



## Introduction

In this chapter we discuss the basic assumptions (axioms) about real numbers, together with certain definitions, and derive other properties of real numbers.

The set of *natural numbers*, denoted by  $\mathbb{N}$ , which consists of the ordinary counting numbers 1, 2, 3, . . . may be considered as the simplest real numbers. The counting numbers are also called *positive integers*. The natural numbers together with 0 and their negatives:  $-1, -2, -3, \dots$  comprise the *integers* and the set consisting these numbers is denoted by  $\mathbb{Z}$ . The set of *rational numbers*, denoted by  $\mathbb{Q}$ , consists of numbers which can be written in the form  $\frac{m}{n}$ , where  $m$  and  $n$  are integers and  $n \neq 0$ . Observe that any integer  $m$  can be written as  $\frac{m}{1}$ , thus the set of rational numbers include the set of integers as a subset. Numbers that are not rationals are called *irrationals*. The set of irrationals is denoted by  $\overline{\mathbb{Q}}$ . Examples of irrationals are

$$\sqrt{2}, \pi, \sqrt[3]{6}.$$

The rational numbers together with the irrational numbers form the *set of real numbers*, denoted by  $\mathbb{R}$ .

**Notations:** One may visualize the set of real numbers using a *number line*, as shown in the Figure 2.1 below.

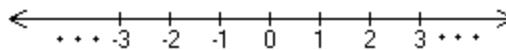


Figure 2.1 Real number line

Select a point as the *origin* to correspond to 0 and another point to the right of 0 to correspond to 1. A scale is now determined. To the left of zero the negative real numbers will be located while to the right are the positive real numbers. Using the concept of fractions we can easily locate the point corresponding to a rational number. For example, to locate the point corresponding to  $\frac{3}{4}$ , divide the line segment from 0 to 1 into 4 equal parts, then the right endpoint of the 3<sup>rd</sup> division would be the desired point. A point corresponding to an irrational number is more difficult to locate than that of a rational number. The irrationals, however, can always be approximated by rationals. For example,  $\sqrt{2}$  is approximately 1.414.

### 2.1 Decimal and Fractional Representation of Rational Numbers

A rational number  $\frac{a}{b}$ ,  $b \neq 0$ , can be expressed as a decimal by dividing  $a$  by  $b$ . The rational numbers have two forms when expressed as decimal expansions: (1) terminating such as 0.75 and (2) repeating such as  $0.3333\dots = 0.\overline{3}$ , where the bar above 3 means that 3 is the repeating digit.



**Illustration 1.**

a)  $\frac{4}{5} = .8$

*Solution:*

$$\begin{array}{r} 5 \overline{)4.0} \\ \underline{4.0} \\ 0 \end{array}$$

b)  $\frac{2}{9} = 0.\overline{2}$

*Solution:*

$$\begin{array}{r} 9 \overline{)2.000} \quad \text{or } 0.\overline{2} \\ \underline{1.8} \\ 20 \\ \underline{18} \\ 20 \\ \underline{18} \\ 2 \end{array}$$

On the other hand, a rational number expressed as a decimal expansion can be expressed as a fraction  $\frac{a}{b}$ ,  $b$

$\neq 0$ : if the decimal is a terminating decimal, we multiply it by  $\frac{10^k}{10^k}$ , where  $k$  is the number of digits after the decimal. The expression  $10^k$  means that we multiply the number 10  $k$  times.

**Illustration 2.**

a)  $0.75 = 0.75 \left( \frac{10^2}{10^2} \right) = \frac{75}{100}$

b)  $0.6 = 0.6 \left( \frac{10}{10} \right) = \frac{6}{10}$

Before we discuss the case where the decimal is a repeating decimal, we pause for a moment to discuss properties of equality. In mathematics, the symbol  $A = B$  means  $A$  and  $B$  are two names for the same thing. Thus, we may replace  $B$  by  $A$  or  $A$  by  $B$  in any expression involving either symbol. Thus, equality means 'is identical to'. The following are the properties of equality.

1. *Reflexive Property.* If  $a \in \mathbb{R}$ , then  $a = a$ .
2. *Symmetric Property.* If  $a, b \in \mathbb{R}$  and  $a = b$ , then  $b = a$ .
3. *Transitive Property.* If  $a, b, c \in \mathbb{R}$  and  $a = b, b = c$ , then  $a = c$ .
4. *Addition Property.* If  $a = b$ , then  $a + c = b + c$ .
5. *Multiplication Property.* If  $a = b$ , then  $ac = bc$ .
6. *Cancellation Laws.* (i) If  $ac = bc$  and  $c \neq 0$ , then  $a = b$ .  
(ii) If  $a + c = b + c$ , then  $a = b$ .



Properties 4 and 5 are our assurance that if the same number is added or multiplied to both sides of equality, the equality is maintained.

To express a repeating decimal to a fraction we suggest the following procedure.

Step 1. Let  $x =$  repeating decimal.

Step 2. Multiply both sides of the equality in (1) by  $10^m$ , where  $m$  is the number of digits after the decimal before the repeating block.

Step 3. Multiply the resulting equality in Step 2 by  $10^n$ , where  $n$  is the number of digits in the repeating block.

Step 4. Subtract the equality obtained in Step 2 from that obtained in Step 3.

Step 5. Divide both sides of the equality obtained in Step 4 by the coefficient of  $x$  to obtain the desired fraction.

**Example.**

a) Express  $0.\overline{24}$  as a fraction.

*Solution:*

$$\begin{aligned}x &= 0.\overline{24} \quad ; \quad m = 0, \quad n = 2. \\x &= 0.\overline{24} = 0.24\overline{24} \\10^2(x &= 0.24) \\100x &= 24.\overline{24} \\- \quad x &= 0.\overline{24} \\99x &= 24 \\x &= \frac{24}{99}.\end{aligned}$$

b) Express  $0.1\overline{23}$  as a fraction.

*Solution:*

$$\begin{aligned}x &= 0.1\overline{23} \quad ; \quad m = 1, \quad n = 2. \\10x &= 1.\overline{23} = 1.23\overline{23} \\10^2(10x &= 1.23\overline{23}) \\1000x &= 123.\overline{23} \\- \quad 10x &= 1.\overline{23} \\990x &= 122 \\x &= \frac{122}{990}.\end{aligned}$$

c) Express  $2.1\overline{34}$  as a fraction.

*Solution:*

$$\begin{aligned}x &= 2.1\overline{34} \quad ; \quad m = 2, \quad n = 1. \\100x &= 213.\overline{4} = 213.4\overline{4} \\10(100x &= 213.4\overline{4}) \\1000x &= 2134.\overline{4} \\- \quad 100x &= 213.\overline{4} \\900x &= 1921 \\x &= \frac{1921}{900}.\end{aligned}$$



**Exercise**

A. Change the following fraction to decimal.

- |                   |                     |                    |                     |
|-------------------|---------------------|--------------------|---------------------|
| 1. $\frac{2}{15}$ | 6. $\frac{5}{11}$   | 11. $\frac{6}{7}$  | 16. $\frac{8}{9}$   |
| 2. $\frac{3}{8}$  | 7. $\frac{3}{24}$   | 12. $\frac{5}{17}$ | 17. $\frac{19}{36}$ |
| 3. $\frac{2}{37}$ | 8. $\frac{6}{13}$   | 13. $\frac{26}{5}$ | 18. $\frac{33}{4}$  |
| 4. $\frac{8}{15}$ | 9. $\frac{13}{15}$  | 14. $\frac{9}{7}$  | 19. $\frac{16}{3}$  |
| 5. $\frac{27}{6}$ | 10. $\frac{18}{21}$ | 15. $\frac{1}{9}$  | 20. $\frac{15}{14}$ |

B. Express each terminating decimal number as a fraction.

- |          |            |           |            |
|----------|------------|-----------|------------|
| 1. 0.25  | 6. 3.125   | 11. 1.82  | 16. 0.3750 |
| 2. 3.128 | 7. 20.4    | 12. 0.44  | 17. 2.65   |
| 3. 1.38  | 8. 1.295   | 13. 25.64 | 18. 2.495  |
| 4. 7.24  | 9. 2.33    | 14. 0.284 | 19. 0.12   |
| 5. 2.6   | 10. 2.2580 | 15. 8.125 | 20. 3.24   |

C. Express each repeating decimal number as a fraction.

- |                       |                        |                         |                        |
|-----------------------|------------------------|-------------------------|------------------------|
| 1. $0.\overline{8}$   | 6. $2.01\overline{32}$ | 11. $4.\overline{13}$   | 16. $0.\overline{125}$ |
| 2. $2.53\overline{9}$ | 7. $0.\overline{54}$   | 12. $2.01\overline{3}$  | 17. $0.2\overline{5}$  |
| 3. $4.\overline{15}$  | 8. $0.\overline{2}$    | 13. $1.30\overline{51}$ | 18. $3.\overline{41}$  |
| 4. $2.15\overline{3}$ | 9. $0.\overline{83}$   | 14. $2.\overline{113}$  | 19. $1.\overline{24}$  |
| 5. $1.\overline{25}$  | 10. $3.01\overline{2}$ | 15. $0.10\overline{23}$ | 20. $3.15\overline{1}$ |

**Percentage, Ratio and Proportion**

**Percentage**, fraction of a whole number expressed in hundredths. The term is derived from the Latin per centum, meaning “per hundred”, as the term represents fractions with the denominator 100. Thus, 20 per cent means 20/100. Usually represented by the symbol %, percentage computations are commonly used in finance and industry, and by scientists to evaluate data.

**Example**

To calculate the percentage relationship of one number (4) to another (16), the former is divided by the latter and multiplied by 100; thus

$$\left(\frac{4}{16}\right) \times 100 = 0.25 \times 100 = 25\% \text{ ; that is, } 4 \text{ is } 25\% \text{ of } 16$$

**Example:** In a class of 20, only 8 students did not agree with the policy. How many percent of the class did not agree with the policy?

**Solution:**  $\left(\frac{8}{20}\right) \times 100 = 0.40 \times 100 = 40\%$  ; thus 40% of the class did not agree with the policy



**Example:** In the College of Nursing, 85% of the 200 first year students attended the orientation. How many first year students attended the orientation?

**Solution:** Let  $n$  = number of 1st yr student who attended the orientation , thus

$$\frac{n}{200} \times 100 = 85\% \text{ therefore } \frac{n}{200} = \frac{85}{100} = 0.85 \text{ and } n = 0.85 \times 200 = 170 \text{ students}$$

**Example:** In the Mat 1N class, 12 students is 30% of the total number of students . How many students are there in this class?

**Solution:** Let  $m$  = total number of students in the class , thus

$$\frac{12}{m} \times 100 = 30\%, \text{ therefore } \frac{12}{m} = 0.30, \text{ thus } m = \frac{12}{0.30} = 40 \text{ students}$$

**Ratio** is a relation between two numbers, defined as the division of one number by the other. Thus, the ratio of 5 and 6 is  $\frac{5}{6}$  and we also write this as 5:6 (read as 5 is to 6). **Proportion** is the equality of ratios.

Thus,  $\frac{5}{6} = \frac{10}{12}$  is a proportion and we sometimes write this as 5:6 = 10:12 (read as 5 is to 6 is equal to 10 is to 12).

**Example:** The ratio of the males and females in a class is 4:5. If there were 12 males, then how many were females?

**Solution:** Let  $f$  = number of females , then the proportion would be 4:5=12:f. Thus, we can write this

as  $\frac{4}{5} = \frac{12}{f}$  and then solve for  $f$  . So, multiply both sides by  $f \left( \frac{5}{4} \right)$  we have

$$f = 12 \times \frac{5}{4} = 15 \text{ females .}$$

### Exercises:

Find the value of  $x$  .

1.  $7 : 3 = x : 15$
2.  $2 : x = 6 : 9$
3.  $x : 9 = 100 : 18$

### Basic Properties of Real Numbers

The following are the basic properties of the real numbers involving the four fundamental operations: addition, subtraction, multiplication, and division. These properties are called *field properties* of  $\mathbb{R}$  .

1. *Closure Property:* The set  $\mathbb{R}$  is *closed* with respect to addition and multiplication. That is, both  $a + b$  and  $a \cdot b$  are in  $\mathbb{R}$  .
2. *Commutative Property:* Addition and multiplication in  $\mathbb{R}$  are *commutative*. That is,  $a + b = b + a$  and  $a \cdot b = b \cdot a$ .



- 3. Associative Property:** Addition and multiplication in  $\mathbb{R}$  are *associative*. That is,  $a + (b + c) = (a + b) + c$  and  $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ .
- 4. Distributive Property of Multiplication over Addition (DPMA):** Multiplication is *distributive* over addition. That is,  $a(b + c) = ab + ac$ .
- 5. Existence of Identity Element:** The number 0 is the *additive identity* in  $\mathbb{R}$ , and the number 1 is the *multiplicative identity* in  $\mathbb{R}$ . That is,  $a + 0 = 0 + a = a$  and  $a \cdot 1 = 1 \cdot a = a$ .
- 6. Existence of Inverse Element:** For each element  $a$  in  $\mathbb{R}$  there is a unique *additive inverse* in  $\mathbb{R}$ , designated by  $-a$ , and for each  $a \neq 0$ , there is a unique *multiplicative inverse* (or *reciprocal*) in  $\mathbb{R}$ , designated by  $\frac{1}{a}$ . Thus,  $a + (-a) = (-a) + a = 0$  and  $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$ .

### Other Properties of $\mathbb{R}$ :

1. The sum of two positive real numbers is positive.
2. The product of two positive real numbers is positive.

Using Property 6, the difference and quotient of two real numbers are defined as follows:

$$a - b = a + (-b) \quad ; \quad \frac{a}{b} = a \cdot \frac{1}{b}, \quad b \neq 0.$$

It should be noted that  $\frac{a}{0}$  is not defined because 0 has no multiplicative inverse. This is why the denominator of a fraction cannot be 0; that is, we cannot divide by 0.

### Further Properties of $\mathbb{R}$ :

1.  $a \cdot 0 = 0$
2.  $-a = (-1)a$
3.  $(-a)(b) = -(ab)$
4.  $(-a)(-b) = ab$
5. If  $ab = 0$ , then either  $a = 0$  or  $b = 0$ .

The preceding properties can be derived from the basic properties of  $\mathbb{R}$ .

**Example 1.** Show that  $a \cdot 0 = 0$ .

*Solution:*

$$\begin{aligned} a \cdot 0 &= a \cdot (0 + 0) \\ a \cdot 0 &= a \cdot 0 + a \cdot 0, \text{ by DPMA} \\ (a \cdot 0) - (a \cdot 0) &= a \cdot 0 \\ 0 &= a \cdot 0. \end{aligned}$$

**Example 2.** Show that  $(-a)(b) = -(ab)$ .

*Solution:*

$$\begin{aligned} a + (-a) &= 0 \\ [a + (-a)] \cdot b &= 0 \cdot b \\ ab + (-a)b &= 0 \end{aligned}$$

The last equation says that  $(-a)b$  is the additive inverse of  $ab$ . Thus,  $(-a)b = -(ab)$ .



### Exercise

Identify the property which justifies the given statement. Let the variables represent real numbers.

- $xy + 0 = xy$ .
- $a + b = a + b$ .
- $3(x + 5) = 3x + 15$ .
- $\left(-\frac{4}{5}\right)\left(-\frac{5}{4}\right) = 1$ .
- $(wy)z + 5 = w(yz) + 5$ .
- $(wy)z + 5 = (yw)z + 5$ .
- $-\sqrt{3} + \sqrt{3} = 0$ .
- If  $x = 2$  and  $2 = y$ , then  $x = y$ .
- If  $ac + d = t$ , then  $t = ac + d$ .
- $uv + 7 = 7 + uv$ .
- $(2 + 4) \cdot 5 = 5 \cdot (2 + 4)$
- $3 + \sqrt{3}$  is a real number.
- $-\frac{1}{2} + \frac{1}{2} = 0$ .
- $2(3x + 4) = 6x + 8$ .
- $3 + (6 + 5) = (3 + 6) + 5$ .
- $2\left(\frac{1}{2}\right) = 1$ .
- $\frac{1}{3} = \frac{1}{3}$ .
- If  $x = a$  and  $a = 4$ , then  $x = 4$ .
- If  $5x = 20$ , then  $x = 4$ .
- $6 \cdot (2 \cdot 5) = (6 \cdot 2) \cdot 5$ .

### Factorization of positive Integers and the Concepts of GCF and LCM

**Definition** If  $n, m, s$  are positive integers and  $m \cdot s = n$ , then  $m$  and  $s$  are said to be *factors* or *divisors* of  $n$  and  $n$  is said to be a multiple of  $m$  and  $s$ .

**Illustration 1.**  $21 = 7 \cdot 3$  so that 7 and 3 are factors of 21 and 21 is a multiple of 3 and 7.

Since  $n = n \cdot 1$  for any integer  $n$ , 1 is a factor of any integer  $n$ . The integer 0 is a multiple of any integer  $n$  since  $0 = n \cdot 0$ .

All positive integers except 1 may be classified as either *composite* or *prime*. A positive integer is called composite if it is different from 1 and has factors other than 1 and itself. A positive integer is called prime if it is not 1 and its factors are only 1 and itself.

**Illustration 2.** Since  $6 = 3 \cdot 2$ ,  $9 = 3 \cdot 3$ ,  $12 = 4 \cdot 3$ , 6, 9, and 12 are composite; while 2, 3, 5, and 7 are prime numbers.

The *Fundamental Theorem of Arithmetic* which states ‘Any positive integer greater than 1 can be written as a product of primes in a unique way except for the order of the factors,’ and which is proved in higher mathematics, establishes the relationship between primes and composites.

**Illustration 3.** a)  $20 = 10(2) = 5(2)(2) = 5 \cdot 2^2$   
b)  $72 = 8(9) = 2(2)(2)(3)(3) = 2^3 \cdot 3^2$   
c)  $120 = 6(20) = 2(3)(10)(2) = 2(3)(5)(2)(2) = 2^3 \cdot 3 \cdot 5$

A positive integer may also be classified as *even* or *odd*. A positive integer is said to be *even* if it is a multiple of 2. That is, if  $x$  is an even integer,



$$x = 2k, \text{ where } k \text{ is an integer.}$$

For example 2, 4, 6, 8 are even integers. 2 is the only even integer that is prime. All other even integers are composite. A positive integer that is not even is said to be *odd*. In general, an odd integer can be written as

$$y = 2k + 1, \text{ where } k \text{ is an integer.}$$

For example 1, 3, 5, 7, 9 are odd integers. Many odd integers are prime like 3, 5, 7, 11.

Two integers are said to be *relatively prime* if they have no common prime factors. For example 15 and 32 are relatively prime, but 15 and 18 are not relatively prime because 3 is a factor of both. We may also say that two integers are relatively prime if their *greatest common factor* (gcf) is 1. To obtain the greatest common factor of two integers express each integer as a product of primes. The gcf is the product of all their common factors.

**Example 1.** Find a) gcf (8 , 24)                      b) gcf (42 , 48)

*Solution:*

$$\begin{aligned} \text{a) } 8 &= 2(2)(2) \\ 24 &= 2(2)(2)(3) \\ \text{gcf}(8, 24) &= 2(2)(2) = 8. \end{aligned}$$

$$\begin{aligned} \text{b) } 42 &= 2(3)(7) \\ 48 &= 2(2)(2)(2)(3) \\ \text{gcf}(42, 48) &= 2(3) = 6. \end{aligned}$$

The *least common multiple* (*lcm*) of two integers is the least integer which is a multiple of both integers. The concept of *lcm* is important in addition of fraction which is discussed in the next section. To find the *lcm* ( $k, n$ ) where  $k$  and  $n$  are integers, express  $k$  and  $n$  as a product of prime factors. The *lcm* is the product of distinct primes with the highest power appearing in the prime factorization of  $k$  and  $n$ .

**Example 2.** Find a) *lcm* (15 , 20)                      b) *lcm* (8 , 12)                      c) *lcm* (45 , 72)

*Solution:*

$$\begin{aligned} \text{a) } 15 &= 5(3) \\ 20 &= 5(2)(2) = 5(2^2) \\ \text{lcm}(15, 20) &= 5(3)(2^2) = 60 \end{aligned}$$

$$\begin{aligned} \text{b) } 8 &= 4(2) = 2^3 \\ 12 &= 4(3) = 2^2(3) \\ \text{lcm}(8, 12) &= 2^3(3) = 24 \end{aligned}$$

$$\begin{aligned} \text{c) } 45 &= 9(5) = 3^2(5) \\ 72 &= 9(8) = 3^2(2^3) \\ \text{lcm}(45, 72) &= 3^2(5)(2^3) = 360 \end{aligned}$$



### Exercises

A. Find the prime factorization of the following.

- |              |         |         |         |
|--------------|---------|---------|---------|
| 1. 121       | 6. 136  | 11. 275 | 16. 250 |
| 2. 360       | 7. 340  | 12. 570 | 17. 625 |
| 3. 1,001     | 8. 230  | 13. 890 | 18. 415 |
| 4. 2,185     | 9. 624  | 14. 160 | 19. 218 |
| 5. 1,000,000 | 10. 410 | 15. 720 | 20. 116 |

B. For each pair of numbers find a) *gcf*

- (15, 45)
- (9, 14)
- (21, 49)
- (48, 56)
- (15, 21)
- (20, 90)
- (15, 25)
- (24, 32)
- (35, 49)
- (18, 64)

b.) *lcm*.

- (8, 14)
- (15, 24)
- (12, 30)
- (11, 20)
- (6, 13)
- (16, 21, 25)
- (13, 18, 21)
- (9, 12, 15)
- (3, 11, 26)
- (9, 14, 20)

### The Arithmetic of Fractions

The primary goal in this section is to derive the rules for adding and multiplying fractions. Let  $m$  and  $n$  designate arbitrary positive integers. Then  $-m$  and  $-n$  are negative integers, and the rules listed below apply.

- $m + (-n) = \begin{cases} +(m-n) & \text{if } m \geq n \\ -(n-m) & \text{if } n > m \end{cases}$
- $m + (-n) = \begin{cases} +(m-n) & \text{if } m \geq n \\ -(n-m) & \text{if } n > m \end{cases}$
- $(-m) + (-n) = -(m+n)$
- $m(-n) = -(mn)$
- $(-m)(-n) = mn$

Now let  $a, b, c, d$  denote integers with  $b$  and  $d$  not equal to 0. Then  $\frac{a}{b}$  and  $\frac{c}{d}$  are rational numbers. A fraction  $\frac{a}{b}$  is said to be in *lowest terms* if  $a$  and  $b$  are relatively prime. To reduce fractions to lowest terms, express both the numerator and denominator as product of its prime factors, then cancel common factors.

**Illustration 1.** a)  $\frac{16}{12} = \frac{2^4}{2^2 \cdot 3} = \frac{2^2 \cdot 2^2}{2^2 \cdot 3} = \frac{4}{3}$



$$b) \frac{72}{108} = \frac{9 \cdot 8}{2 \cdot 54} = \frac{9 \cdot 8}{2 \cdot 9 \cdot 6} = \frac{2^2 \cdot 2^3}{2^2 \cdot 2^2 \cdot 3} = \frac{2}{3}$$

The product of two fractions is obtained using the rule:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

That is, the product of two fractions is the product of their numerators over the product of their denominators and reducing answer to lowest terms.

**Illustration 2.** a)  $\frac{3}{4} \cdot \frac{8}{15} = \frac{2^1 \cdot 3^1}{2^2 \cdot 3^1} = \frac{2}{5}$

b)  $\frac{24}{45} \cdot \frac{12}{40} = \frac{2^3 \cdot 3^1 \cdot 2^2}{3^2 \cdot 5^1} \cdot \frac{2^2 \cdot 3^1}{2^3 \cdot 5^1} = \frac{2^4}{7 \cdot 5} = \frac{4}{25}$

Division of fraction follows the rule:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

That is, to divide  $\frac{a}{b}$  by  $\frac{c}{d}$ , change the division sign to multiplication and invert the divisor.

**Illustration 3.** a)  $\frac{5}{6} \div \frac{25}{8} = \frac{5^1}{2 \cdot 3} \cdot \frac{2^3}{5^2} = \frac{4}{15}$

b)  $\frac{26}{42} \div \frac{13}{14} = \frac{2 \cdot 13}{2 \cdot 3 \cdot 7} \cdot \frac{2 \cdot 7}{13} = \frac{2}{3}$

Addition of fractions with the same denominator follows the rule:

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$$

When the fractions have different denominators, addition is done using the rule:

$$\frac{a}{b} + \frac{c}{d} = \frac{a \left[ \frac{lcm(b,d)}{b} \right] + c \left[ \frac{lcm(b,d)}{d} \right]}{lcm(b,d)}$$

The  $lcm(b, d)$  is also known as the *least common denominator (lcd)*.



**Illustration 4.** a)  $\frac{1}{3} + \frac{4}{3} = \frac{1+4}{3} = \frac{5}{3}$

b)  $\frac{3}{4} + \frac{5}{6} = \frac{3}{2^2} + \frac{5}{2(3)} = \frac{3(3) + 5(2)}{2^2 \cdot 3} = \frac{19}{12}$

c)  $\frac{5}{8} - \frac{3}{20} = \frac{5}{8} + \frac{-3}{20} = \frac{5(5) - 3(2)}{4 \cdot 2 \cdot 5} = \frac{25-6}{40} = \frac{19}{40}$

## Exercises

Perform the indicated operations.

1.  $5 + 4$

2.  $9 + 3$

3.  $7 + 6$

4.  $8 + 7$

5.  $11 + 15$

6.  $(-5) + (-4)$

7.  $(-9) + (-3)$

8.  $(-7) + (-6)$

9.  $(-8) + (-7)$

10.  $(-11) + (-15)$

11.  $9 + (-5)$

12.  $11 + (-8)$

13.  $16 + (-9)$

14.  $12 + (-4)$

15.  $\frac{28}{4}$

16.  $\frac{36}{-9}$

17.  $\frac{5}{6} - \frac{3}{7}$

18.  $\frac{3}{5} + \frac{1}{4}$

19.  $\frac{5}{9} \cdot \frac{2}{3}$

20.  $\frac{2}{7} \cdot \frac{3}{5}$

21.  $-15 - (-5)$

22.  $-8 - (-4)$

23.  $10 - (-3)$

24.  $14 - (-17)$

25.  $-9 + 5 - (-4)$

26.  $3(8)$

27.  $9(-4)$

28.  $-7(6)$

29.  $(-4)(-7)$

30.  $\frac{3}{5} + \frac{7}{15}$

31.  $\frac{5}{8} + \frac{3}{20}$

32.  $\frac{-4+5}{-7+7}$

33.  $\frac{5}{4} + \frac{4}{3} - \frac{3}{2}$

34.  $\frac{3(-2)(1)(-2)}{-4+7}$

35.  $\frac{4(-5)(-2)(-1)}{2(-1)(3)(-2)}$

36.  $\frac{4(-2)(0)(-3)}{-4-2}$

37.  $\frac{-24+(-16)}{-5-3} + \frac{8-(-6)}{-1-6}$

38.  $\frac{2+3}{3(-2)+4} \div \frac{10+3(-4-1)}{2(5-3)}$

39.  $\frac{25-(-3)}{3+(-2)} \cdot \frac{14-2(2+4)}{3-2(5-2)}$

40.  $\frac{21(33-30)}{3(8-15)} \div \frac{-5-(-7)}{-7+5}$



# Algebraic Expressions

## Positive Integer Exponents

Positive integers are used as exponents to indicate repeated multiplication. Suppose we have a product of three factors, say  $(4)(4)(4)$ . Then we can write the product as  $4^3$ , where the “power 3” indicates that 4 is to be multiplied 3 times.

In general, if  $n$  is a positive integer and  $a$  is any real number, then

$$a^n = a \cdot a \cdot a \cdot \dots \cdot a \quad (n \text{ factors of } a)$$

where  $n$  is called the *exponent* and  $a$  is the *base*. The expression  $a^n$  is read as “ $a$  to the  $n$ th power”. If the exponent is 2, then  $a^2$  is read as  $a$  to the second power, or “ $a$  squared”, and if the exponent is 3,  $a^3$  is read as  $a$  to the third power, or “ $a$  cubed”. An exponent of 1 is usually not written, so that  $a^1 = a$ .

### Illustration 1.

- a)  $3^3 = 3 \cdot 3 \cdot 3 = 27$
- b)  $(-5)^2 = (-5)(-5) = 25$
- c)  $-4^3 = -(4 \cdot 4 \cdot 4) = -64$

## Properties of Exponents

If  $a$  and  $b$  are real numbers and  $m$  and  $n$  are positive integers, then

Property 1.  $a^n a^m = a^{n+m}$

Property 2.  $(a^n)^m = a^{nm}$

Property 3.  $(ab)^n = a^n b^n$

Property 4.  $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$

The above properties can be verified easily using the definition of  $a^n$  :

$$a^n \cdot a^m = \underbrace{(a \cdot a \cdot a \cdots a)}_{n \text{ factors}} \underbrace{(a \cdot a \cdot a \cdots a)}_{m \text{ factors}} = \underbrace{a \cdot a \cdot a \cdots a}_{n+m \text{ factors}} = a^{n+m}$$

$$(a^n)^m = \underbrace{a^n \cdot a^n \cdot a^n \cdots a^n}_{m \text{ factors}} = a^{nm}$$

$$(ab)^n = \underbrace{(ab)(ab)(ab) \cdots (ab)}_{n \text{ factors}} = \underbrace{(a \cdot a \cdot a \cdots a)}_{n \text{ factors}} \underbrace{(b \cdot b \cdot b \cdots b)}_{n \text{ factors}} = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdots \frac{a}{b} = \frac{a \cdot a \cdot a \cdots a}{b \cdot b \cdot b \cdots b} = \frac{a^n}{b^n}$$



**Illustration 1.**

- a)  $x^4 \cdot x^5 = x^{4+5} = x^9$
- b)  $(3x^4)(5x^3) = 3 \cdot 5 x^{3+4} = 15 x^7$
- c)  $(3x^2)^3 = 3^3 \cdot x^{2(3)} = 27 x^6$
- d)  $(2^3 \cdot 3^{-2})^{-2} = (2^3)^{-2} (3^{-2})^{-2} = 2^{-6} \cdot 3^4 = \frac{1}{2^6} \cdot 3^4 = \frac{81}{64}$
- e)  $\left(\frac{1}{4}\right)^2 = \frac{1^2}{4^2} = \frac{1}{16}$
- f)  $\left(\frac{3^{-1}}{2^{-3}}\right)^{-2} = \frac{(3^{-1})^{-2}}{(2^{-3})^{-2}} = \frac{3^2}{2^6} = \frac{9}{64}$

**Zero and Negative Integer Exponents**

In this section, we will extend the concept of an exponent to include the use of zero and negative integers.

Let us consider first the use of zero as an exponent in such a way that the definition of a positive integer exponent and the properties of exponents discussed above will still hold. For example, if Property 1 is to hold for a zero exponent, then

$$x^4 \cdot x^0 = x^{4+0} = x^4.$$

In other words,  $x^0$  acts like 1 because  $x^4 \cdot x^0 = x^4$ .

Now suppose that  $n$  is a positive integer, and if Property 1 is to hold for negative integer exponents then,  $a^n \cdot a^{-n} = a^0 = 1$ . Therefore,  $a^{-n}$  must be the reciprocal of  $a^n$ , since their product is 1. That is,  $a^{-n} = \frac{1}{a^n}$ .

**Illustration 2.**

- a)  $3^0 = 1$
- b)  $(2y^3)^0 = 1$
- c)  $x^{-6} = \frac{1}{x^6}$

*Property 5:* If  $n$  and  $m$  are positive integers and  $a$  is a real number where  $a \neq 0$ , then

$$\frac{a^n}{a^m} = \begin{cases} a^{n-m} & \text{if } n > m \\ 1 & \text{if } n < m \\ a^{m-n} & \text{if } n = m \end{cases}$$

Property 5 can be verified in a similar manner as in Properties 1 – 4. For instance, if  $n > m$ , then

$$\frac{a^n}{a^m} = \frac{\overbrace{a \cdot a \cdot a \cdot \dots \cdot a}^{n-m \text{ factors}} \cdot a \cdot a \cdot a \cdot \dots \cdot a}{a \cdot a \cdot a \cdot \dots \cdot a} = a^{n-m}.$$



**Illustration 3.**

a)  $\frac{x^7}{x^3} = x^{7-3} = x^4$     b)  $\frac{24z^2}{6z^5} = \frac{4}{z^{5-2}} = \frac{4}{z^3}$     c)  $\frac{x^3y^5}{x^3y^2} = 1 \cdot y^{5-2} = y^3$

**Example 1.** Simplify the given expressions. Express the results with positive integer exponents only.

a)  $\left(\frac{3x^2y}{4a^{-1}b^{-3}}\right)^{-1}$     b)  $\left(\frac{5x^{-3}y^2}{25x^2y^{-1}}\right)^2$     c)  $\left(\frac{9x^5y^4}{3x^2y^3}\right)^{-2}$

*Solution:*

a)  $\left(\frac{3x^2y}{4a^{-1}b^{-3}}\right)^{-1} = \frac{3^{-1}x^{-2}y^{-1}}{4^{-1}ab^3} = \frac{4}{3ab^3x^2y}$   
b)  $\left(\frac{5x^{-3}y^2}{25x^2y^{-1}}\right)^2 = \frac{5^2x^{-6}y^4}{(5^2)^2x^4y^{-2}} = \frac{5^2y^{4+2}}{5^4x^{4+6}} = \frac{y^6}{5^{4-2}x^{10}} = \frac{y^6}{5^2x^{10}} = \frac{y^6}{25x^{10}}$   
c)  $\left(\frac{9x^5y^4}{3x^2y^3}\right)^{-2} = \left(\frac{9}{3}x^{5-2}y^{4-3}\right)^{-2} = (3x^3y^1)^{-2} = \frac{1}{(3x^3y^1)^2} = \frac{1}{3^2x^6y^2}$

**Exercise 3.1**

A. Evaluate each of the following numerical expressions.

- |   |  |   |
|---|--|---|
| 1. $2^{-3}$                                 | 8. $(3^{-2})^{-1}(2^2)^{-2}$               | 15. $(2^5)(2^{-3})(2^{-4})$                       |
| 2. $\left(\frac{3}{5}\right)^0$             | 9. $\left(\frac{5}{6}\right)^{-2}$         | 16. $\left(\frac{3^2}{4^{-2}}\right)^{-2}$        |
| 3. $3^4$                                    | 10. $(3^{-1})(2^2)^{-1}$                   | 17. $\frac{2^{-2}+3^{-1}}{4^{-1}}$                |
| 4. $(-4)^3$                                 | 11. $\frac{1}{3^{-3}}$                     | 18. $\frac{2^0+3^{-1}}{4^{-2}}$                   |
| 5. $\left(\frac{1}{2}\right)^4$             | 12. $\left(\frac{3^{-2}}{8^{-1}}\right)^2$ | 19. $\left(\frac{3^2+2^{-1}}{3^{-2}}\right)^{-2}$ |
| 6. $\left(\frac{2^{-1}}{3^{-3}}\right)^2$   | 13. $(-2)^4$                               | 20. $\frac{2}{3^{-1}}$                            |
| 7. $\left(\frac{2+3^0}{2^{-1}}\right)^{-3}$ | 14. $(3^{-2})^2(2^{-1})^3$                 |   |



B. Use the properties of exponents to simplify the expression and write it with positive exponents only.

1.  $x^4 \cdot x^{-6}$

2.  $(a^{-3})^7$

3.  $\frac{x^{-5}}{x^{-7}}$

4.  $(2x^{-1}y^2)(3x^{-2}y^{-3})$

5.  $(8a^{-4}b^{-5})(-6a^{-1}b^8)$

6.  $(2x^{-3})(3x^4)$

7.  $\left(\frac{8x^4yz^2}{2x^3y^2z^{-1}}\right)^{-1}$

8.  $\frac{a^2b^{-3}}{a^{-1}b^{-2}}$

9.  $\frac{7x^5y}{8x^2y^4z^0}$

10.  $\frac{8^{-1}s^{-3}t^0}{(2s^{-1}t)^{-5}}$

11.  $\left(\frac{14x^{-2}y^{-4}}{7x^{-3}y^{-6}}\right)^{-2}$

12.  $\left(\frac{3^{-2}m^{-2}n^{-4}}{3^{-4}m^{-3}n^{-6}}\right)^{-2}$

13.  $\frac{x^{12}}{x^{-5}}$

14.  $(3x^{-2})^4$

15.  $\frac{(8^{-1}s^{-3}t^0)^{-1}}{(4s^{-1}t^2)^{-5}}$

16.  $\frac{(a^{-3}b^5)^{-2}}{(4^{-2}a^{-1}b^2)^{-1}}$

17.  $\frac{(-3x^3yz^2)(-2x^4y^2z^5)}{-3x^4yz^4}$

18.  $\frac{(3^0w^3x^{-1}z^2)^{-2}(2w^4x^2z^5)^2}{4w^4x^{-2}z^4}$

19.  $\frac{(x^{-2}y^{-3}z^2)^2(-2x^4y^2z^5)^{-1}}{(x^4y^{-2}z^4)^{-2}}$

20.  $(2x^3yz^2)(xy^3z^0)$

## Polynomials

A *constant* is any symbol whether a number or a letter which is used to represent a fixed value. In contrast, a *variable* is any symbol used to represent a number whose value is not fixed. For example, 3,  $-1/3$ ,  $\pi$  are constants while  $l$  and  $w$  are variables which represent the length and width of a rectangle in the formula  $A = l \cdot w$  of the area of a rectangle.

An expression involving constants and/or variables with all or some of the four algebraic operations of addition, subtraction, multiplication and division, raising to a power, is called an *algebraic expression*. For example,

$$x, 6, 2xy + \frac{3}{x},$$

are algebraic expressions. An algebraic expression involving only non-negative integer powers of one or more variables is called a *polynomial*. For example,

$$x, x^2 + 2y, \frac{x}{4} - 3xy + 5$$

are polynomials. A *term* of a polynomial is a product of a constant and/or variables the sum of which form the polynomial. In a term, we call the constant as the *numerical coefficient* or simply *coefficient* and the variable(s) as the *literal factors* or *literal parts*. For example,  $\frac{x}{4}$ ,  $3xy$ ,  $5$



are the terms of the polynomial  $\frac{x}{4} - 3xy + 5$ . The numerical coefficient of  $x$  in the first term is  $\frac{1}{4}$ . In the second term,  $x$  and  $y$  are literal factors of  $-3$ . A polynomial with only one term is called a *monomial*; *binomial* if it has two terms; and *trinomial* if it has 3 terms.

- Illustration 1.**
- a)  $6xz^4$  is a monomial.
  - b)  $4ab^3 - 3c$  is a binomial.
  - c)  $4ab - 3c + 1$  is a trinomial.

The *degree* of a monomial in one variable is the exponent of that variable. If a monomial has more than one variable, its degree is the sum of the exponents of all the variables that appear. The *degree* of a polynomial is the same as the degree of the term with the highest degree.

**Illustration 2.**

- a) The degree of  $7x^2$  is 2.
- b) The monomial  $3a^2 b^3 c$  is of degree 6 since the sum of the exponents of its variables is  $2 + 3 + 1 = 6$ .
- c) The binomial  $-2u^3 v^2 w^3 + 3u^4 v w^4$  is of degree 9; the degree of the first term is 8 and the degree of the second term is 9; 9 is greater than 8.
- d) The monomial  $2^4 x^2 y^3$  is of degree 5. (Only the exponents of the variables determine the degree of the term)
- e) The trinomial  $3^4 x^3 y + 5x^2 y^3 + 8$  is of degree 5 since  $5x^2 y^3$  is the term with the highest degree and its degree is 5.

## Operations on Polynomials

### Addition and Subtraction of Polynomials

*Similar terms or like terms* are those with the same literal factors or some literal parts. For example,

$$x^2y, -3x^2y, \frac{1}{2}x^2y$$

are similar terms. All of them have  $x^2$  and  $y$  as literal parts. However,

$$x^2y, -3xy, \frac{1}{2}xy^2$$

are not similar since they do not have common literal parts. All of them contain  $x$  and  $y$ , but they occur with different powers. To be similar, it is necessary that the terms must have the same powers of the literal factors present.

In adding polynomials, we simply add the numerical coefficients of like terms. To subtract a polynomial from another, we use the rule

$$A - B = A + (-B)$$

where  $A$  and  $B$  represent polynomials.

**Example 1.** Find the sum.

- a)  $(5x^3 + 7y - 3) + (4x^3 - 8y + 4)$
- b)  $(-3x^2 + 24x - 5) + (-3 + 6x - 3x^2)$

*Solution:*



a) Adding the coefficients of like terms we have,

$$(5 + 4)x^3 + (7 - 8)y + (-3 + 4) = 9x^3 - y + 1$$

We may also put the expressions in a vertical column of like terms:

$$\begin{array}{r} 5x^3 + 7y - 3 \\ + 4x^3 - 8y + 4 \\ \hline 9x^3 - y + 1 \end{array}$$

$$\begin{aligned} \text{b) } (-3x^2 + 24x - 5) + (-3 + 6x - 3x^2) &= (-3 - 3)x^2 + (24 + 6)x + (-5 - 3) \\ &= -6x^2 + 30x - 8 \end{aligned}$$

**Example 2.** Find the difference.

$$\text{(a) } (3x^2 + 5x - 4) - (4x^3 - 4x^2 + x - 2)$$

$$\text{(b) } (-3x^4 - 2x^3 + 5) - (4x^4 + 5x^3 - 4x + 7)$$

$$\begin{aligned} \text{a) } (3x^2 + 5x - 4) - (4x^3 - 4x^2 + x - 2) &= (3x^2 + 5x - 4) + (-4x^3 + 4x^2 - x + 2) \\ &= -4x^3 + (3 + 4)x^2 + (5 - 1)x + (-4 + 2) \\ &= -4x^3 + 7x^2 + 4x - 2 \end{aligned}$$

$$\begin{aligned} \text{b) } (-3x^4 - 2x^3 + 5) - (4x^4 + 5x^3 - 4x + 7) &= (-3x^4 + 2x^3 + 5) + (-4x^4 - 5x^3 + 4x - 7) \\ &= (-3 - 4)x^4 + (2 - 5)x^3 + 4x + (5 - 7) \\ &= -7x^4 - 3x^3 + 4x - 2 \end{aligned}$$

**Remark:** It is also convenient to use vertical arrangement for subtraction problems.

**Example 3.** Use vertical subtraction to subtract  $(4x^3 + 5x^2 - 2x + 1)$  from  $(3x^2 + 5x - 6)$ .

*Solution:* First, write the polynomials with like terms in the same vertical line.

$$\begin{array}{r} 4x^3 + 5x^2 - 2x + 1 \\ - \underline{\quad 3x^2 + 5x - 6} \end{array}$$

Next, change the subtraction sign to an addition sign and change the sign of all the terms in the polynomial to be subtracted. Then add the resulting terms in each vertical line.

$$\begin{array}{r} 4x^3 + 5x^2 - 2x + 1 \\ + \underline{\quad -3x^2 - 5x + 6} \\ 4x^3 + 2x^2 - 7x + 7. \end{array}$$