CALCULUS 2

THREE DIMENSIONAL CO-ORDINATE SYSTEMS

Good to know

Distance formula: The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$$

Equation of a sphere: The equation of the sphere with canter at c(h, k, l) and radius r is:

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

VECTORS

Good to know

Addition of Vectors

If $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_1, b_2, b_3 >$

then $\vec{a}+\vec{b} =< a_1+b_1, a_2+b_2, a_3+b_3 >$

and $\vec{a}-\vec{b} =< a_1-b_1, a_2-b_2, a_3-b_3 >$

The Dot Product

If $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_1, b_2, b_3 >$

then the dot product of $\vec{a}$ and $\vec{b}$ is

$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Projections

Scalar projection of $\vec{b}$ and $\vec{a}$ is :

$$Comp_{\vec{a}}\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{a}|}$$

Vector projection of $\vec{b}$ and $\vec{a}$ is :

$$\Pr oj_{\vec{a}}\vec{b} = \left(\frac{\vec{a}.\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|} = \frac{(\vec{a}.\vec{b})}{|\vec{a}|^2}\vec{a}$$

The Cross Product

If $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_1, b_2, b_3 >$ then

the cross product of $\vec{a}$ and $\vec{b}$ is a vector given by

$\vec{a}.\vec{b} =< a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 >$

Theorem

If θ is the angle between $\vec{a}$ and $\vec{b}$ then

$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

Properties of Dot Product

* $\vec{a}.\vec{b} = |\vec{a}|^2$ where $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

  when $\vec{a} =< a_1, a_2, a_3 >$
* $\vec{a}.\vec{b} = \vec{b}.\vec{a}$
* $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$
* $(c.\vec{a}).\vec{b} = c.(\vec{a}.\vec{b}) = \vec{a}.(c.\vec{b})$
* $(\vec{0}.\vec{a}) = \vec{a}.\vec{0}) = 0$
* If θ is the angle between the vector $\vec{a}$ and $\vec{b}$ then $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$
* The vectors $\vec{a}$ and $\vec{b}$ are orthogonal if $\vec{a}.\vec{b} = 0$
* $\vec{a}$ and $\vec{b}$ are parallel and directed in same direction if $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|$
* $\vec{a}$ and $\vec{b}$ are parallel and directed in opposite direction if $\vec{a}.\vec{b} = -|\vec{a}||\vec{b}|$

Results

* Two non-zero vectors $\vec{a}$ and $\vec{b}$ are parallel if $\vec{a}\times\vec{b} = 0$
* The length of cross product $\vec{a}\times\vec{b}$ is equal to the area of parallelogram determined by vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is:

  $v = |\vec{a}.(\vec{b}\times\vec{c})|$

VECTORS

Important Properties

$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$	$(k\vec{a})\times\vec{b} = k(\vec{a}\times\vec{b}) = \vec{a}\times(k\vec{b})$	$\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$
$(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c}+\vec{b}\times\vec{c}$	$\vec{a}.(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b}).\vec{c}$	$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})b - (\vec{a}.\vec{b})\vec{c}$

EQUATION OF LINES AND PLANES

Vector Equation of a Line

$\vec{r} = \vec{r}_0 + t\vec{\upsilon}$ where $\vec{r}_0$ is the position vector of a point on line, $\vec{\upsilon}$ is the vector parallel to line and t is the parameter.

Parametric Equation of Line

$x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$

where (x_0, y_0, z_0) is a point on line and $< a,b,c >$ are the direction number of the line

Symmetric Equations of Line

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

Equation of Plane

Vector equation: If $\vec{n}$ is the normal vector of plane and is the $\vec{r}_0$ position vector of a point in the plane then its vector equation is: $\vec{n}.(\vec{r}-\vec{r}_0) = 0$

Scalar equation: If (x_0, y_0, z_0) is a point in plane and $< a,b,c >$ are the direction number of normal to the plane then the equation of plane is:

$$a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$$

or $ax+by+cz+d = 0$ where $d = -(ax_0+by_0+cz_0)$

Perpendicular distance of a point (x_1, y_1, z_1) from the plane $ax+by+cz = d$

$$D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

Equation you will need

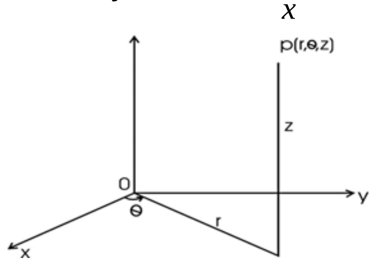
Quadric Surface	Equation	Quadric Surface	Equation
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$	Elliptic paraboloid	$\frac{z}{c} = \frac{x^2}{a^2}+\frac{y^2}{b^2}$
Hyperbolic paraboloid	$\frac{z}{c} = \frac{x^2}{a^2}-\frac{y^2}{b^2}$	Cone	$\frac{z^2}{c^2} = \frac{x^2}{a^2}+\frac{y^2}{b^2}$
Hyperboloid of one sheet	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = 1$	Hyperboloid of two sheet	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$

Cylindrical co-ordinates

$x = r\cos\theta$, $y = r\sin\theta$, z = z

where $r^2 = x^2+y^2$, $\tan\theta = \frac{y}{x}$, z = z

Diagram

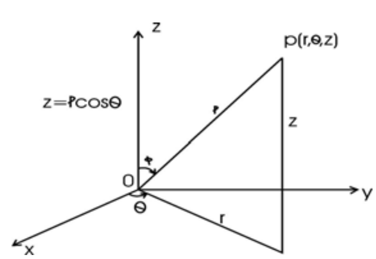


Spherical Co-ordinates

$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$,

$z = \rho\cos\phi$ where $\rho^2 = x^2+y^2+z^2$

Diagram



PARAMETRIC EQUATIONS AND POLAR CO-ORDINATES

Arc Length and Surface Area

When equation of the curve is y = f(x), $a \le x \le b$ then

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \;,$$

$$S = \int_a^b 2\pi f\left(x\right)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

When equation of the curve is x = f(t), y = f(t), $\alpha \le t \le \beta$ then

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \;,$$

$$S = \int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

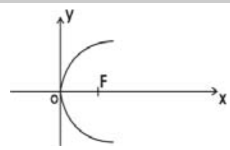
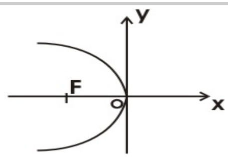
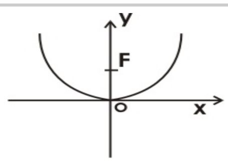
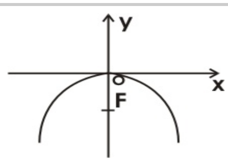
When equation of the curve is $r = f\left(\theta\right)$, $a \le \theta \le b$ then

$$L = \int_a^b \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta \;,$$

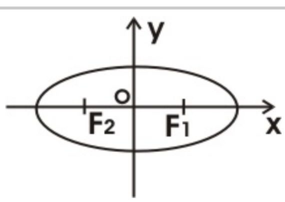
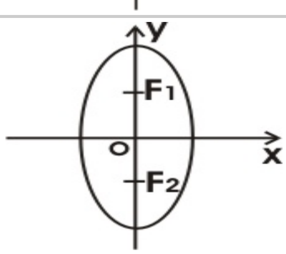
$$S = \int_a^b 2\pi r Sin\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

CONIC SECTION

Parabola

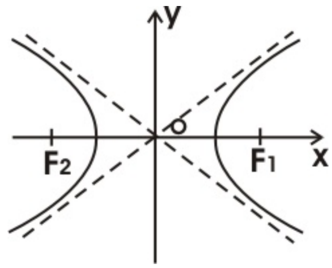
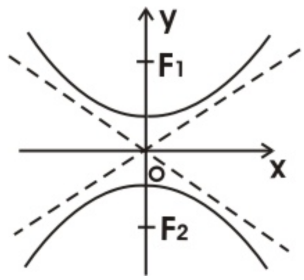
Equation	Graph	Directrix	Vertex	Axis	Focus
$y^2 = 4ax$		x = -a	(0, 0)	y = 0	(4a, 0)
$y^2 = -4ax$		x = a	(0, 0)	y = 0	(-4a, 0)
$x^2 = 4ay$		y = -a	(0, 0)	x = 0	(0, -4a)
$x^2 = -4ay$		y = a	(0, 0)	x = 0	(0, -4a)

Ellipse

Equation	Graph	Foci	Vertices	Length Major	Length Minor	Eccentricity
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a>b)		$\left(\pm ae,0\right)$	$\left(\pm a,0\right)$ $\left(0,\pm b\right)$	2a	2b	$e=\sqrt{1-\frac{b^2}{a^2}}$
$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ (a>b)		$\left(0,\pm ae\right)$	$\left(0,\pm a\right)$ $\left(\pm b,0\right)$	2a	2b	$e=\sqrt{1-\frac{b^2}{a^2}}$

CONIC SECTION

Hyperbola

Equation	Graph	Foci	Vertices	Eccentricity	Asymptotes
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (a>b)		$(\pm ae,0)$	$(\pm a,0)$	$e=\sqrt{\frac{b^2}{a^2}+1}$	$y=\pm\frac{b}{a}x$
$\frac{x^2}{b^2}-\frac{y^2}{a^2}=1$ (a>b)		$(0,\pm ae)$	$(0,\pm a)$	$e=\sqrt{\frac{b^2}{a^2}+1}$	$y=\pm\frac{a}{b}x$

Polar form of the equation of conics

A polar equation of the form $r=\frac{ed}{1\pm eCos\theta}$ or $r=\frac{ed}{1\pm eSin\theta}$ represents a conic section with eccentricity e.

The conic is ellipse if e < 1
The conic is a parabola if e = 1
The conic is a hyperbola if e > 1

DERIVATIVES AND INTERNALS OF VECTOR FUNCTIONS

Differentiation Rules

If and are differentiable vectors, c scalar and *f* a real valued function, then

1. $\frac{d}{dt}[\vec{u}(t)+\vec{\upsilon}(t)]=\vec{u}'(t)+\vec{\upsilon}'(t)$

2. $\frac{d}{dt}[c\vec{u}(t)]=c\vec{u}'(t)$

3. $\frac{d}{dt}[f(t)+\vec{u}(t)]=f'(t)\vec{u}(t)+f(t)\vec{u}'(t)$

4. $\frac{d}{dt}[\vec{u}(t).\vec{\upsilon}(t)]=\vec{u}'(t).\vec{\upsilon}(t)+\vec{u}(t).\vec{\upsilon}'(t)$

5. $\frac{d}{dt}[\vec{u}(t)\times\vec{\upsilon}(t)]=\vec{u}'(t)\times\vec{\upsilon}(t)+\vec{u}(t)\times\vec{\upsilon}'(t)$

6. $\frac{d}{dt}[\vec{u}(f(t))]=f'(t)\vec{u}'(f(t))$

The Definite Integral of a Continuous Vector Function

If $\vec{r}(t)=<f(t),g(t),h(t)>$ then

$$\int_a^b \vec{r}(t)=(\int_a^b f(t)dt)\hat{i}+(\int_a^b g(t)dt)\hat{j}+(\int_a^b h(t)dt)\hat{k}$$

* Curvature of a curve: $K(t)=\frac{|\vec{T}'(t)|}{|\vec{r}(t)|}$ where

$$T'(t)=\frac{\vec{r}'(t)}{|\vec{r}(t)|}$$

Also

$$K(t)=\frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

* Normal vector: $\vec{N}(t)=\frac{\vec{T}'(t)}{|T'(t)|}$

* Binomial vector: $\vec{B}(t)=\vec{T}(t)\times\vec{N}(t)$

PARTIAL DERIVATIVES

Good to know

If $f=f(x, y)$ then

* partial derivative of f with respect to x :

$$fx(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

* partial derivative of f with respect to y:

$$fy(x,y)=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

Directional Derivative

If f is differentiable function of x and y then directional derivative of f in the direction of unit vector $\vec{u}=<a,b>$ is $D\vec{u}f(x,y)=fx(x,y)a+fy(x,y)b$

or $D\vec{u}f(x,y)=\vec{\nabla}f(x,y).\vec{u}$

where $\vec{\nabla}f(x,y)=\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}$ called the gradient vector of f.

Higher Derivatives

* $f(x)x = fxx = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}$
* $f(x)y = fxy = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\partial x}$
* $f(y)x = fyx = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y\partial x}$
* $f(y)y = fyy = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}$

Total Differential

$$dz = fx(x,y)dx + fy(x,y)dy$$
$$=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$$

The Chain Rule

1. If z=f(x, y) and x=x(t), y=y(t) then

$$\frac{dz}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$$

2. If z=f(x, y) where x=g(s, t) and y=h(s, t), then

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}.\frac{\partial y}{\partial t}$$

Clairaut's Theorem

Suppose f is defined on a disk D that contains the point (a, b). if both the functions f_{xy} and f_{yx} are continuous on D, then $f_{xy}(a, b)=f_{yx}(a, b)$

Method of Lagrange's Multiplies

To find maximum and minimum values of $f(x, y, z)$ subjects to the constraint $g(x, y, z)=k$ [assuming that these extreme values exist and $\nabla g\neq 0$ on the surface $g(x,y,z)=k$]

(a). Find all values of x, y, z and λ such that
$\nabla f(x,y,z)=\lambda\nabla g(x,y,z)$ and $g(x,y,z)=k$

(b). Evaluate f at all the points (x, y, z) that result from part(a). The largest of these is maximum value of f and smallest of these is minimum value of f.

Second Derivative Test

Let the second derivative partial derivatives of f are continuous on a disk with centre (a, b) and let fx(a, b)=0 and fy(a, b)=0 i.e. (a, b) is the critical point of f. Let

$$D=D(a,b)=fx(a,b)\,fyy(a,b)-[fxy(a,b)]^2$$

(a). If D > 0 and $fxx(a, b) > 0$ then $f(a, b)$ is local minimum

(b). If D > 0 and $fxx(a, b) < 0$ then $f(a, b)$ is local maximum

(c). If D < 0 then $f(a, b)$ is neither local maximum nor local minimum and (a, b) is called a saddle point.

DOUBLE INTEGRALS

Definition to know

The double integral of f over the rectangle R is defined as: $\iint_R f(x,y)dA=\lim_{m,n\to\infty}\sum_{c=1}^{m}\sum_{d=1}^{n}f(x_{ij},y_{ij})\Delta A$

If this limit exists.

[where (x_{ij}, y_{ij}) is a point in each rectangle R_{ij} obtained by dividing R into sub rectangles and ΔA is the area of each sub rectangle.]

DOUBLE INTEGRALS

Mid – Point Rule for Double Integrals

Given a rectangle R=[a, b] x [c, d]. Divide R into sub rectangles R_{ij} obtained by dividing [a, b] into m – subintervals and [c, d] into n – subintervals.

Then $\iint_R f(x,y)dA = \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i,\bar{y}_j)\Delta A$

where $\bar{x}_i$ is the mid – point of $[x_{i-1}, x_i]$ and $\bar{y}$ is the mid-point of $[y_{j-1}, y_j]$ and ΔA is the area of each sub rectangle.

Equation you will need

Description	Equation
Mass of the lamina with density f(x, y) which occupies region D is given by	$m = \iint_D f(x,y)dA$
The co-ordinates of centre of mass $(\bar{x},\bar{y})$ are	$\bar{x} = \frac{My}{m} = \frac{1}{m}\iint_D xf(x,y)dA$ $\bar{y} = \frac{Mx}{m} = \frac{1}{m}\iint_D yf(x,y)dA$
Surface area : The area of the surface with equation z=f(x, y), (x, y) ϵ D where fx and fy are continuous	$A(s) = \iint_D \sqrt{[fx(x,y)]^2 + [fy(x,y)]^2 + 1}$ dA $= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$ dA
Moment of lamina about x-axis	$Mx = \iint_D yf(x,y)dA$
Moment of lamina about y-axis	$My = \iint_D xf(x,y)dA$
Moment of inertia about x-axis	$Ix = \iint_D y^2 f(x,y)dA$
Moment of inertia about y-axis	$Iy = \iint_D x^2 f(x,y)dA$
Moment of inertia about origin:	$Io = \iint_D (x^2+y^2) f(x,y)dA$
Radius of gyration $\bar{\bar{y}}$ with respect to x-axis	$m\bar{\bar{y}}^2 = Ix$
Radius of gyration $\bar{\bar{x}}$ with respect to y-axis	$m\bar{\bar{x}}^2 = Iy$

Fubini's Theorem

If f is continuous on the rectangle

$R = \{(x,y) : a \le x \le b, c \le y \le d\}$, then

$$\int_R\int f(x,y)dA = \int_a^b\int_c^d f(x,y)dydx = \int_c^d\int_a^b f(x,y)dxdy$$

Note :

* If f(x, y) = g(x)h(y)

then $\int_R\int f(x,y)dA = \iint_R g(x)h(y)dA$

$= \int_a^b g(x)dx\int_c^d h(y)dy$

where R=[a, b] x [c, d]

* If f is continuous on region D and D is type I region i.e.

$D = \{(x,y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}$

then $\int_D\int f(x,y)dA = \int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)dydx$

* If f is continuous on region D and D is type II region i.e.

$D = \{(x,y) : c \le y \le d, h_1(y) \le x \le h_2(y)\}$

then $\int_D\int f(x,y)dA = \int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$

* If $D = D_1 U D_2$ where D_1 and D_2 do not overlap except perhaps on their boundaries, then

$$\int_D\int f(x,y)dA = \int_{D_1}\int f(x,y)dA + \int_{D_2}\int f(x,y)dA$$

* $\int_D\int 1dA = A(D)$ = Area of region D.

* If $m \le f(x,y) \le M$ for all (x, y) in D, then

$mA(D) \le \int_D\int f(x,y)dA \le MA(D)$

* Double integral in polar co-ordinates : If f is continuous on a polar rectangle R given by $o \le a \le r \le b$, $\alpha \le \theta \le \beta$ where $o \le \beta - \alpha \le 2\pi$, then

$$\int_R\int f(x,y)dA = \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta$$

TRIPLE INTEGRALS

Definition to know

Let a function of f is defined on a rectangular box B given by: $B = \{(x, y, z) : a \le x \le b, c \le y \le d, r \le z \le s\}$ Divided B into sub-boxes by dividing the intervals [a, b] into l subintervals $[x_{i-1}, x_i]$ of equal width Δx, dividing [c, d] into m subintervals of width Δy and dividing [r, s] into m subintervals of width Δz.

Then Bijk = $[x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ and each sub box has volume Δv = Δx Δy Δz then the triple integral of f over box B is

$$\iiint_B f(x, y, z)dv = \int_r^s \int_c^d \int_a^b f(x, y, z)dxdydz$$

Triple Integrals in Cylindrical and Spherical Co-ordinates

If f is continuous on a cylindrical region E given by:

$$\alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta), u_1(r\cos\theta, r\sin\theta) \le z \le u_2(r\cos\theta, r\sin\theta)$$

then

$$\iiint_E f(x, y, z)dv = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r dz dr d\theta$$

If f is continuous on a spherical region E given by:

$$E = \{(\rho, \theta, \phi) : a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$$

then

$$\iiint_E f(x, y, z)dv = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$