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**Calculus 1**
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## CALCULUS 1
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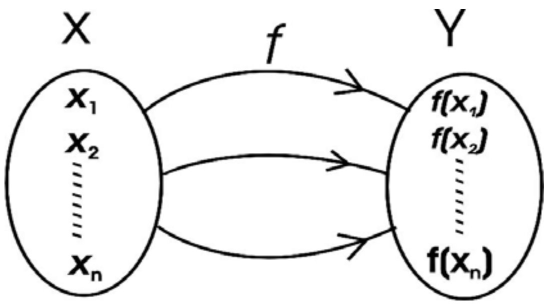
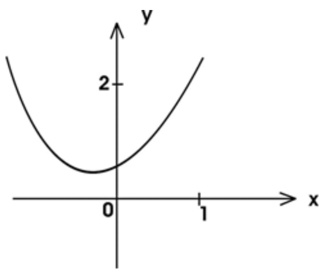
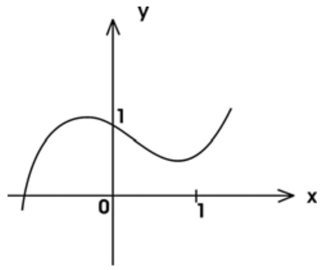
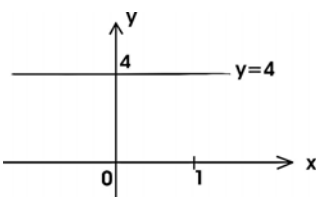
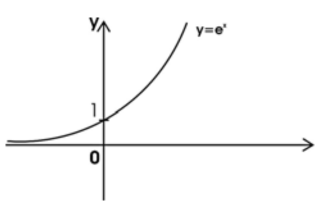
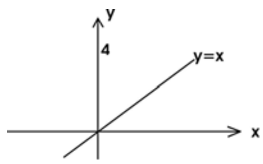
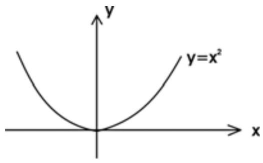
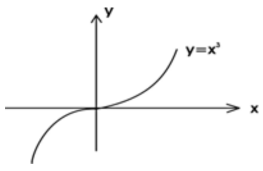
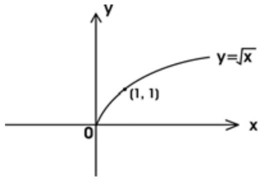
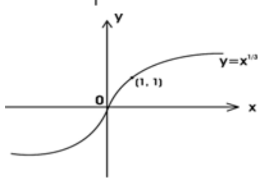
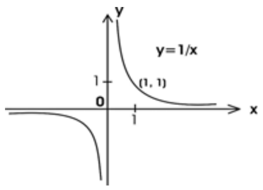
### FUNCTIONS AND THEIR TYPES

A function f is a rule that associates each element of a set X to a unique element of set Y. If x is an element of set x then *f(x)* is an element of set Y. The set *x* is called the domain of function *f* and the set of all possible values of *f(x)* for all x in the domain set is called the range of *f*.

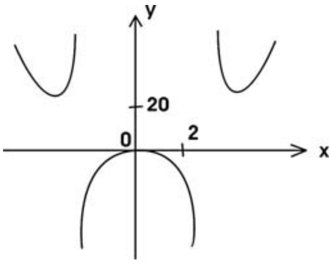
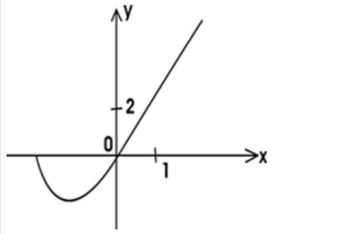
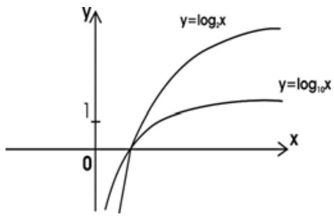


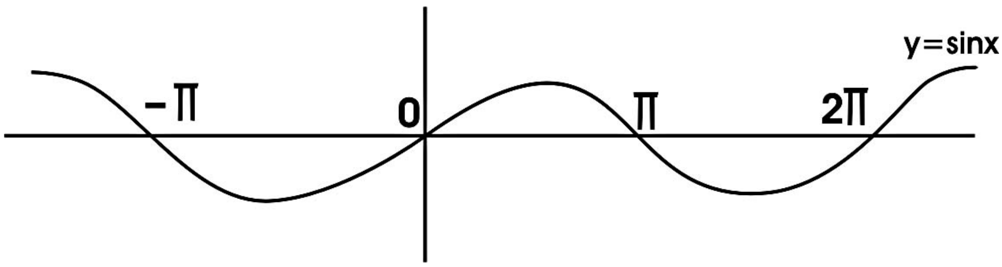
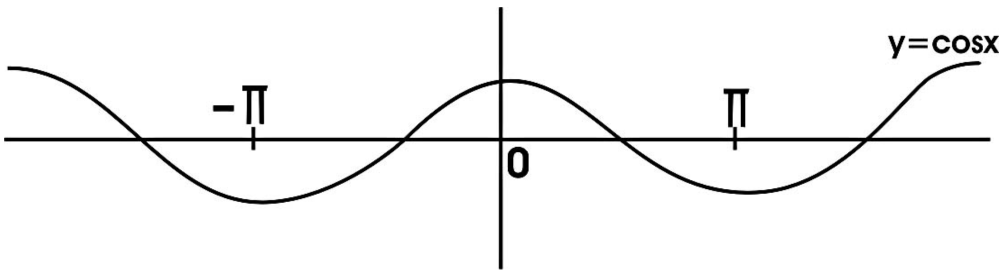

### TYPES OF FUNCTIONS:

| Type | Diagram |
|---|---|
| **Polynomials :** A function is called a polynomial if $f(x)= a_nx^n + a_{n-1}x^{n-1}$ ----------+ $a_2x^2 + a_1x + a_0$ where n is non-negative integer and $a_0, a_1, a_2$,-------$a_n$ are all constants with $a_n \neq 0$. Here n is called the degree of polynomial.<br>For example:<br>$f(x)=x^2+x+1$ |  |
| $f(x)=x^3-x+1$ |  |
| **Constant function :** A function f is called a constant function if $f(x)$=c(a constant) for all $x$ in the domain.<br>For example:<br>$f(x)=4$ |  |
| **Exponential functions :** the functions of the form $f(x)=a^x$, where a is a positive constant, are called exponential functions.<br>For example : $f(x)=2^x$, $f(x)=e^x$ |  |

| Type | Diagram |
|---|---|
| **Power function :** A function is called a power function if it is of the form $f(x)=x^a$ , where a is a constant. |  |
| * When a=n, where n is a positive integer.<br>E.g. $f(x)=x$, $f(x)=x^2$, |  |
| $f(x)=x^3$ |  |
| *Where a=1/n , where n is a positive integer<br>E.g. $f(x)=\sqrt{x}$, |  |
| $f(x)=x^{1/3}$ |  |
| *Where a=-1<br>E.g. $f(x)=x^{-1}$ |  |

## TYPES OF FUNCTIONS:

| Type | Diagram | Type | Diagram |
|---|---|---|---|
| **Rational function :** If a function f is the ratio of two polynomial function, then it is called a rational function.<br><br>i.e. $f(x)=h(x)/g(x)$ , where h(x) and $g(x)$ are polynomials with $g(x) \neq 0$ in the domain.<br><br>For example: $f(x)=2x^4-x^2+1/x^2-4$ is a rational function with domain not containing $\pm$ 2. |   | **Algebraic functions :** If a function f can be obtained using algebraic operations on polynomials, then it is called an algebraic function.<br>For example : $f(x)=\sqrt{x^2+1}$ |   |
| | | **Logarithmic functions :** The functions of the form $f(x)=\log_a x$, where base a is a positive number are called Logarithmic functions.<br>The domain of logarithmic functions is $(0,\infty)$ and the range is$(-\infty,\infty)$.<br><br>For example : $f(x)=\log_2 x$, $f(x)=\log_{10} x$ |   |

| Type | Diagram |
|---|---|
| **Trigonometric functions :** The functions involving trigonometric ratios like sinx, cosx, tanx etc. are called trigonometric functions.<br><br>For example : $f(x)=\sin x$,<br><br>$f(x)=\cos x$ |     |

## LIMITS AND CONTINUITY

### Definitions to know

**LIMITS**

| Name | Definition |
|---|---|
| Limit of a function | A function f is said to have limit l at x=a if for every e>0 there exist a $\delta$ >0 such that $\mid fx-\ell \mid < \varepsilon$ whenever 0<\|x-a\|<. $\delta$ And the limit is denoted by $\frac{f(b)-f(a)}{g(b)-g(a)}$ |
| Left– hand limit | A function f is said to have left – hand limit l at x=a if for every $\varepsilon>0$ there exists a $\delta>0$ such that $\mid f(x)-l \mid<\varepsilon$ whenever a- $\delta$<x<a and it is denoted by $\lim_{x \to a^-} f(x)=l$ |
| Right– hand limit | A function f is said to have right – hand limit l at x=a if for every $\varepsilon>0$ there exists a $\delta>0$ such that $\mid f(x)-l \mid<\varepsilon$ whenever $a<x<a+\delta$ and it is denoted by $\lim_{x \to a^+} f(x)=l$<br><br>Note : $\lim_{x \to a} f(x)=l$ if and only if $\lim_{x \to a^-} f(x)=l = \lim_{x \to a^+} f(x)=l$ |
| Infinite limits | A function f defined on some open interval that contains number a , is said to have infinite limit ∞ if for every positive number M, there exist a positive number $\delta$ such that *f(x)*> M whenever And it is denoted by $\lim_{x \to a} f(x)=\infty$<br><br>A function f defined on some open interval that contains number a , is said infinite limit -∞ number such that $f(x)<N$ whenever $0<\mid x-a \mid<\delta$ and it is denoted by $\lim_{x \to a} f(x)=-\infty$ |

**CONTINUITY**

| Name | Definition |
|---|---|
| Continuous function | A function f is continuous at a number a If $\lim_{x \to a} f(x) = f(a)$ |
| Discontinuous function | A function is said to be discontinuous at a if it is not continuous at a. |
| Continuity on an interval | A function is a continuous on an interval if it is continuous at every point in the interval.<br><br>Note : The following functions are continuous at every point in their domain :<br>i. Polynomials<br>ii. Rational functions<br>iii. Root functions<br>iv. Trigonometric functions<br><br>Note : If g is continuous at a and f is continuous at g(a) then the composite function fog is continuous at a |
| The intermediate value theorem | If f is continuous on the closed interval [a, b] and if M is a number such that f (a)<M<f(b) where f(a)≠ f(b) , then there exist a number c $\varepsilon$ (a, b) such that f(c)=M. |
| Cauchy's mean value theorem | If the functions f and g are continuous on [a, b] and differentiable on (a, b) and g'(x)≠ 0 for all x in (a, b). Then there is a number c in (a, b) such that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$ |

## LIMITS AND CONTINUITY

### LIMIT LAWS

Suppose that the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and c is a constant, then

1. $\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a}[f(x) \pm \lim_{x \to a} g(x)$

2. $\lim_{x \to a} cf(x) = c\lim_{x \to a} f(x)$

3. $\lim_{x \to a}[f(x)g(x)] = \lim_{x \to a}[f(x) \text{ x } \lim_{x \to a} g(x)$

4. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ $for \lim_{x \to a} g(x) \neq 0$

5. $\lim_{x \to a}[f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$

6. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$

7. $\lim_{x \to a} c = a$

8. $\lim_{x \to a} x = a$

9. $\lim_{x \to a} x^n = a^n$, where n is a positive integer, a>0

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$, where n is a positive integer, a>0

11. $\lim_{x \to a}(1+x)^{1/x} = e$

12. $\lim_{x \to \infty}(1+\frac{1}{x})^x = e$

13. $\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a$

14. $\lim_{x \to 0} \frac{\sin x}{x} = 1$

15. $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$

### CONTINUITY

| Name | Definition |
|---|---|
| L'Hospital's rule' | If f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a), and if $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ **or** $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$ then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ if the limit on right side exist (or ∞or -∞)<br>Note: Indeterminate forms when we can use L'Hospital's rule:<br>* $\frac{0}{0}$ * $0.\infty$<br>* $0^0$ * $1^\infty$<br>* $\frac{\infty}{\infty}$ * $\infty - \infty$<br>* $\infty^0$ |

<u>The squeeze theorem</u> If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$

then $\lim_{x \to a} g(x) = L$

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### INDEFINITE INTEGRALS

#### BASIC FORMULAS

| | | |
|---|---|---|
| * $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ | * $\int \sin x dx = -\cos x + c$ | * $\int \cos x dx = \sin x + c$ |
| * $\int \sec^2 x dx = \tan x + c$ | * $\int csc^2 x dx = -\cot x + c$ | * $\int \sec x \tan x dx = -\csc x + c$ |
| * $\int \frac{dx}{x} = ln\,\lvert x \rvert + c$ | * $\int e^x dx = e^x + c$ | * $\int a^x dx = \frac{a^x}{ln\,a} + c$ |
| * $\int \tan x dx = ln\,\lvert \sec x \rvert + c$ | * $\int cot x dx = ln\,\lvert \sin x \rvert + c$ | * $\int \sec x dx = ln\,\lvert \sec x + \tan x \rvert + c$ |
| * $\int \csc x dx = ln\,\lvert \csc x + cot\, x \rvert + c$ | * $\int \frac{dx}{\sqrt{a^2+x^2}} = \sin^{-1}\frac{x}{a} + c$ | * $\int \frac{dx}{a^2+x^2} = \sin^{-1}\frac{x}{a} + c$ |
| * $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$ | * $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left\lvert \frac{x+a}{x-a} \right\rvert + c$ | * $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left\lvert \frac{x-a}{x+a} \right\rvert + c$ |

**Note :** c denotes the arbitrary constant of integration

#### Exponential and Logarithmic Formulas

| | | |
|---|---|---|
| * $\int xe^{ax} dx = \frac{1}{a^2}(ax-1)e^{ax} + c$ | * $\int ln\, x dx = x\,ln\, x - x + c$ | * $\int \frac{1}{x\,ln\,x} dx = \ln\lvert ln\, x \rvert + c$ |

| | |
|---|---|
| * $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$ | * $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx - b\sin bx) + c$ |

#### Inverse Trigonometric Formulas

| | |
|---|---|
| * $\int \sin^{-1} x dx = x\sin^{-1} x + \sqrt{1-x^2} + c$ | * $\int \cos^{-1} x dx = x\cos^{-1} x - \sqrt{1-x^2} + c$ |
| * $\int \tan^{-1} x dx = x\tan^{-1} x - \frac{1}{2}\left(1-x^2\right) + c$ | * $\int x\sin^{-1} x dx = \frac{2x^2-1}{4}\sin^{-1} x + \frac{x\sqrt{1-x^2}}{4} + c$ |
| * $\int x\cos^{-1} x dx = \frac{2x^2-1}{4}\cos^{-1} x - \frac{x\sqrt{1-x^2}}{4} + c$ | * $\int x\tan^{-1} x dx = \frac{x^2+1}{2}\tan^{-1} x - \frac{x}{2} + c$ |

#### Hyperbolic Formulas

| | | |
|---|---|---|
| * $\int \sinh x dx = \cosh x + c$ | * $\int \cosh x dx = \sinh x + c$ | * $\int \tanh x dx = ln\,\lvert \cosh x \rvert + c$ |
| * $\int \coth x dx = ln\,\lvert \sinh x \rvert + c$ | * $\int \text{sech}\, x dx = yan^{-1}\lvert \sinh x \rvert + c$ | * $\int cs\text{ch}\, x dx = ln\,\left\lvert \tanh\frac{x}{2} \right\rvert + c$ |

**Note :** c is arbitrary constant of integration $\lim_{n\to\infty} a_n = 0$

# INDEFINITE INTEGRALS

## Reduction Formulas

* $\int \sin^n xdx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} xdx$

* $\int \cos^n xdx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} xdx$

* $\int \tan^n xdx = \frac{1}{n-1}\tan^{n-1} x\int \tan^{n-2} xdx$

* $\int \cot^n xdx = \frac{-1}{n-1}\cot^{n-1} x\int \cot^{n-2} xdx$

* $\int \sec^n xdx = \frac{1}{n-1}\tan x\sec^{n-2} x\frac{n-2}{n-1}\int \sec^{n-2} xdx$

* $\int \csc^n xdx = \frac{-1}{n-1}\cot x\csc^{n-2} x\frac{n-2}{n-1}\int \csc^{n-2} xdx$

* $\int x^n \ln xdx = \frac{x^{n+1}}{(n+1)^2}[(n+1)\ln x - 1] + c$

* $\int x^n e^{ax} dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax} dx$

* $\int \sin^n x\cos^m xdx$

$$= \frac{-\sin^{n-1} x\cos^{m+1} x}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} x\cos^m xdx = \frac{\sin^{n-1} x\cos^{m+1} x}{n+m} + \frac{m-1}{n+m}\int \sin^n x\cos^{m-2} xdx$$

## Special Techniques for Solving Problems

If the integrand is a function of :

| | | |
|---|---|---|
| * $ax+b$ Substitute $ax+b=t$ | * $\sqrt{ax+b}$ Substitute $\sqrt{ax+b} = t$ | * $\sqrt{x^2 + a^2}$ Substitute $x = a\tan t$ |
| * $\sqrt{x^2 - a^2}$ Substitute $x = a\sec t$ | * $\sqrt{a^2 - x^2}$ Substitute $x = a\sin t$ | * $e^{ax}$ Substitute $e^{ax} = t$ |
| * $\ln x$ Substitute $\ln x = t$ | * $\sin x, \cos x$ Substitute $\tan\frac{x}{2} = t$ | |

## Integration of Rational Functions by Partial Fractions

To integrate the rational function $\frac{f(x)}{g(x)}$ when:

1. When $g(x)$ is a product of distinct linear factors, we can write $g(x) = (a_1x+b_1)(a_2x+b_2)............(a_nx+b_n)$ then

$$\frac{f(x)}{g(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + ------- + \frac{A_n}{a_nx+b_n}$$

2. When g(x) contains irreducible quadratic factors i.e.

$g(x) = ax^2 + bx + c$ where $b^2 - 4ac < 0$ , then

$$\frac{f(x)}{g(x)} = \frac{Ax+B}{ax^2+bx+c}$$

3. When $g(x)$ is a product of linear factors, some of which are repeated, i.e.

$g(x) = (a_1x+b_1)^k(a_2x+b_2)............(a_nx+b_n)$ then

$$\frac{f(x)}{g(x)} = \frac{A_1^{(1)}}{a_1x+b_1} + \frac{A_2^{(2)}}{(a_1x+b_1)^2} + ------- + \frac{A_k^{(k)}}{(a_1x+b_1)^k} + \frac{A_2}{a_2x+b_2} + ----- + + \frac{A_n}{a_nx+b_n}$$

4. When g(x) contains repeated irreducible factors i.e.

$g(x) = (ax^2 + bx + c)^k$ where $b^2 - 4ac < 0$ , then

$$\frac{f(x)}{g(x)} = \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + ------ + \frac{A_kx+B_kx}{(ax^2+bx+c)^k}$$

# DEFINITE INTEGRALS

## BASIC PROPERTIES

* $\int_a^b c\,dx = c(b-a)$
* $\int_a^b [f(x) \pm g(x) = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
* $\int_a^b f(x)dx = -\int_b^a f(x)dx$
* If $f$ is an even function i.e. $f(-x)=f(x)$ then $\int_{-a}^{b} f(x)dx = 2\int_0^a f(x)dx$
* If $f$ is an odd function i.e. $f(-x)=-f(x)$ then $\int_{-a}^{b} f(x)dx = 0$
* $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
* $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_0^c f(x)dx$

## Comparison Properties

* If $f(x)\geq 0$ for $a \leq x \leq b$ then $\int_a^b f(x)dx \geq 0$
* If $f(x)\geq g(x)$ for $a \leq x \leq b$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
* If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

## Fundamental Theorem of Calculus

Let $f$ is a continuous function on closed interval [a, b]

1. .If $h(x) = \int_a^x f(y)$ , then $h'(x) = f(x)$
2. $\int_a^b f(X)dx = F(b) - F(a)$ , where F is an anti-derivative of $f$ i.e. $F'(x)=f(x)$

## Mid Point Rule

$$\int_a^b f(X)dx \approx Mn = \Delta x[f(\bar{x}_1) + f(\bar{x}_2) + - - - - - + f(\bar{x}_k)]$$

where $\Delta x = \frac{b-a}{k}$ and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$

## Trapezoidal Rule

$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + - - - + 2f(x_{k-1}) + f(x_k)]$$

where $\Delta x = \frac{b-a}{k}$ and $x_i = a + i\Delta x$

## Simpson's Rule

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) - - - +2f(x_{k-2}) + 4f(x_{k-1}) + f(x_k)]$$

where k is even and $\Delta x = \frac{b-a}{k}$

# DIFFERENTIATION

## Definition

The derivative of a function *f* at a number a, denoted by *f'(a)*, is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if the limit exists.

The derivative of function *f* at any number x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ if the limit exists.

**Different notations for derivatives**
If *y=f(x)* then the derivative of then the derivative of *y* with respect to x is:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

## Differentiation Formulas

* Derivative of a constant function : $\frac{d}{dx}(c) = 0$

* If $n$ is any real number :

$$\frac{d}{dx}(x^n) = nx^{n-1}$$ [power Rule]

* $$\frac{d}{dx}[cf(x)] = c\frac{d}{dx} f(x)$$

* Sum rule : If *f* and *g* are differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

* Difference rule :

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

* Product rule :

$$\frac{d}{dx}[f(x).g(x)] = f(x)\frac{d}{dx} g(x) + g(x)\frac{d}{dx} f(x)$$

(or $(uv)' = uv' + u'v$ )

## Derivatives of Trigonometric Functions

| | |
|---|---|
| * $\frac{d}{dx}(\sin x) = \cos x$ | * $\frac{d}{dx}(\cos x) = -\sin x$ |
| * $\frac{d}{dx}(\tan x) = \sec^2 x$ | * $\frac{d}{dx}(\csc x) = -\csc x \cot x$ |
| * $\frac{d}{dx}(\sec x) = \sec x \tan x$ | * $\frac{d}{dx}(\cot x) = -\csc^2 x$ |

## Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx} f(x) - f(x)\frac{d}{dx} g(x)}{[g(x)]^2}$$

(or $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$ )

## The Chain Rule

If *f* and *g* are both differentiable functions and $F(x) = f(g(x))$ is a composite function, then F is differentiable and $F'(x) = f'(g(x)g'(x))$

**OR: If** *y=f(u)* and *u=g(x)* are both differentiable functions

then, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

## Differentiation of Implicit Functions

<u>Technique:</u> Differentiate both sides of equation wrt x and then solve the resulting equation for y'

E.g. If $x^3 + y^3 = 3$ , find $\frac{dy}{dx}$ , The given equation is $x^3 + y^3 = 3$

Differentiating both sides with respect to x:

$3x^2 + 3y^2\frac{dy}{dx} = 0$ Or $3y^2\frac{dy}{dx} = -3x^2$

Or $\frac{dy}{dx} = \frac{-x^2}{y^2}$

## Derivatives of Exponential and Logarithmic Functions

| | |
|---|---|
| * $\frac{d}{dx}(e^x) = e^x$ | * $\frac{d}{dx} \ln \lvert x \rvert = \frac{1}{x}$ |
| * $\frac{d}{dx}(a^x) = a^x \ln a$ | * $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$ |

# DIFFERENTIATION

## Derivatives of Inverse Trigonometric Function

* $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ ,$-1 < x < 1$
* $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$ , $-1 < x < 1$
* $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
* $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$ ,$|x| > 1$
* $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

## Higher Order Derivatives

Second derivative of $y=f(x)$ is the derivative of the derivative of $f$ and is denoted by:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2 y}{dx^2} = f^{11}(x) = D^2 f(x)$$

Third derivative of y=f(x) is :

$$y^{(1)} = f^{(1)}(x) = \frac{d}{dx}\left(\frac{d^2 y}{dx^2}\right) = \frac{d^3 y}{dx^2} = D^3 f(x)$$

n$^{th}$ order derivative :

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n} = D^n f(x)$$

## The First Derivative Test

Let *c* is a critical point of a continuous function *f*,

(a ). If *f* ‘changes sign from positive to negative at *c*, then *f* has a local maximum at *c*.

(b ). If *f* ‘changes sign from negative to positive at *c*, then *f* has a local minimum at *c*.

(c ). If *f* ‘ does not change sign at *c*, then *f* has no local maximum or minimum at *c*.

## Second Derivative Test

Let *f’’* is continuous near *c*.

(a ). If *f’(c)=0* and *f’’(c)>0*, then *f* has a local maximum at *c*.

If *f’(c)=0* and *f’’(c)<0*, then *f* has a local minimum at *c*.

## Derivatives of Hyperbolic Functions

* $\frac{d}{dx}(\sinh x) = \cosh x$
* $\frac{d}{dx}(\cosh x) = \sinh x$
* $\frac{d}{dx}(\tanh x) = \text{sech}^2\, x$
* $\frac{d}{dx}(\text{csch}\, x) = -\csc hx \coth x$
* $\frac{d}{dx}(\text{sech}\, x) = -\sec hx \tanh x$
* $\frac{d}{dx}(\coth x) = -\csc h^2 x$

## Derivatives of Inverse Hyperbolic Functions

* $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$
* $\frac{d}{dx}(\cosh^{-1} x) = -\frac{1}{\sqrt{x^2-1}}$ , $|x| > 1$
* $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1+x^2}$ , $|x| \neq 1$
* $\frac{d}{dx}(\csc h^{-1} x) = -\frac{1}{|x|\sqrt{x^2+1}}$
* $\frac{d}{dx}(\text{sech}\, x) = -\frac{1}{x\sqrt{1-x^2}}$ , $-1 < x < 1$
* $\frac{d}{dx}(\cot h\, x) = -\frac{1}{1-x^2}$ , $|x| \neq 1$

## Concavity Test

(a ). If *f’’(x)>0* for all *x* in I, then the graph of *f* is concave upward on I.

(b ). If *f’’(x)<0* for all *x* in I, then the graph of *f* is concave downward on I.

## Rolle’s Theorem

Let *f* is a function such that

1. *f* is continuous on [a, b]
2. *f* is differentiable on (a, b)
3. *f(a)=f(b)*

then there exists a number $c\varepsilon(a,b)$ such that *f’(c)=0*

# DIFFERENTIATION

## The Mean Value Theorem

Let $f$ is a function such that

1. $f$ is continuous on[a, b]
2. $f$ is differentiable on (a, b)
3. $f(a)=f(b)$

then there exists a number $c\varepsilon(a,b)$ such that $f'(c)=0$

## Increasing / Decreasing Test

(a ). If $f'(x)>0$ on an interval, then $f$ is increasing on that interval.

(b ). If $f'(x)<0$ on an interval, then $f$ is decreasing on that interval.

## Scientific Applications of Derivatives

* If s=f(t) is the position function of the particle moving in straight line then:

velocity $(v) = \dfrac{ds}{dt}$ is the rate of change of displacement with time.

Acceleration $a = \dfrac{dv}{dt}\left( = \dfrac{d^2s}{dt^2}\right)$ is the rate of change of velocity with time.

* If $m=f(x)$ is the mass of a rod or piece of wire, then linear density $(\rho) = \dfrac{dm}{dx}$ is the rate of change of mass with length.

* If Q=$f(t)$ is the charge passing through the surface at time t, then current $(I) = \dfrac{dq}{dt}$ is the rate of change of charge with time.

* If $w=f(t)$ is the work done in time t then Power

$(P) = \dfrac{dw}{dt}$ is the rate of change of work done with time.

## Applications of Differentiation

Maximum and Minimum values/Extreme values:

* Absolute maximum: A function $f$ has absolute maximum at c if $f(c) \geq f(x)$ for all $x$ in domain of $f$.

* Absolute minimum: A function $f$ has absolute maximum at c if $f(c) \leq f(x)$ for all $x$ in domain of $f$.

* Local maximum: A function $f$ has local maximum at c if $f(c) \geq f(x)$ in some open interval containing $c$.

* Local minimum: A function $f$ has local maximum at c if $f(c) \leq f(x)$ in some open interval containing $c$.

* The extreme value theorem: If $f$ is continuous on a closed interval [a, b], then $f$ attains on absolute maximum $f(d)$ at some $c$, $d$ in [a, b].

* Fermat's theorem: If $f$ has local maximum or minimum at $c$ and if $f'(c)$ exists, then $f'(c)=0$

* Critical number: A critical number of function $f$ is a number $c$ in the domain of $f$ such that $f'(c)=0$ or $f'(c)$ does not exist.

# SEQUENCES AND SERIES

## Definitions to know

### Limit of a Sequence

A sequence $\{a_n\}$ is said to have limit $l$ if for every e>0 there exists a positive integer N such that $|a_n - l| < e$ whenever n>N And limit is denoted by $\lim_{n\to\infty} a_n = l$ or $a_n \to l$ as $n \to \infty$

* If $\lim_{n\to\infty}$ an exists then we say that the sequence $\{a_n\}$ is convergent or otherwise we say that the sequence $\{a_n\}$ is divergent.

### Increasing / Decreasing Sequence

A sequence $\{a_n\}$ is called increasing

if $a_n < a_{n+1}$ for all n $\geq$ 1 i.e. $\int_1^\infty f(x)\,dx$

A sequence $\{a_n\}$ is called decreasing

if $a_n > a_{n+1}$ for all n $\geq$ 1 i.e. $a_1 < a_2 < a_3 < ----$

# SEQUENCES AND SERIES

## Definitions to know

### Limit Laws of Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and $c$ is a constant, then

* $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

* $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$

* $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$

* $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n . \lim_{n\to\infty} b_n$

* $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$

* $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$ if p>0 and $a_n$>0

### Convergence / Divergence of a series

For a given series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + - - - -$$

let $S_n$ denotes the partial sum

$$s_n = a_1 + a_2 + a_3 + - - - - + a_n$$

If the sequence $\{s_n\}$ converges to a real number s then the series $\sum_{n=1}^{\infty} a_n$ is also convergent and it converges to s.

Otherwise the series is called divergent.

### Theorem

If $\sum a_n$ and $\sum b_n$ are convergent series then $\sum ca_n$ , (c a constant),

$\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ are also series and

* $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$ * $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

* $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

### Monotonic Sequence

A sequence is called monotonic if it is either increasing or decreasing.

### Bounded Above / Bounded Below

A sequence $\{a_n\}$ is bounded above if there is a number M such that $a_n \leq$ M for all n $\geq$ 1

A sequence $\{a_n\}$ is bounded below if there is a number M such that M < $a_n$ for all n $\geq$ 1

### Bounded Sequence

A sequence is called bounded if it is bounded above and below.

### Monotonic Sequence theorem

Every bounded, monotonic sequence is convergent.

### Theorem

If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

### Theorem

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

### Test for Divergence

If $\lim_{n\to\infty} a_n$ does not exists or if $\lim_{n\to\infty} a_n \neq 0$, then the $\sum_{n=1}^{\infty} a_n$ is divergent.

### Convergence of Geometric series

The geometric series

$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + - - - -$ converges if |r| < 1 and diverges if |r| $\geq$ 1

## SEQUENCES AND SERIES

### Test for the convergence of series

* The alternating series test: If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + ----(a_n > 0)$$

is monotonically decreasing and $\lim_{n\to\infty} a_n = 0$

then the series is convergent.

* The ratio-test:

(i). If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = l < 1$

then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii). If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = l > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$

then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

* The root-test:

(i). If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = l < 1$ then the series $\sum_{n=1}^{\infty} a_n$

is convergent.

(ii). If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = l > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$ ,

then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

### Definitions to know

### Absolutely Convergent

A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

### Conditionally Convergent

A series $\sum_{n=1}^{\infty} a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

### Test for the convergence of series

* The integral test: Suppose *f* is a continuous, positive, decreasing function on [1, ∞] and let $a_n$=if (*x*). Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent.

**Or**

(i). If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii). If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

* The P-test: The p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent if p > 1 and divergent if p ≤ 1

* The comparison test: Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

(i). If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent

(ii). If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent

* The limit comparison test: Let $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n\to\infty}\frac{a_n}{b_n} = c$ where c is a finite number and c > 0, then either both the series converge or both diverge.

## SEQUENCES AND SERIES

### Good to know

#### Theorem

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

#### Theorem

If the power series $\sum c_n(x-a)^n$ has radius of convergence R > 0, then the function *f* defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + ---- = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable and continuous on the interval (a - R, a + R) and

(i).

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + ---- = \sum_{n=0}^{\infty} c_n(x-a)^{n-1}$$

(ii).

$$\int f(x)dx = c + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} ---- = c + \sum_{n=0}^{\infty}\frac{c_n(x-a)^{n+1}}{n+1}$$

the radii of convergence of power series in equation (i) and (ii) are both R.

## TAYLOR AND MACLAURIN SERIES

### Taylor series of a function f at a

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$+ \frac{f'''(a)}{3!}(x-a)^3 + ----$$

### Binomial Series

If k is any real number and |x| < 1, then

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + ---$$

$$= \sum_{n=0}^{\infty} {}^k c_n x^n$$

Where ${}^k c_n = \dfrac{k!}{n!(k-n)!}$ and ${}^k c_0 = 1$

### Maclaurin's series of a function f

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

$$= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + ----$$

### Some important Maclaurin series

* $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{2!} + ----$ , $x\varepsilon(-\infty,\infty)$
* $\sin x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + ----$ , $x\varepsilon(-\infty,\infty)$
* $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + ----$ , $x\varepsilon(-\infty,\infty)$
* $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + ----$ , $x\varepsilon(-\infty,\infty)$
* $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + ----$ , $x\varepsilon(-1,1)$
* $\tan^{-1} x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + ----$ , $x\varepsilon(-1,1)$