\pi x)}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \text{Sin} \frac{(n\pi x)}{L} dx$$

The Fourier series converges to  $f(x)$  at all points where  $f$  is continuous and to  $[f(x+) + f(x-)]/2$  at all points where  $f$  is discontinuous.

**Even and Odd Functions**

A function  $f$  is an even function if its domain contain the point  $-x$  wherever it contains the point  $x$  and if  $f(-x) = f(x)$ , for each  $x$  in the domain of  $f$ .

A function  $f$  is an odd function if its domain contains  $-x$  wherever it contains  $x$  and if  $f(-x) = -f(x)$  for each  $x$  in the domain of  $f$ .

**Fourier cosine series:** The Fourier series of any even function consists only of the even trigonometric functions  $\text{Cos}(n\pi x/L)$  and the constant term, such a series is called Fourier cosine series. The series is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \text{Cos} \frac{n\pi x}{L}$$

where  $a_n = \frac{2}{L} \int_0^L f(x) \text{Cos} \frac{n\pi x}{L} dx, n = 0, 1, 2, \dots$

**Fourier series:** The Fourier series of any odd function consists only of the trigonometric functions  $\text{Sin}(n\pi x/L)$  such a series is called Fourier sine series and it is given by:

$$f(x) = \sum_{n=1}^{\infty} b_n \text{Sin} \frac{n\pi x}{L}$$

where  $b_n = \frac{2}{L} \int_0^L f(x) \text{Sin} \frac{n\pi x}{L} dx, n = 1, 2, \dots$

**Heat Conduction Equation**

**Definition to know**

Consider a straight bar of uniform cross section and homogeneous material. Let the  $x$  – axis be taken along the bar and let  $x = 0$  and  $x = L$  denote the ends of the bar. Suppose further that the sides of the cross sectional dimensions are so small that the temperature  $u$  can be considered constant on any given cross section. Then  $u$  is a function of  $x$  and time  $t$ . The variation of temperature in the bar is governed by a partial differential equation,  $\alpha^2 u_{xx} = u_t$ ,  $0 < x < L$

(where  $\alpha^2$  is a constant known as thermal diffusivity)  
This equation is known as the heat conduction equation.

**Solution of Heat Conduction Equation**

The solution of heat conduction equation  $\alpha^2 u_{xx} = u_t$ ,  $0 < x < L$ ,  $t > 0$

with boundary conditions

$$u(0, t) = 0, u(L, t) = 0, t > 0$$

and the initial condition

$$u(x, 0) = f(x), 0 \leq x \leq L$$

is given by

$$u(x, t) = \sum_{n=1}^{\infty} C_n e^{-n^2 \pi^2 \alpha^2 t / L^2} \text{Sin} \frac{n\pi x}{L}$$

where  $c_n = \int_0^L f(x) \text{Sin} \frac{n\pi x}{L} dx$

and  $f(x) = \sum_{n=1}^{\infty} c_n \text{Sin} \frac{n\pi x}{L}$

**Special cases:**

1. Non – homogeneous boundary conditions:

If the boundary conditions are:  $u(0, t) = T_1$ ,  $u(L, t) = T_2$ ,  $t > 0$  then the solution is given by:

$$u(x, t) = (T_2 - T_1) \frac{x}{L} + T_1 + \sum_{n=1}^{\infty} c_n e^{-n^2 \pi^2 \alpha^2 t / L^2} \text{Sin} \frac{n\pi x}{L}$$

where  $c_n = \frac{2}{L} \int_0^L \left[ f(x) - (T_2 - T_1) \frac{x}{L} - T_1 \right] \text{Sin} \frac{n\pi x}{L} dx$

2. Bar with insulated ends:

If the bar has insulated ends, then in this case the boundary conditions are:  $u_x(0, t) = 0$ ,  $u_x(L, t) = 0$ ,  $t > 0$

Then the solution is given by:

$$u(x, t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2 \pi^2 \alpha^2 t / L^2} \text{Cos} \frac{n\pi x}{L}$$

where  $c_n = \frac{2}{L} \int_0^L f(x) \text{Cos} \frac{n\pi x}{L} dx$ ,  $n = 0, 1, 2, \dots$

**The Wave Equation**

**Definition to know**

Suppose that an elastic string of length  $L$  is tightly stretched between two supports at the same horizontal level, so that the  $x$ -axis lies along the string.

Suppose that the string is set in motion so that it vibrates in vertical plane. Let  $u(x, t)$  denotes at the point  $x$  at time  $t$ . If the damping effects are neglected and if amplitude of motion is not so large then  $u(x, t)$  satisfies the partial differential equation.  $a^2 u_{xx} = u_{tt}$  domain  $0 < x < L, t > 0$

This equation is known as one – dimensional wave equation.

The constant coefficient  $a^2$  is given by  $a^2 = \frac{T}{\rho}$  where  $T$

is the tension in the string and  $\rho$  is mass per unit length.

**Solution of Wave Equation**

The wave equation is:  $a^2 u_{xx} = u_{tt}$

with boundary conditions  $u(0, t) = 0, u(L, t) = 0, t \geq 0$

and initial conditions

Initial position:  $u(x, 0) = f(x), 0 \leq x \leq L$

Initial velocity:  $u_t(x, 0) = g(x), 0 \leq x \leq L$

Where  $f$  and  $g$  are given functions with  $f(0) = f(L) = 0, g(0) = g(L) = 0$

1. If the initial displacement is non – zero but initial velocity is zero then the initial conditions are:

$$u(x, 0) = f(x), u_t(x, 0) = 0, 0 \leq x \leq L$$

Then the solution of the wave equation is given by:

$$u(x, t) = \sum_{n=0}^{\infty} c_n \text{Sin} \frac{n\pi x}{L} \text{Cos} \frac{n\pi at}{L}$$

where  $c_n = \frac{2}{L} \int_0^L f(x) \text{Sin} \frac{(n\pi x)}{L} dx$ ,  $n = 1, 2, \dots$

2. If the initial conditions are:

$$u(x, 0) = 0, u_t(x, 0) = g(x), 0 \leq x \leq L$$

then the solution of wave equation is given by:

$$u(x, t) = \sum_{n=1}^{\infty} k_n \text{Sin} \frac{n\pi x}{L} \text{Sin} \frac{n\pi at}{L}$$

where  $k_n = \frac{2}{n\pi a} \int_0^L g(x) \text{Sin} \frac{n\pi x}{L} dx$ ,  $n = 1, 2, \dots$

LAPLACE'S EQUATION

Laplace's Equation

The Laplace's equation is given by:

$$u_{xx} + u_{yy} = 0 \text{ (in two dimensions)}$$

and  $u_{xx} + u_{yy} + u_{zz} = 0$  (in three dimensions)

Dirichlet Problem for a Rectangle

If the function  $u$  satisfies the Laplace's equation  $u_{xx} + u_{yy} = 0$  in the rectangle  $0 < x < a, 0 < y < b$  and also satisfying boundary conditions

$$u(x, 0) = 0, u(x, b) = 0, 0 < x < a$$

$$u(0, y) = 0, u(a, y) = f(y), 0 \leq y \leq b$$

where  $f$  is a given function on  $0 \leq y \leq b$  then its solution is given by

$$u(x, y) = \sum_{n=1}^{\infty} c_n \text{Sinh} \frac{n\pi a}{b} \text{Sin} \frac{n\pi y}{b}$$

where  $c_n \text{Sinh} \frac{n\pi a}{b} = \frac{2}{b} \int_0^b f(y) \text{Sin} \frac{n\pi y}{b} dy$

Normalized Eigenfunctions

The Eigenfunction  $\phi_n(x)$  associated with Eigen values  $\lambda_n$  of Sturm Liouville problem are said to be normalized if they satisfy the condition

$$\int_0^1 r(x) \phi_n^2 dx = 1, n = 1, 2, \dots$$

Sturm Liouville's Boundary Value Problem

It consists of a differential equation of the form

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0$$

on the interval  $0 < x, 1$ , together with boundary conditions

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0$$

$$\beta_1 y(1) + \beta_2 y'(1) = 0$$

**Note:** If we introduce the linear homogeneous differential operator defined by  $L[y] = -[p(x)y']' + q(x)y$  then the differential equation

$$[p(x)y']' + q(x)y + \lambda r(x)y = 0$$

can be written as  $L[y] = \lambda r(x)y$

Dirichlet's Problem for a Circle

If  $u$  is function satisfying Laplace's equation in a circular region  $r < a$  subject to boundary condition  $u(a, \theta) = f(\theta)$  where  $f$  is a given function on  $0 \leq \theta \leq 2\pi$ , then Laplace's equation in polar co-ordinates is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

and its solution is given by:

$$u(r, \theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n (c_n \text{Cos}\theta + k_n \text{Sinn}\theta)$$

where  $a^n c_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \text{Cos}n\theta d\theta, n = 0, 1, 2, \dots$

and  $a^n k_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \text{Sinn}\theta d\theta, n = 1, 2, \dots$

Self-Adjoint Boundary Value Problem

A boundary value problem is said to be self-adjoint if every pair of functions say  $u$  and  $v$  satisfying boundary conditions satisfy the condition

$$(L[u], v) - (u, L[v]) = 0$$

(where  $(u, v) = \int_a^b u(x)v(x) dx$ )

Lagrange's Identity

If  $u$  and  $v$  be functions having continuous second derivatives on the interval  $0 \leq x \leq 1$  and satisfying boundary conditions  $\alpha_1 u(0) + \alpha_2 u'(0) = 0, \beta_1 u(1) + \beta_2 u'(1) = 0$

and  $\alpha_1 v(0) + \alpha_2 v'(0) = 0, \beta_1 v(1) + \beta_2 v'(1) = 0$

then the Lagrange's identity is

$$\int_0^1 \{L[u]v - uL[v]\} dx = 0$$

Results to know:

- All the Eigen values of Sturm - Liouville problem are real.
- If  $\phi_1$  and  $\phi_2$  are two Eigen functions of the Sturm - Liouville problem corresponding to Eigen values  $\lambda_1$  and  $\lambda_2$  respectively, and if  $\lambda_1 \neq \lambda_2$  then

$$\int_0^1 r(x) \phi_1(x) \phi_2(x) dx = 0$$

- The Eigen values of the Sturm - Liouville problem are all simple, i.e. to each Eigen value there corresponds only one linearly independent Eigen function.