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Algebra II
Lesson 6.6
Fundamental Theorem of Algebra

In previous lessons we called the answers to polynomials: x-intercepts, roots, zeros, and factors. Let's summarize:

- A real number, x is a root of a polynomial F(x), if, F(x)=0
- A root, r, is an x-intercept of F(x)
- $x - r$ is a factor of F(x)
- When $(x - r)$ is divided into a polynomial, F(x), the remainder=0
- r is a "zero" of F(x)

Sometimes you have to work "backwards" to get the desired answer. That is you may be given a set of answers and your job will be to reconstruct the polynomial that gave you the roots/answers. Building a polynomial takes 3 steps:

1. set each solution =0
2. set up one equation with factors equal to zero and multiply the factors together (using FOIL)
3. combine like terms and write the terms in descending order

Example: Write a polynomial function with the roots x= 3, -1, 2

$x = 3$ $\quad x = -1$ $\quad x = 2$

$(x - 3) = 0$ $\quad (x + 1) = 0$ $\quad (x - 2) = 0$ Set the solutions equal to zero

$(x - 3)(x + 1)(x - 2) = 0$ Multiply the factors out

$(x - 3)(x^2 - x - 2) = 0$

$x^3 - x^2 - 2x - 3x^2 + 3x + 6$ Combine like terms

$x^3 - 4x^2 + x + 6 = 0$ Arrange in descending order

In 1797 Carl Gauss proved what is known as the **Fundamental Theorem of Algebra.** It states that *a polynomial of degree* $n \geq 1$ *has at least one complex zero. In essence, t*he fundamental theorem of algebra guarantees that every polynomial has a complete factorization, if we are allowed to use complex numbers (a+bi). Remember that a real number may be written as a complex number.

Recap: Solving polynomial equations can be put into a procedure:

- Find all possible roots of the polynomial (Rational Root Theorem – 6.5)
- Test the possible roots using synthetic division; reduce the polynomial
- Check for multiplicities of roots (would show up as another factor)
- Once the polynomial is reduced to a quadratic $ax^2 + bx + c = 0$, factor or use the Quadratic Formula to solve
- Use a graphing calculator to verify/check your answers.

Example: Solve $x^4 - 3x^3 + 5x^2 - 27x - 36 = 0$

Find all possible roots

p=-36, q=1

$$x = \frac{p}{q} = \frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36}{1}$$

$$= \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36$$

Use Rational root Theorem

$x = 1$: $1^4 - 3(1)^3 + 5(1)^2 - 27(1) - 36 = 60$ $\quad No$

$\boldsymbol{x = -1}$ $(-1)^4 - 3(-1)^3 + 5(-1)^2 - 27(-1) - 36 = 0$ $\quad Yes$

Test the roots

Use synthetic division to reduce the polyn.

-1	1	-3	5	-27	-36
	↓	-1	4	-9	36
	1	-4	9	-36	0

$x^3 - 4x^2 + 9x - 36$

Write polynomial

Continue using the rational root theorem to solve for roots

$x = 2$ $\quad (2)^3 - 4(2)^2 + 9(2) - 36 = -26 \neq 0$ $\quad No$

$x = -2$ $\quad (-2)^3 - 4(-2)^2 + 9(2) - 36 = -78 \neq$ $\quad No$

$x = 3$ $\quad (3)^3 - 4(3)^2 + 9(3) - 36 = -18 \neq 0$ $\quad No$

$x = -3$ $\quad (-3)^3 - 4(-3)^2 + 9(-3) - 36 = -126 \neq$ $\quad No$

$\boldsymbol{x = 4}$ $\quad (4)^3 - 4(4)^2 + 9(4) - 36 = 0$ $\quad Yes$

Use synthetic division to reduce the polyn.

4	1	-4	9	-36
	↓	4	0	36
	1	0	9	0

$x^2 + 9 = 0$

$x^2 = -9$

$x = \sqrt{-9}$

$\boldsymbol{x = \pm 3i}$

Write polynomial

Solve for x

Since the polynomial is a 4th degree there will be at most FOUR roots or zeros (The degree of the polynomial equals the number of complex solutions, where real are complex). We stop our search for roots as we have reached 4 roots; our polynomial has the roots of: $x = -1, 4, \pm 3i$. Hence, our solution may be written as: $\boldsymbol{(x+1)(x+4)(x+3i)(x-3i) = 0}$ Checking with the graphing calculator (complex roots cannot be graphed, use the original equation. :