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Algebra II

Lesson 6-5

Theorems about Roots of Polynomial Equations

Is there a simple way to predict how many roots/zeros/answers are possible? Yes! It is called the **Rational Root Theorem**. This theorem basically states that if we factor the constant term (p) and the leading coefficient(q), make a ratio of all possible combinations of the factors: $\frac{\text{factor of constant term}}{\text{factor of leading coefficient}} = \frac{p}{q}$, then the ratios that result in a zero when put into the equation ARE ROOTS. To better understand this process, let's look at an example:

Identify all of the real roots of: $x^4 - 3x^2 - 4 = 0$

p=4, q=1,

factorization of p: $\pm 1, \pm 2, \pm 4$; factorization of q: ± 1

so $\frac{p}{q} = \pm \frac{1}{1}, \pm \frac{2}{1}, \pm \frac{4}{1}$,

$$\begin{aligned}1^4 - 3(1)^2 - 4 &= -6 \neq 0 \therefore \text{not a root} \\ -1^4 - 3(-1)^2 - 4 &= -6 \neq 0 \therefore \text{not a root} \\ 2^4 - 3(2)^2 - 4 &= 0 \therefore \text{root} \\ (-2)^4 - 3(-2)^2 - 4 &= 0 \therefore \text{root} \\ (4)^4 - 3(4)^2 - 4 &= 204 \neq 0 \therefore \text{not a root} \\ (-4)^4 - 3(-4)^2 - 4 &= 204 \neq 0 \therefore \text{not a root}\end{aligned}$$

Identify the constant term and the leading coefficient

Factor each term

List all ration combinations

Plug ratios into the polynomial equation and evaluate

Root: x=2

Root: x=-2

Notice that this theorem is called the Rational Root Theorem; this is a means of finding *rational* roots. Irrational roots are possible as well as complex roots. To find these roots, you must reduce the polynomial down to a quadratic by dividing the real factors into the original polynomial, and use the Quadratic Formula to solve.

Summary:

When given a polynomial to solve, we can find all **real roots** by

1. using the Rational Root Theorem to find the rational roots,
2. use synthetic division to reduce the polynomial,
3. and use the quadratic formula to find the remaining roots.

Always check the graph to make sure the roots match the graph.

Example: Identify all roots for

$$3x^3 - 18x^2 - 9x + 132 = 0$$

$$x^3 - 6x^2 - 3x + 44 = 0$$

1.

p=44, q=1

factorization: p: $\pm 1, \pm 2, \pm 4, \pm 11, \pm 22, 44$ q: ± 1

$$\frac{p}{q} = \pm \frac{1}{1}, \pm \frac{2}{1}, \pm \frac{4}{1}, \pm \frac{11}{1}, \pm \frac{22}{1}, \pm \frac{44}{1}$$

$$1^3 - 6(1)^2 - 3(1) + 44 = 36, \text{not a root}$$

$$-1^3 - 6(-1)^2 - 3(-1) + 44 = 40, \text{not a root}$$

$$(2)^3 - 6(2)^2 - 3(2) + 44 = 22, \text{not a root}$$

$$(-2)^3 - 6(-2)^2 - 3(-2) + 44 = 18, \text{not a root}$$

$$(4)^3 - 6(4)^2 - 3(4) + 44 = 0 \quad \text{Root}$$

$$(-4)^3 - 6(-4)^2 - 3(-4) + 44 = -104, \text{not a root}$$

Is there any number/variable in common with all 4 terms? Yes. Factor out.

Use Rational Root Theorem: Constant term and leading coefficient

Factor terms

List ratio combinations

Plug ratios into the polynomial and evaluate

Root x = 4 ; Gives a factor of (x - 4)

Testing the remaining values will show that none of the remaining ratios will be roots

2. Now do synthetic division to reduce the polynomial:

$$x^3 - 6x^2 - 3x + 44 = 0$$

$$\begin{array}{r|rrrr} 4 & 1 & -6 & -3 & 44 \\ & & & & \\ \hline & 1 & 2 & -11 & 0 \end{array}$$

$$x^2 - 6x - 11 = 0$$

Factored, we get: $(x - 4)(x^2 - 6x - 11) = 0$

3.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-11)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{4 + 44}}{2}$$

$$x = \frac{2 \pm \sqrt{48}}{2} = \frac{2 \pm \sqrt{16}\sqrt{3}}{2}$$

$$x = \frac{2 \pm 4\sqrt{3}}{2}$$

$$x = 1 \pm 2\sqrt{3}$$

Now use the quadratic formula to solve for the remaining solutions using: $a=1, b=-2, c=-11$

So all the possible roots for this polynomial are

$$x = 4, 1 \pm 2\sqrt{3}$$

Remember: If you have one irrational root, there will be a second irrational root. The **Irrational Root Theorem** states that if you have a root of $a + \sqrt{b}$, then you will also have the root of $a - \sqrt{b}$. These are called **conjugates**. Likewise for imaginary roots we always have **conjugate solutions** as well: if you have the root of $a + bi$, then you will also have the root $a - bi$. (These are known as complex conjugates.)