Technical Appendix to: Exclusive dealing with imperfect downstream competition

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This technical appendix provides the proof for Proposition 4 in Abito and Wright (2005). Similar to the main paper, we assume without loss of generality that in the $S = 1$ subgame, retailer 1 signs the contract while retailer 2 is the free retailer.

The first lemma shows that when g is close to 1 (but not equal to 1), the entrant finds it profitable to induce its retailer to monopolize the market. In equilibrium, the incumbent sets the signed retailer's wholesale price equal to the former's marginal cost (c_I) while the entrant necessarily offers the free retailer a wholesale price strictly less than the incumbent's marginal cost. The free retailer buys from the entrant and monopolizes the market.

Lemma 1 *Consider the subgame* $S = 1$ *with entry and linear wholesale prices, where retailer 1 signs with the incumbent. For sufficiently little retailer differentiation, the unique equilibrium is characterized by* $w_1 = c_I$ *and* $w_2 < c_I$*, with retailer 2 (and the entrant) monopolizing the market.*

Proof. *For any wholesale price* $w_1 \geq c_I$ *set by the incumbent, if the entrant monopolizes the market it will set the highest possible wholesale price such that retailer 2 will still choose to monopolize the market. Denote this wholesale price* $w_2(w_1)$*. Given retailer 1 is willing to price down to* w_1*, retailer 2 has to price at* $(-a(1-g)+w_1)/g$ *to monopolize, which implies* $w_2(w_1) = \left(w_1\left(2-g^2\right) - a(1-g)(2+g)\right)/g$*. Given these wholesale prices, if retailer 2 increases its price by* $\Delta > 0$*, it will share the market with retailer 1 and its profits will decrease by* $\Delta^2/b\left(1-g^2\right)$*. If the entrant increases its wholesale price by*

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$\Delta > 0$ *then it will induce retailer 2 to share the market, and the entrant's profit will increase by*

$$\frac{\Delta}{g}\frac{(a - c_E + \Delta)\left(-2g + g^3\right) + \left(8 - 9g^2 + 2g^4\right)(a - w_1)}{b(1+g)(1-g)(4-g^2)}.$$

Taking the limit as $g \to 1$*, the change in profits is negative since* $w_1 + \Delta > c_I > c_E$*. This implies the entrant always prefers to price so as to induce monopolization if* g *is sufficiently close to* 1*. That is, the equilibrium must involve monopolization. Moreover, the entrant cannot do better by decreasing* w_2 *below* $w_2(w_1)$ *since it already monopolizes the whole market at a retail price that does not depend on its wholesale price, so decreasing* w_2 *will lead to lower margins and no increase in demand.*[1] *If* $w_1 > c_I$*, then the incumbent can obtain positive profits by lowering* w_1 *to induce market sharing, while if it sets* $w_1 < c_I$ *it obtains a loss. Thus, any equilibrium must be characterized by* $w_1 = c_I$ *and* $w_2 = w_2(c_I) < c_I$ *for* $0 \leq g < 1$*. Clearly, the incumbent cannot do better by increasing* w_1 *since it already receives no demand at* $w_1 = c_I$*, while lowering* w_1 *generates a loss. We have already shown that the entrant does not want to deviate from* $w_2 = w_2(c_I)$ *for* g *close enough to 1, so this equilibrium indeed exists.* ■

Because of the lack of smoothness in the retailers' reaction functions[2] in the $S = 1$ subgame, we can identify regions defined by critical values of g such that in each region, a certain type of equilibrium occurs. Before deriving these critical values, let us revisit the optimization problem faced by retailers. Since lemma 1 covers the case where the entrant's retailer monopolizes the market, we now consider equilibrium when both retailers are active. Given that retailers share the market, the appropriate demand for each retailer is the one given by equation (2) in Abito and Wright (2005). Taking wholesale prices as given, price equilibrium in the retail pricing stage is given by

$$p_i(w_i, w_j) = \frac{(2+g)(1-g)a + 2w_i + gw_j}{4-g^2}.$$

Substituting these prices back into the demand and profit functions of each retailer yields

$$q_i(w_i, w_j) = \frac{(2+g)(1-g)a - \left(2-g^2\right)w_i + gw_j}{b(4-g^2)(1+g)(1-g)}$$

and

$$\pi_i(w_i, w_j) = \frac{\left((2+g)(1-g)a - \left(2-g^2\right)w_i + gw_j\right)^2}{b(4-g^2)^2(1+g)(1-g)}.$$

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[1] The only exception to this logic is if the entrant lowers w_2 so much that retailer 2 takes the whole market by setting its monopoly price given w_2. Given our assumption that the entrant's cost advantage is non-drastic, we can also rule out this case since it requires the entrant sets w_2 below its own marginal cost c_E, which implies it makes a loss.

[2] This is a consequence of the demand specified in Abito and Wright (2005).

In the wholesale pricing stage, recall that the entrant and the incumbent respectively solve the following *basic* optimization problems:

$$\max_{c_E \leq w_2 \leq c_I} \left(w_2 - c_E\right) q_2\left(w_2,\ w_1\right)$$

and

$$\max_{w_1 \geq c_I} \left(w_1 - c_I\right) q_1\left(w_1,\ w_2\right)$$

where $q_i\left(w_i, w_j\right) \geq 0$ is given above (retail pricing stage). These problems yield the following best response functions for the upstream firms:

$$\begin{aligned} w_2\left(w_1\right) &= \frac{a\left(1-g\right)\left(2+g\right) + c_E\left(2-g^2\right) + g w_1}{4-2g^2} \\ w_1\left(w_2\right) &= \frac{a\left(1-g\right)\left(2+g\right) + c_I\left(2-g^2\right) + g w_2}{4-2g^2}. \end{aligned}$$

Note that these optimization problems and best response functions are only applicable if the constraints $(c_E \leq)\ w_2 \leq c_I$, $w_1 \geq c_I$ and $q_i\left(\cdot\right) \geq 0$ are satisfied. In particular, we can derive critical values of g denoted by g^L and g^H such that (i) if $g < g^L$, then the constraint $w_2 \leq c_I$ is binding and thus the above response functions are inapplicable; (ii) if $g^L \leq g \leq g^H$, then constraints are non-binding and thus the optimal wholesale prices are given by the unique intersection of the above response functions; and finally (iii) if $g > g^H$, then the constraints $w_1 \geq c_I$ and $q_1\left(\cdot\right) \geq 0$ are binding and the result of lemma 1 follows. The following lemma characterizes these critical values and optimal wholesales prices.

Lemma 2 *There exist critical values g^L and g^H such that*

1. *if $g < g^L$, then $w_1 = \left(\left(2+g\right)\left(1-g\right)a + \left(2-g\right)\left(1+g\right)c_I\right) / \left(4-2g^2\right)$ and $w_2 = c_I$ are the optimal wholesale prices chosen by the incumbent and the entrant respectively;*
2. *if $g^L \leq g \leq g^H$, then the optimal w_1 $(\geq c_I)$ and w_2 $(\leq c_I)$ are given by the intersection of the above response functions; and finally*
3. *if $g > g^H$, then $w_1 = c_I$ and $w_2 = \left(c_I\left(2-g^2\right) - a\left(1-g\right)\left(2+g\right)\right)/g$ are the optimal wholesale prices chosen by the incumbent and the entrant respectively.*

The analytical forms of g^L and g^H are as follows:

$$\begin{aligned} g^L \quad &= \quad -\frac{2-\lambda+\sqrt{36-52\lambda+17\lambda^2}}{16\left(1-\lambda\right)} + \\ &\frac{1}{2}\sqrt{4 + \frac{18-17\lambda}{4\left(1-\lambda\right)} + \frac{\left(2-\lambda\right)^2}{32\left(1-\lambda\right)^2} + \frac{2\left(1-\lambda\right)\left(-\frac{(2-\lambda)}{1-\lambda} + \frac{(18-17\lambda)(2-\lambda)}{4(1-\lambda)^2} + \frac{(2-\lambda)^3}{64(1-\lambda)^3}\right)}{\sqrt{36-52\lambda+17\lambda^2}}} \end{aligned}$$

and

$$\begin{aligned} g^{H} &= -\frac{1+\sqrt{9-8\lambda+2\lambda^{2}}}{4\left(2-\lambda\right)}+ \\ & \frac{1}{2}\sqrt{\frac{9}{2}+\frac{1}{2\left(2-\lambda\right)^{2}}+\frac{16-8\lambda}{2\left(2-\lambda\right)}+\frac{\left(2-\lambda\right)\left(\frac{1}{\left(2-\lambda\right)^{3}}+\frac{2}{2-\lambda}\right)}{2\sqrt{9-8\lambda+2\lambda^{2}}}} \end{aligned}$$

where both g^L *(bounded by* 0 *and* 1*) and* g^H *(bounded by* 0.8875 *and* 1*) are decreasing in* $\lambda \in (0,1)$ *and* $g^H \geq g^L$*.*

Proof. *To derive* g^L *and* g^H *we need to consider the basic optimization given above. The critical value* g^L *is the greatest value of* g *whereby the solution* w_2 *given optimal* w_1 *is greater than or equal to* c_I*. In other words, for* $g < g^L$*, the constraint on the entrant's optimization problem is binding hence the entrant and incumbent chooses* $w_2 = c_I$ *and*

$$\begin{aligned} w_1 &= \arg\max_{w_1 \geq c_I} \left(w_1 - c_I\right) q_1\left(w_1,\, c_I\right) \\ &= \left(\left(2+g\right)\left(1-g\right)a + \left(2-g\right)\left(1+g\right)c_I\right) / \left(4-2g^2\right) \end{aligned}$$

respectively, otherwise, the equilibrium pair of wholesale prices is given by the intersection of the above best response functions and it must be that the optimal w_2 *is less than* c_I*. Therefore* g^L *solves* $w_2^* - c_I = 0$ *where*

$$w_2^* = \frac{a\left(1-g\right)\left(2+g\right)\left(4+g-2g^2\right)+\left(2-g^2\right)\left(c_I g + 2c_E\left(2-g^2\right)\right)}{\left(4+g-2g^2\right)\left(4-g-2g^2\right)}$$

is derived from the intersection of the above reaction functions.

Next, recall that when $g > g^H$*, the entrant induces its retailer to monopolize the downstream market. A simple way to derive* g^H *is to find the value of* g *such that the optimal* w_1 *in the above basic optimization problem (again using the reaction functions given above) is equal to* c_I*. With* $w_1 = c_I$*, the optimal* w_2 *would then be equal to* $\left(c_I\left(2-g^2\right) - a\left(1-g\right)\left(2+g\right)\right)/g$ *using lemma 1. Intuitively,* g^H *is the lowest value for which the incumbent decides not to sell to its retailer since the constraints* $w_1 \geq c_I$ *and* $q_1(\cdot) \geq 0$ *are binding. Thus* g^H *solves* $w_1^* - c_I = 0$ *where*

$$w_1^* = \frac{a\left(1-g\right)\left(2+g\right)\left(4+g-2g^2\right)+\left(2-g^2\right)\left(c_E g + 2c_I\left(2-g^2\right)\right)}{\left(4+g-2g^2\right)\left(4-g-2g^2\right)}$$

is derived from the intersection of the above reaction functions.

Thus for $g^L \leq g \leq g^H$*, the incumbent and entrant optimally and uniquely sets* w_1^* *and* w_2^* *since none of the constraints are binding.*

The exact analytical forms of g^L and g^H are derived by solving for the roots of $w_1^* - c_I = 0$ and $w_2^* - c_I = 0$, respectively. The unique analytical forms are as given above. It can be confirmed numerically, that both g^L (bounded by 0 and 1) and g^H (bounded by 0.8875 and 1) are decreasing, and that $g^H \geq g^L$ in $\lambda \in (0,1)$. ■

Lemma 3 *Assuming the entrant enters in the $S = 1$ subgame, equilibrium profits are given as follows:*

1. *if $g < g^L$, then $\pi_1^{(i)} = \pi_{i|S=2}$,*

$$\pi_2^{(i)} = \frac{\left(4+g-2g^2\right)^2 (1-g)(a-c_I)^2}{4b(1+g)(2-g)^2(2-g^2)^2},$$

$$\Pi_I^{(i)} = \frac{(2+g)(1-g)(a-c_I)^2}{4b(1+g)(2-g)(2-g^2)}$$

and

$$\Pi_E^{(i)} = \frac{\left(4+g-2g^2\right)(c_I-c_E)(a-c_I)}{2b(1+g)(2-g)(2-g^2)};$$

2. *if $g^L \leq g \leq g^H$, then*

$$\begin{aligned}\pi_1^{(ii)} &= \frac{\left(2-g^2\right)^2\left[c_E g\left(2-g^2\right) - c_I\left(8-9g^2+2g^4\right) + a(1-g)(2+g)\left(4+g-2g^2\right)\right]^2}{b(1-g)(1+g)(4-g^2)^2\left[(4+g-2g^2)(4-g-2g^2)\right]^2},\\ \pi_2^{(ii)} &= \frac{\left(2-g^2\right)^2\left[c_I g\left(2-g^2\right) - c_E\left(8-9g^2+2g^4\right) + a(1-g)(2+g)\left(4+g-2g^2\right)\right]^2}{b(1-g)(1+g)(4-g^2)^2\left[(4+g-2g^2)(4-g-2g^2)\right]^2},\end{aligned}$$

$\Pi_I^{(ii)} = \left(\pi_1^{(ii)}\right)\left(4-g^2\right)/\left(2-g^2\right)$ *and* $\Pi_E^{(ii)} = \left(\pi_2^{(ii)}\right)\left(4-g^2\right)/\left(2-g^2\right)$*; and finally*

3. *if $g > g^H$, then both the incumbent and retailer 1 get nil, while*

$$\pi_2^{(iii)} = \frac{(a-c_I)^2\left(1-g^2\right)}{bg^2}$$

and

$$\Pi_E^{(iii)} = \frac{(a-c_I)\left[c_I\left(2-g^2\right) - a(1-g)(2+g) - c_E g\right]}{bg^2}.$$

Furthermore, $\pi_2^{(iii)} \geq \pi_2^{(i)}$ for all g and $\pi_2^{(ii)} \geq \pi_2^{(i)}$ if $g < g^L$.

Proof. *The profit functions easily follow by using the optimal wholesale prices given by lemma 2 in the computation of profits. Straightforward algebra confirms $\pi_2^{(iii)} \geq \pi_2^{(i)}$. Comparing $\pi_2^{(ii)}$ and $\pi_2^{(i)}$ analytically is difficult but notice that for $g < g^L$, the entrant would rather price above c_I if there are no constraints. This simply implies $\pi_2^{(ii)} \geq \pi_2^{(i)}$ if $g < g^L$.* ■

Lemma 4 *Define* g^* *as the value of* g *such that in the region* $g^L \leq g \leq g^H$*, half of the incumbent's profit when both retailers sign is just equal to the required compensation in order to attract a deviating retailer given one retailer already signs. The condition that* $\lambda \leq 2/3$ *is sufficient to establish that the required compensation is feasible in the region* $g^L \leq g \leq g^H$*.*

Proof. *With* $g^L \leq g \leq g^H$*, define* $x^{*(ii)}$ *as the required compensation in order to attract a deviating retailer given one retailer already signs. Note that* $x^{*(ii)}$ *is the solution to* $\pi_{i|S=2} + x = \pi_2^{(ii)}$*. We are primarily concerned with the equation* $\Pi_{I|S=2} - x^{*(ii)} =$

$$\frac{(a-c_E)^2}{16b}\left[\frac{1}{1+g}\left(\frac{(2-\lambda)^2}{2-g}+\frac{(1-g)(2-\lambda)^2}{(2-g)^2}-\frac{4\left(2-g^2\right)^2\left(16-18g^2+4g^4-2g(2-\lambda)+g^3(2-\lambda)\right)^2}{(1-g)\left(4-g^2\right)^2\left(16-17g^2+4g^4\right)^2}\right)\right]$$

Consider $\Gamma = \left(\Pi_{I|S=2} - x^{*(ii)}\right) b/\left(a-c_E\right)^2$*. By definition,* g^* *is the value of* g *such that* $\Gamma = 0$*. We need to determine the properties of* Γ *for* $\lambda \leq 2/3$ *and* $g \in [\ g^L,\ g^H]$*. Note that* Γ *is continuous in* $g \in (0,1)$ *and as* $g \to 1$*,* $\Gamma \to -\infty$*. A numerical examination confirms that provided* $\lambda \leq 2/3$*, the following results hold: (i) a unique root* g^* *exists; (ii) for* $g \leq g^*$*,* $\Gamma \geq 0$*; and (iii)* $g^* > g^H$*. Together these results imply that* $\Gamma \geq 0$ *and thus compensation is feasible for* $g^L \leq g \leq g^H$ *and* $\lambda \leq 2/3$*.* ■

Now we are ready to prove our main proposition.

Proposition 1 (Proposition 4 in Abito and Wright, 2005) *Suppose the fixed costs of entry are low enough so that the entrant can enter even if just one retailer is available. Provided intrabrand competition is not too strong (*$g < \left(\sqrt{17}-1\right)/4$ *is sufficient), then we have a unique entry equilibrium. For strong intrabrand competition (*$g > \left(\sqrt{17}-1\right)/4$*), then we have a unique exclusion equilibrium provided the entrant's cost advantage is not too strong (*$0 < \lambda \leq 2/3$ *is sufficient).*

Proof. *Define* $\underline{F}$ *as*

$$\begin{aligned}\underline{F} &= \Pi_E^{(i)} \ if\ 0 < g < g^L \\ &= \Pi_E^{(ii)} \ if\ g^L \leq g \leq g^H \\ &= \Pi_E^{(iii)} \ if\ g^H < g < 1.\end{aligned}$$

If $F < \underline{F}$*, the entrant can enter even with one retailer. Consider first the case where* $g < g^L$*. Note that in order to attract a deviating retailer given that the other retailer does not sign, the compensation* x *must satisfy* $\pi_1^{(i)} + x \geq \pi_{i|S=0}$*. This translates to the condition* $x \geq 3\pi_{i|S=2}$*. If only one retailer* *signs, the maximum compensation that the incumbent can afford is only* $\Pi_I^{(i)}$*, which is always less than* $3\pi^s_{I|S=2}$ *regardless of* g*. Thus, an equilibrium with* $S = 1$ *and exclusion does not exist. Furthermore, in order to attract a deviating retailer given that the other retailer signs, the compensation* x *must satisfy* $\pi_{i|S=2} + x \geq \pi_2^{(i)}$*, which occurs if and only if*

$$x \geq x^{*(i)} \equiv \frac{(a - c_I)^2 (1 - g) \left(6 + g - 3g^2\right)}{4b (2 - g) (2 - g^2)^2}.$$

The condition for which compensation is feasible for the incumbent ($\Pi_{I|S=2}/2 \geq x^{*(i)}$) *occurs if and only if* $g \geq \left(\sqrt{17} - 1\right)/4$*. Since* $x^{*(i)} > 3\pi^s_{I|S=2}$*, if* $g \geq \left(\sqrt{17} - 1\right)/4$*, the incumbent can offer a compensation of* $x^{*(i)}$ *and each retailer will want to sign the contract regardless of what the other retailer does. If* $g < \left(\sqrt{17} - 1\right)/4$ *we have a unique entry equilibrium with both retailers buying from the entrant and the incumbent setting* $x = 0$*.*

Next consider the case where $g^L \leq g < g^H$*. Intuitively,* $\pi_1^{(i)} \geq \pi_1^{(ii)}$ *since* g *is higher and retailers' prices are now lower than in the* $g < g^L$ *case due to lower wholesale prices–which also implies* $\Pi_I^{(i)} \geq \Pi_I^{(ii)}$*. Therefore similar to the previous case, an equilibrium with* $S = 1$ *and exclusion does not exist. From the previous lemma, recall that we defined* g^* *as the value of* g *such that in the region* $g^L \leq g < g^H$*,* $\Pi_{I|S=2}/2 = x^{*(ii)}$ *where* $x^{*(ii)}$ *is the compensation needed in order for* $\pi_{i|S=2} + x = \pi_2^{(ii)}$*. Furthermore, recall that* $g < g^*$ *implies* $\Pi_{I|S=2}/2 \geq x^{*(ii)}$*. With our assumption that* $\lambda \leq 2/3$*,* $g^* \geq g^H \geq 0.9414$ *and therefore we have unique exclusion in the region* $g^L \leq g < g^H$*.*

Lastly consider the case where $g \geq g^H$*. Obviously* $\Pi_I^{(i)} \geq \Pi_I^{(iii)} = 0$ *and* $\pi_1^{(i)} \geq \pi_1^{(iii)} = 0$ *and therefore an equilibrium with* $S = 1$ *and exclusion does not exist. Defining* $x^{*(iii)}$ *as the level of compensation such that* $\pi_{i|S=2} + x \geq \pi_2^{(iii)}$ *(required compensation to attract a deviating retailer given the other signs), we find that in order for* $\Pi_{I|S=2}/2 \geq x^{*(iii)}$*,* $g \geq 0.9414$ *which is equal to* g^H *when* $\lambda \cong 0.6851 > 2/3$*. Therefore since* g^H *is decreasing in* λ *(by lemma 2),* $\Pi_{I|S=2}/2 \geq x^{*(iii)}$ *holds for* $\lambda \leq 2/3$*.*

Combining these results, we conclude that $\lambda \leq 2/3$ *is sufficient for a unique exclusion equilibrium to prevail whenever* $g \geq \left(\sqrt{17} - 1\right)/4$*.*

To check $g < \left(\sqrt{17} - 1\right)/4$ *is sufficient for unique entry, recall that* $\pi_2^{(iii)} \geq \pi_2^{(i)}$ *for all* g *and* $\pi_2^{(ii)} \geq \pi_2^{(i)}$ *if* $g < g^L$ *(lemma 3). Thus if compensation is not feasible when the deviating retailer earns* $\pi_2^{(i)}$ *(*$g < \left(\sqrt{17} - 1\right)/4$*), then it must also be that compensation is not feasible when deviating profit is equal to* $\pi_2^{(iii)}$ *for* $g < \left(\sqrt{17} - 1\right)/4$ *and when deviating profit is equal to* $\pi_2^{(ii)}$ *if* $g < g^L$*. It remains to see what happens when* $g^L < g < \left(\sqrt{17} - 1\right)/4$*. Numerical examination shows that* $g^L < \left(\sqrt{17} - 1\right)/4$ *implies*

$g^* < g^L$ and therefore compensation is not feasible in the $g^L < \left(\sqrt{17}-1\right)/4 < g^H$ region. Therefore $g < \left(\sqrt{17}-1\right)/4$ is sufficient for unique entry to occur in equilibrium. ■

Remark 1 *When $\lambda > 2/3$, $g^* > g^H$ does not necessarily hold and thus we have unique entry equilibrium for $g \in (g^*, g^H]$ if $g^* < g^H$.*

The figure below illustrates the typical case when $\lambda \leq 2/3$.

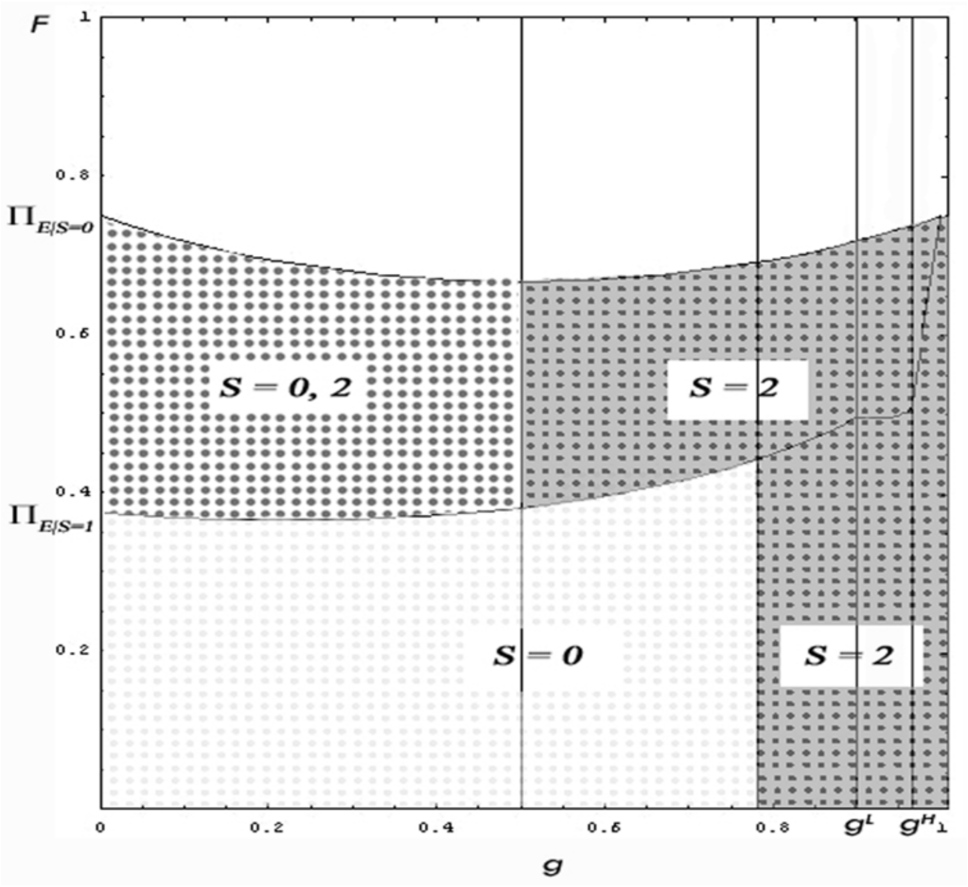

Figure 1. Typical Case ($\lambda = 2$)

1 References

Abito, J. M. and J. Wright (2005) “Exclusive dealing with imperfect downstream competition,” mimeo. National University of Singapore.