Chapter 3
Algebraic Expressions

3.1 Positive Integer Exponents

Positive integers are used as exponents to indicate repeated multiplication. Suppose we have a product of three factors, say (4)(4)(4). Then we can write the product as 4^3, where the "power 3" indicates that 4 is to be multiplied 3 times.

In general, if n is a positive integer and a is any real number, then

$$a^n = a \cdot a \cdot a \cdot \ldots \cdot a \quad (n \text{ factors of } a)$$

where n is called the *exponent* and a is the *base*. The expression a^n is read as " *a to the nth power*". If the exponent is 2, then a^2 is read as a to the second power, or "*a squared*", and if the exponent is 3, a^3 is read as a to the third power, or "*a cubed*". An exponent of 1 is usually not written, so that $a^1 = a$.

Illustration 1.

a) $3^3 = 3 \cdot 3 \cdot 3 = 27$
b) $(-5)^2 = (-5)(-5) = 25$
c) $-4^3 = -(4 \cdot 4 \cdot 4) = -64$

Properties of Exponents

If a and b are real numbers and m and n are positive integers, then

Property 1. $a^n a^m = a^{n+m}$

Property 2. $(a^n)^m = a^{nm}$

Property 3. $(ab)^n = a^n b^n$

Property 4. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$, $b \neq 0$

The above properties can be verified easily using the definition of a^n:

$$a^n \cdot a^m = \underbrace{(a \cdot a \cdot a \cdots a)}_{n\ factors}\ \underbrace{(a \cdot a \cdot a \cdots a)}_{m\ factors} = \underbrace{a \cdot a \cdot a \cdot a \cdots a}_{n\,+\,m\ factors} = a^{n+m}$$

$$(a^n)^m = \underbrace{a^n \cdot a^n \cdot a^n \cdot a^n \cdots a^n}_{m\ factors} = a^{nm}$$

$$(ab)^n = \underbrace{(ab)(ab)(ab) \cdots (ab)}_{n\ factors} = \underbrace{(a \cdot a \cdot a \cdot a \cdots a)}_{n\ factors}\underbrace{(b \cdot b \cdot b \cdots b)}_{n\ factors} = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdots \frac{a}{b} = \frac{a \cdot a \cdot a \cdots a}{b \cdot b \cdot b \cdots b} = \frac{a^n}{b^n}.$$

Illustration 1.

a) $x^4 \cdot x^5 = x^{4+5} = x^9$

b) $(3x^4)(5x^3) = 3 \cdot 5\, x^{3+4} = 15\, x^7$

c) $(3x^2)^3 = 3^3 \cdot x^{2(3)} = 27\, x^6$

d) $(2^3 \cdot 3^{-2})^{-2} = (2^3)^{-2}(3^{-2})^{-2} = 2^{-6} \cdot 3^4 = \frac{1}{2^6} \cdot 3^4 = \frac{81}{64}$

e) $\left(\frac{1}{4}\right)^2 = \frac{1^2}{4^2} = \frac{1}{16}$

f) $\left(\frac{3^{-1}}{2^{-3}}\right)^{-2} = \frac{(3^{-1})^{-2}}{(2^{-3})^{-2}} = \frac{3^2}{2^6} = \frac{9}{64}$

Zero and Negative Integer Exponents

In this section, we will extend the concept of an exponent to include the use of zero and negative integers.

Let us consider first the use of zero as an exponent in such a way that the definition of a positive integer exponent and the properties of exponents discussed above will still hold. For example, if Property 1 is to hold for a zero exponent, then

$$x^4 \cdot x^0 = x^{4+0} = x^4.$$

In other words, x^0 acts like 1 because $x^4 \cdot x^0 = x^4$.

Now suppose that n is a positive integer, and if Property 1 is to hold for negative integer exponents then, $a^n \cdot a^{-n} = a^0 = 1$. Therefore, a^{-n} must be the reciprocal of a^n, since their product is 1. That is, $a^{-n} = \frac{1}{a^n}$.

Illustration 2.

a) $3^0 = 1$ b) $(2y^3)^0 = 1$ c) $x^{-6} = \frac{1}{x^6}$

Property 5: If n and m are positive integers and a is a real number where $a \neq 0$, then

$$\frac{a^n}{a^m} = \begin{cases} a^{n-m} & \text{if } n > m \\ \dfrac{1}{a^{m-n}} & \text{if } n < m \\ 1 & \text{if } n = m \end{cases}$$

Property 5 can be verified in a similar manner as in Properties 1 – 4. For instance, if $n > m$, then

$$\frac{a^n}{a^m} = \frac{\not{a} \cdot \not{a} \cdot \not{a} \cdot \cdot \cdot \not{a} \cdot \overbrace{a \cdot a \cdot a \cdot \cdot \cdot a}^{n-m \text{ factors}}}{\not{a} \cdot \not{a} \cdot \not{a} \cdot \cdot \cdot \not{a}} = a^{n-m} .$$

Illustration 3.

a) $\dfrac{x^7}{x^3} = x^{7-3} = x^4$ b) $\dfrac{24z^2}{6z^5} = \dfrac{4}{z^{5-2}} = \dfrac{4}{z^3}$ c) $\dfrac{x^3 y^5}{x^3 y^2} = 1 \cdot y^{5-2} = y^3$

Example 1. Simplify the given expressions. Express the results with positive integer exponents only.

a) $\left(\dfrac{3x^2 y}{4a^{-1}b^{-3}}\right)^{-1}$ b) $\left(\dfrac{5x^{-3}y^2}{25x^2 y^{-1}}\right)^2$ c) $\left(\dfrac{9x^5 y^4}{3x^2 y^3}\right)^{-2}$

Solution:

a) $\left(\dfrac{3x^2 y}{4a^{-1}b^{-3}}\right)^{-1} = \dfrac{3^{-1}x^{-2}y^{-1}}{4^{-1}ab^3} = \dfrac{4}{3ab^3x^2y}$

b) $\left(\dfrac{5x^{-3}y^2}{25x^2 y^{-1}}\right)^2 = \dfrac{5^2 x^{-6} y^4}{(5^2)^2 x^4 y^{-2}} = \dfrac{5^2 y^{4+2}}{5^4 x^{4+6}} = \dfrac{y^6}{5^{4-2} x^{10}} = \dfrac{y^6}{5^2 x^{10}} = \dfrac{y^6}{25x^{10}}$

c) $\left(\dfrac{9x^5 y^4}{3x^2 y^3}\right)^{-2} = \left(\dfrac{9}{3} x^{5-2} \; y^{4-3}\right)^{-2} = \left(3 \; x^3 \; y^1\right)^{-2} = \dfrac{1}{\left(3x^3 \; y^1\right)^2} = \dfrac{1}{3^2 x^6 y^2}$

Exercise 3.1

A. Evaluate each of the following numerical expressions.

1. 2^{-3}
2. $\left(\frac{3}{5}\right)^0$
3. 3^4
4. $(-4)^3$
5. $\left(\frac{1}{2}\right)^4$
6. $\left(\frac{2^{-1}}{3^{-3}}\right)^2$
7. $\left(\frac{2+3^0}{2^{-1}}\right)^{-3}$
8. $\left(3^{-2}\right)^{-1}\left(2^2\right)^{-2}$
9. $\left(\frac{5}{6}\right)^{-2}$
10. $(3^{-1})(2^2)^{-1}$
11. $\frac{1}{3^{-3}}$
12. $\left(\frac{3^{-2}}{8^{-1}}\right)^2$
13. $(-2)^4$
14. $(3^{-2})^2(2^{-1})^3$
15. $(2^5)(2^{-3})(2^{-4})$
16. $\left(\frac{3^2}{4^{-2}}\right)^{-2}$
17. $\frac{2^{-2}+3^{-1}}{4^{-1}}$
18. $\frac{2^0+3^{-1}}{4^{-2}}$
19. $\left(\frac{3^2+2^{-1}}{3^{-2}}\right)^{-2}$
20. $\frac{2}{3^{-1}}$

B. Use the properties of exponents to simplify the expression and write it with positive exponents only.

1. $x^4 \cdot x^{-6}$
2. $(a^{-3})^7$
3. $\frac{x^{-5}}{x^{-7}}$
4. $(2x^{-1}y^2)(3x^{-2}y^{-3})$
5. $(8a^{-4}b^{-5})(-6a^{-1}b^8)$
6. $(2x^{-3})(3x^4)$
7.
8. $\frac{a^2b^{-3}}{a^{-1}b^{-2}}$
9. $\frac{7x^5y}{8x^2y^4z^0}$
10. $\frac{8^{-1}s^{-3}t^0}{(2s^{-1}t)^{-5}}$
11. $\left(\frac{14x^{-2}y^{-4}}{7x^{-3}y^{-6}}\right)^{-2}$
12. $\left(\frac{3^{-2}m^{-2}n^{-4}}{3^{-4}m^{-3}n^{-6}}\right)^{-2}$
13. $\frac{x^{12}}{x^{-5}}$
14.
15. $\frac{(8^{-1}s^{-3}t^0)^{-1}}{(4s^{-1}t^2)^{-5}}$
16. $\frac{(a^{-3}b^5)^{-2}}{(4^{-2}a^{-1}b^2)^{-1}}$
17. $\frac{(-3x^3yz^2)(-2x^4y^2z^5)}{-3x^4yz^4}$
18. $\frac{(3^0w^3x^{-1}z^2)^{-2}(2w^4x^2z^5)^2}{4w^4x^{-2}z^4}$
19. $\frac{(x^{-2}y^{-3}z^2)^2(-2x^4y^2z^5)^{-1}}{(x^4y^{-2}z^4)^{-2}}$
20. $\left(2x^3yz^2\right)\left(xy^3z^0\right)$

7. $\left(\dfrac{8x^4yz^2}{2x^3y^2z^{-1}}\right)^{-1}$ 14. $\left(3x^{-2}\right)^4$

3.2 Polynomials

A *constant* is any symbol whether a number or a letter which is used to represent a fixed value. In contrast, a *variable* is any symbol used to represent a number whose value is not fixed. For example, 3, $-1/3$, π are constants while l and w are variables which represent the length and width of a rectangle in the formula $A = l \cdot w$ of the area of a rectangle.

An expression involving constants and/or variables with all or some of the four algebraic operations of addition, subtraction, multiplication and division, raising to a power and extraction of roots, is called an *algebraic expression*. For example,

$$x,\ 6,\ 2xy + \frac{3}{x},\ -2x^3 + 2\sqrt{x} - 1$$

are algebraic expressions. An algebraic expression involving only non-negative integer powers of one or more variables is called a *polynomial*. For example,

$$x\ ,\ x^2 + 2y\ ,\ \frac{x}{4} - 3xy + 5$$

are polynomials. A *term* of a polynomial is a product of a constant and/or variables the sum of which form the polynomial. In a term, we call the constant as the *numerical coefficient* or simply *coefficient* and the variable(s) as the *literal factors* or *literal parts*. For example, $\frac{x}{4}$, $3xy$, 5 are the terms of the polynomial $\frac{x}{4} - 3xy + 5$. The numerical coefficient of x in the first term is ¼. In the second term, x and y are literal factors of – 3. A polynomial with only one term is called a *monomial*; *binomial* if it has two terms; and *trinomial* if it has 3 terms.

Illustration 1. a) $6xz^4$ is a monomial.
b) $4ab^3 - 3c$ is a binomial.
c) $4ab - 3c + 1$ is a trinomial.

The *degree* of a monomial in one variable is the exponent of that variable. If a monomial has more than one variable, its degree is the sum of the exponents of all the variables that appear. The *degree* of a polynomial is the same as the degree of the term with the highest degree.

Illustration 2.

a) The degree of $7x^2$ is 2.
b) The monomial $3a^2b^3c$ is of degree 6 since the sum of the exponents of its variables is 2 + 3 + 1= 6.
c) The binomial $-2u^3v^2w^3 + 3u^4vw^4$ is of degree 9; the degree of the first term is 8 and the degree of the second term is 9; 9 is greater than 8.
d) The monomial $2^4x^2y^3$ is of degree 5. (Only the exponents of the variables determine the degree of the term)

e) The trinomial $3^4x^3y + 5x^2y^3 + 8$ is of degree 5 since $5x^2y^3$ is the term with the highest degree and its degree is 5.

3.3 Operations on Polynomials

Addition and Subtraction of Polynomials

Similar terms or *like terms* are those with the same literal factors or some literal parts. For example,

$$x^2y,\ -3x^2y,\ \frac{1}{2}x^2y$$

are similar terms. All of them have x^2 and y as literal parts. However,

$$x^2y,\ -3xy,\ \frac{1}{2}xy^2$$

are not similar since they do not have common literal parts. All of them contain x and y, but they occur with different powers. To be similar, it is necessary that the terms must have the same powers of the literal factors present.

In adding polynomials, we simply add the numerical coefficients of like terms. To subtract a polynomial from another, we use the rule

$$A - B = A + (-B)$$

where A and B represent polynomials.

Example 1. Find the sum.

a) $(5x^3 + 7y - 3) + (4x^3 - 8y + 4)$
b) $(-3x^2 + 24x - 5) + (-3 + 6x - 3x^2)$

Solution:

a) Adding the coefficients of like terms we have,

$$(5 + 4)x^3 + (7 - 8)y + (-3 + 4) = 9x^3 - y + 1$$

We may also put the expressions in a vertical column of like terms:

$$\begin{array}{rl} & 5x^3 + 7y - 3 \\ + & \underline{4x^3 - 8y + 4} \\ & 9x^3 - y \ \ + 1 \end{array}$$

b) $(-3x^2 + 24x - 5) + (-3 + 6x - 3x^2) = (-3 - 3)x^2 + (24 + 6)x + (-5 - 3)$
$= -6x^2 + 30x - 8$

Example 2. Find the difference.

(a) $(3x^2 + 5x - 4) - (4x^3 - 4x^2 + x - 2)$
(b) $(-3x^4 - 2x^3 + 5) - (4x^4 + 5x^3 - 4x + 7)$

a) $(3x^2 + 5x - 4) - (4x^3 - 4x^2 + x - 2) = (3x^2 + 5x - 4) + (-4x^3 + 4x^2 - x + 2)$
$= -4x^3 + (3 + 4)x^2 + (5 - 1)x + (-4 + 2)$
$= -4x^3 + 7x^2 + 4x - 2$

b) $(-3x^4 - 2x^3 + 5) - (4x^4 + 5x^3 - 4x + 7) = (-3x^4 + 2x^3 + 5) + (-4x^4 - 5x^3 + 4x - 7)$
$= (-3 - 4)x^4 + (2 - 5)x^3 + 4x + (5 - 7)$
$= -7x^4 - 3x^3 + 4x - 2$

Remark: It is also convenient to use vertical arrangement for subtraction problems.

Example 3. Use vertical subtraction to subtract $(4x^3 + 5x^2 - 2x + 1)$ from $(3x^2 + 5x - 6)$.

Solution: First, write the polynomials with like terms in the same vertical line.

$$\begin{array}{r} 4x^3 + 5x^2 - 2x + 1 \\ -\underline{\quad\quad 3x^2 + 5x - 6} \end{array}$$

Next, change the subtraction sign to an addition sign and change the sign of all the terms in the polynomial to be subtracted. Then add the resulting terms in each vertical line.

$$\begin{array}{r} 4x^3 + 5x^2 - 2x + 1 \\ +\underline{\quad\quad -3x^2 - 5x + 6} \\ 4x^3 + 2x^2 - 7x + 7. \end{array}$$

Multiplication of polynomials

To determine the product of polynomials, we apply the distributive property and the law of exponents $a^n a^m = a^{n+m}$.

Example 1. Find the product: $3x(4x^2 - x + 1)$.

Solution: $3x(4x^2 - x + 1) = 3x(4x^2) + 3x(-x) + 3x(1)$
$= 12x^3 - 3x^2 + 3x.$

Example 2. Find the product: $(2x + y)(x - 3y + 1)$.

Solution:
$$(x - 3y + 1)(2x + y) = (x)(2x) + xy + (-3y)(2x) + (-3y)(y) + 1(2x) + 1(y)$$
$$= 2x^2 + xy - 6xy - 3y^2 + 2x + y$$
$$= 2x^2 - 5xy - 3y^2 + 2x + y$$

In this example, the product can also be obtained by putting the expression in a vertical column, that is,

$$\begin{array}{lrl} & x - 3y + 1 & \text{(multiplicand)} \\ & 2x + y & \text{(multiplier)} \\ \hline & 2x^2 - 6xy + 2x & \\ + & xy \quad - 3y^2 + y & \\ \hline & 2x^2 - 5xy + 2x - 3y^2 + y & \text{(product)} \end{array}$$

Example 3. Find the product: $-4\,a^n\, b^m\,(3\,a^{2n}\, b^m + 6\, a^{3n}\, b^{3m})$.

Solution:

$$-4\,a^n\, b^m\,(3\,a^{2n}\, b^m + 6\, a^{3n}\, b^{3m}) = -4\,a^n\, b^m\,(3\, a^{2n}\, b^m) + (-4\,a^n\, b^m)(6\,a^{3n}\, b^{3m})$$

$$= -12a^{3n}b^{2m} - 24a^{4n}b^{4m}$$

Division of Polynomials

Let P and Q be polynomials. We want to find $\frac{\text{P}}{\text{Q}}$, $\text{Q} \neq 0$.

Case 1. If the divisor is a monomial, divide each term of the dividend P by the monomial Q. Here, we apply the law of exponents

$$\frac{a^n}{a^m} = a^{n-m},\ a \neq 0.$$

Example 1. Divide $4x^3y^2 - 8x^4y^5 + 12x^2y$ by $2x^2y$.

Solution:

$$(4x^3y^2 - 8x^4y^5 + 12x^2y) \div 2x^2y = \frac{4x^3y^2}{2x^2y} + \frac{-8x^4y^5}{2x^2y} + \frac{12x^2y}{2x^2y}$$

$$= 2xy - 4x^2y^4 + 6$$

Case 2. If the divisor is not a monomial, we follow the procedure below:

Step 1. Arrange the terms of the dividend and the divisor in descending powers of a variable. Insert missing terms (if any) with numerical coefficient 0.
Step 2. Divide the first term of the dividend by the first term of the divisor. This step gives the first term of the quotient.
Step 3. Multiply the divisor by the first term of the quotient and subtract the product from the dividend.
Step 4. Treating the difference obtained in Step 3 as the new dividend, repeat Step 2 to obtain the second term of the quotient.
Step 5. Continue the process until a remainder is obtained which is either 0 or a polynomial of lower degree than the divisor in the variable chosen in Step 1. The last difference is the remainder.

Example 2. Divide $x^3 + 5x^2 + 5x - 2$ by $2 + x$.

Solution:

$x^2 + 3x - 1$ → quotient

divisor ← $x+2 \overline{) x^3 + 5x^2 + 5x - 2}$ → dividend

$-\underline{(x^3 + 2x^2)}$

$3x^2 + 5x - 2$ → new dividend

$-\underline{(3x^2 + 6x)}$

$-x - 2$ → new dividend

$-\underline{(-x - 2)}$

0 → remainder

Example 3. Divide $5y + 6y^3 - 1$ by $2y - 4$.

Solution:

$3y^2 + 6y + \frac{29}{2}$ → quotient

divisor ← $2y - 4 \overline{) 6y^3 + 0y^2 + 5y - 1}$ → dividend

$-\underline{(6y^3 - 12y^2)}$

$12y^2 + 5y - 1$ → new dividend

$-\underline{(12y^2 - 24y)}$

$29y - 1$ → new dividend

$-\underline{(29y - 58)}$

57 → remainder

We may check the result of division by multiplying the quotient and the divisor and adding the remainder to get the original dividend.

Exercise 3.3

A. Perform the indicated operation.

1. $(4a - 5b + 6) + (-2a + 7b - 8)$
2. $(3x^2 + 7xy) + (x^2 - 5xy) + (-3x^2 + xy)$
3. $(11c^3 + b^2 - d) + (-8c^3 - b^2 + 4d) + (c^3 + 3b^2 - 2d)$
4. $(5y - 2) - (2y + 1)$

5. $(3x^{2} + 2xy + y^{2}) + (4x^{2} - 3xy + 4y^{2})$
6. $(6x^{4} + 2xy^{3} + y^{3}) - (5xy^{3} - 3y^{3})$
7. $(4x^2 y^2 + xy - 8) - (-5x^2 y^2 + 3xy + 1)$
8. $(4x + 2y) + (x^{2} - 2x + y)$
9. $(7ab - 3c + 1) - (3ab + 2c - 12) - (-4ab - 6c + 8)$
10. $(5a^{3} b^{2} + 4a^{2} b - 2) - (2a^{3} b^{2} + 2a^{2} + 5)$

11. $(3x - 2y) + (6x - 3y)$
12. $(4x^{2} + 2y) - (6x^{2} - 8y)$
13. $(3a^{3}b - 2ab^{2}) + (4a^3 b + 2)$
14. $(2xy^{3} - 3xy) - (4x^{2} + 6xy - 2)$
15. $(5x^{2} - 5) + (2x^{2} + x)$
16. $(x^{5} + 3x^{4} - y^{2}) + (3x^{5} - 2x^{4} + 1)$
17. $(y^{3} + 2z^{4}) - (2y^{3} + 3z)$
18. $(6a^{4} + 4a^{2} - 2) - (2a^{4} - 3a^{3} - 4a^{2} - 2)$
19. $(2x + 1) + (5x + 2)$
20. $(2xy^{3} - 3xy) + (4xy^{3} + 6xy - 2)$

B. Find the following product:

1. $-4xy(2x^3 + xy^2 - 3)$
2. $(x^2 + 3y^2)(2x^2 - 7y^2)$
3. $(2x - 3y)^2$
4. $5xy^{2}(3x + 2y + 1)$
5. $(4x^{2} - 2x + 1)(3x^{2} - 4x + 2)$
6. $-2xy^{3}(6x^{2}y - 2xy^{2} + 3)$
7. $(2x + 3)(5x - 7)$
8. $(2x^{2} - y^{2})(4x^{2} + y^{2})$
9. $(3x + 2y + 1)(4x + 3y + 2)$
10. $-6x^{2}y^{2}(4xy^{2} - 2xy)$
11. $(5x + 2)(6x + 1)$
12. $(a + 2b)(3a^3 - b^3 + 1)$
13. $(4x^{m} + 3y^{m})(x^{m} - 2y^{m})$
14. $(6x^{2} + 2y)(3x + 2y)$
15. $(2xy + 5)(x^{2} - 3)$
16. $c^{2n}(-3c^{4n} + c^{n} - 5)(5b + 1)(5b - 1)$
17. $(4a + 2b - c)(a - 3b + 2c)$
18. $(2xy + 5)(x^{2} - 3)$
19. $4x^{2}(2x^{3} - 5x^{2} + 2x - 2)$
20. $(3xy^{2} + 2)(5xy - 2x)$

C. Find the quotient of the following:

1. $\dfrac{15x^4 - 20x^3 + 10x^2}{5x}$
2. $\dfrac{24x^6 y^4 - 6x^3 y^2}{6x^3 y^2}$
3. $\dfrac{12x^3 + 6x^2 y + 6xy + 3y^2}{2x + y}$
4. $\dfrac{10x^5 - 5x^3 + 9x^4 - 3x^2 + 4x + 3}{2x^2 + x - 1}$
5. $\dfrac{10x^4 - 15x^3 - 4x^2 - 4x + 3}{2x + 3}$
11. $\dfrac{9a^{4m} - 12a^{6m} + 2a^{3m}}{-3a^{3m}}$
12. $\dfrac{9x^4 y^2 z + 27x^5 y^3 z^2}{9xy^2 z}$
13. $\dfrac{12x^3 y^4 - 4x^2 y^4 - 8xy^3}{4xy}$
14. $\dfrac{6x^4 + 2x^3 + 15x^2 + 11x + 2}{3x^2 y}$
15. $\dfrac{4x^2 + 2xy - 2y^2}{x + y}$

6. $\dfrac{6x^4 + 2x^3 + 15x^2 + 11x + 2}{3x + 1}$

7. $(2x^3 + 3x^2 - 4x - 6) \div (2x + 3)$

8. $(125x^4 - y^4) \div (5x + y)$

9. $(10x^2 + 3xy - y^2) \div (2x + y)$

10. $(15x^2 + 23x + 4) \div (5x + 1)$

16. $\dfrac{10x^5yz^4 - 14x^4y^2z^2 - 2x^2y^3z^4}{2xyz^2}$

17. $(10x^2 + 3xy - y^2) \div (2x + y)$

18. $(3y^2 + 14y - 7) \div (3y - 1)$

19$(6a^3 + 7a^2 + 2a) \div (2a + 1)$

20. $(y^2 + 3y - 10) \div (y + 5)$

3.4 Special Products

There are products of polynomials which have special forms and knowledge of their forms will shorten some algebraic computations. These special products will help a great deal in simplification of expressions where factoring is involved. The most common special types are given below. In the discussion that follows, A and B represent polynomials.

Special Product 1. *The Perfect Square*

(i) $(A + B)^2 = A^2 + 2AB + B^2$

(ii) $(A - B)^2 = A^2 - 2AB + B^2$

Illustration 1. a) $(2x + y)^2 = (2x)^2 + 2(2x)(y) + (y)^2$
$= 4x^2 + 4xy + y^2$

b) $(2y^2 + 3z)^2 = (2y^2)^2 + 2(2y^2)(3z) + (3z)^2$
$= 4y^4 + 12y^2z + 9z^2$

c) $(2x - 5y)^2 = (2x)^2 - 2(2x)(5y) + (5y) = 4x^2 - 20xy + 25y^2$

d) $(2y^2 - 3z)^2 = (2y^2)^2 - 2(2y^2)(3z) + (3z)^2$
$= 4y^4 - 12y^2z + 9z^2$

Special Product 2. *Difference of Two Squares*

$(A + B)(A - B) = A^2 - B^2$

Illustration 2. a) $(2x + 3y)(2x - 3y) = (2x)^2 - (3y)^2 = 4x^2 - 9y^2$

b) $(5a - 4b)(5a + 4b) = (5a)^2 - (4b)^2 = 25a^2 - 16b^2$

c) $(z^2 - 7)(z^2 + 7) = (z^2)^2 - (7)^2 = z^4 - 49$

Special Product 3. *Product of Two Binomials with Similar Terms*

(i) $(x + a)(x + b) = x^2 + (a + b)x + ab$

(ii) $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$

Illustration 3. a) $(x + 3)(x + 4) = x^2 + (3 + 4)x + 3(4) = x^2 + 7x + 12$

b) $(y - 5)(y + 6) = y^2 + (-5 + 6)y + (-5)(6) = y^2 + y - 30$

c) $(z - 7)(z - 5) = z^2 + (-7 - 5)z + (-7)(-5) = z^2 - 12z + 35$

d) $(2x + 3)(3x - 2) = (2)(3)x^2 + [(2)(-2) + (3)(3)]x + (3)(-2)$

$$= 6x^2 + (-4 + 9) - 6$$

$$= 6x^2 + 5x - 6$$

e) $(3y - 2)(4y - 1) = (3)(4)y^2 + [(3)(-1) + (-2)(4)]y + (-2)(-1)$

$$= 12y^2 + (-3 - 8)y + 2$$

$$= 12y^2 - 11y + 2$$

Special Product 6. *The Sum of two Cubes*

$$(A + B)(A^2 - AB + B^2) = A^3 + B^3$$

Illustration 4. a) $(3x + 2y)(9x^2 - 6xy + 4y^2) = (3x + 2y)[(3y)^2 - (3x)(2y) + (2y)^2]$

$$= (3x)^3 + (2y)^3$$

$$= 27x^3 + 8y^3$$

b) $(2x + 1)(4x^2 - 2x + 1) = (2x + 1)[(2x)^2 - (2x)(1) + (1)^2]$

$$= (2x)^3 + (1)^3$$

$$= 8x^3 + 1.$$

Special Product 7. *The Difference of Two Cubes*

$$(A - B)(A^2 + AB + B^2) = A^3 - B^3$$

Illustration 5. a) $(3x - 2y)(9x^2 + 6xy + 4y^2) = (3x - 2y)[(3y)^2 + (3x)(2y) + (2y)^2]$

$$= (3x)^3 - (2y)^3$$

$$= 27x^3 - 8y^3$$

b) $(2x - 1)(4x^2 + 2x + 1) = (2x - 1)[(2x)^2 + (2x)(1) + (1)^2]$

$$= (2x)^3 - (1)^3$$

$$= 8x^3 - 1.$$

Exercises 3.4

Find product of each using the Special Products formulas.

1. $-3a(3b - 7c)$
2. $7xy^2(3x^2 - 5xy^3 + 2xy)$
3. $(7x + 5y)(7x - 5y)$
4. $(2x - y)^2$
5. $(2x - y)(3x - y)$
6. $(4ab^2 - ab)^3$
7. $(3a^2 - 2b^2)^2$
8. $(2xy - 3)(2xy + 3)$
9. $(2x + 1)(3x - 2)$
10. $(3x - 2y)^3$
11. $(2x - 3)(4x^2 + 6x + 9)$
12. $(3w + 5z)(9w^2 - 15wz + 25z^2)$
13. $(2x - 5y)(4x^2 + 10xy + 25y^2)$
14. $(3a^2 + 4b)^2$
15. $(6x - 4y)(6x + 4y)$
16. $(x + 2)^2$
17. $(2a - 1)^2$
18. $(2x + 4)^2$
19. $(-2x + 1)^2$
20. $(1 - x^2)(1 + x^2)$

3.5 Methods of Factoring Polynomials

If a polynomial is the product of other polynomials, then each of the polynomials in the product is called a *factor* of the original polynomial. For instance,

$$x^2 - 25 = (x + 5)(x - 5)$$

hence, $x + 5$ and $x - 5$ are the factors of $x^2 - 25$. The process of finding factors of a given polynomial is called *factoring*.

A polynomial with integer coefficients is said to be *prime* if it has no monomial factor or polynomial factor with integer coefficients, other than one and itself. It is said to be in *completely factored form* when each of its polynomial factors is prime.

Method 1. **Removing the Greatest Common Factor(GCF)**

$$AB + AC + AD = A(B + C + D)$$

Remark: Note that A is a common factor. To obtain the other factor, divide each term of the given polynomial by the common factor A.

Illustration 1. a) $15x^2 + 9x = 3x(5x + 3)$
b) $6x^3y^2 - 3x^2y + 9x^4y^3 = 3x^2y(2xy - 1 + 3x^2y^2)$
c) $x(a + b) - y(a + b) = (a + b)(x - y)$

Method 2. **Factoring Difference of Two Squares**

$$A^2 - B^2 = (A + B)(A - B)$$

Illustration 2. a) $25y^2 - 9z^2 = (5y)^2 - (3z)^2 = (5y + 3z)(5y - 3z)$
b) $49b^2 - 16c^2 = (7b)^2 - (4c)^2 = (7b + 4c)(7b - 4c)$
c) $x^2 - 9 = (x)^2 - (3)^2 = (x + 3)(x - 3)$
d) $16y^4 - 9z^2 = (4y^2)^2 - (3z)^2 = (4y^2 + 3z)(4y^2 - 3z)$.
e) $1 - 4w^2 = (1)^2 - (2w)^2 = (1 + 2w)(1 - 2w)$

Method 3. **Factoring the Sum and Difference of Two Cubes**

(i) $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$
(ii) $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$

Illustration 3. a) $x^3 + 8 = (x)^3 + (2)^3 = (x + 2)(x^2 - 2x + 4)$
b) $y^3 - 27 = (y)^3 - (3)^3 = (y - 3)(y^2 + 3y + 9)$
c) $8a^3 + 1 = (2a)^3 + (1)^3 = (2a + 1)(4a^2 - 2a + 1)$
d) $64c^3 - 27d^3 = (4c)^3 - (3d)^3 = (4c - 3d)(16c^2 + 12cd + 9d^2)$

Method 4. **Factoring Trinomial $ax^2 + bx + c$, where a, b, c are integers and not all equal to zero.**

Some Guidelines:

1. Before factoring, arrange the trinomial in descending powers if the coefficient a is positive and arrange in ascending powers if a is negative.
2. Select a pair of factors of the first term and the last term so that:
   (a) The sum of the inner and outer products is equal to the middle term of the given trinomial.
   (b) The product of the last terms of the factors is equal to the last term of the given trinomial.
   (c) The product of the first terms of the factors is equal to the first term of the given trinomial.
3. When the last term is positive and when the middle term is positive, the factoring is of the form (+)(+).

4. When the last term is positive and when the middle term is negative, the factoring is of the form (–)(–).
5. When the last term is negative and when the middle term is positive, the factoring is of the form (–)(+) or (+)(–).

Remarks: (i) This method is usually called factoring trinomial by "Trial and Error". You have to check several cases to obtain a correct combination.
(ii) You may check your factoring by multiplying the factors. If the resulting product is the given polynomial, then you obtained the correct factors.

Example 1. Factor the following polynomials:
a) $x^2 + 8x + 12$
b) $y^2 - 5y + 6$
c) $6x^2 - 7x - 20$
d) $5x^2 + 2x - 3$

Solution:

a) $x^2 + 8x + 12 = ?$
Since the last term is positive and the middle term is also positive, the factors must be of the form

$$x^2 + 8x + 12 = (\quad + \quad)(\quad + \quad).$$

Since the product of the first terms of the factors is x^2, we must have

$$x^2 + 8x + 12 = (x + \quad)(x + \quad).$$

Now select pair of factors of 12 so that the sum of the inner and outer product equal to the middle term. Since the sum of the inner and outer products of $(x + 6)(x + 2)$ is $8x$, (inner product = $(6)(x) = 6x$, outer product = $(2)(x) = 2x$, and sum = $8x$), we have

$$x^2 + 8x + 12 = (x + 6)(x + 2).$$

b) $y^2 - 5y + 6 = ?$
Since the last term is positive and the middle term is also positive, the factors must be of the form

$$y^2 - 5y + 6 \quad = (\quad - \quad)(\quad - \quad).$$

Since the product of the first terms of the factors is y^2, we must have

$$y^2 - 5y + 6 \quad = (y - \quad)(y - \quad).$$

Now select pair of factors of 6 so that the sum of the inner and outer product equal to the middle term. The correct combination is $(y-3)(y-2)$ because the inner product = $-3y$, outer product = $-2y$, and their sum = $-5y$. Therefore,

$$y^2 - 5y + 6 \quad = (y-3)(y-2).$$

c) $6x^2 - 7x - 20 = ?$
The factors must be one of the following forms :

$(2x - \quad)(3x + \quad)$, or $(2x + \quad)(3x - \quad)$, or $(6x - \quad)(x + \quad)$, or $(6x + \quad)(x - \quad)$. By "trial and error", the correct factors are given

$$6x^2 - 7x - 20 = (2x-5)(3x+4).$$

d) $5x^2 + 2x - 3 = (5x-3)(x+1)$.

Method 5. **Factoring by Grouping**

If a polynomial has more than three terms, it may be possible to group terms in such a way that each group has a common factor.

Example 2. Factor $ax + ay + bx + by$.

Solution:
To factor the given polynomial, we group the first two terms and the last two terms:

$$ax + ay + bx + by. = (ax + ay) + (bx + by)$$

Observe that the first two terms have a common factor a and the last two terms have a common factor b:

$$\begin{aligned} ax + ay + bx + by &= (ax + ay) + (bx + by) \\ &= a(x+y) + b(x+y). \end{aligned}$$

Note that $(x + y)$ is a common factor to each term. Therefore, we have

$$\begin{aligned} ax + ay + bx + by. &= (ax + ay) + (bx + by) \\ &= a(x+y) + b(x+y). \\ &= (x+y)(a+b). \end{aligned}$$

Alternative solution:
We may group the first and third terms and the second and fourth terms.

$$\begin{aligned} ax + ay + bx + by &= (ax + bx) + (ay + by) \\ &= x(a + b) + y(a + b) \\ &= (a + b)(x + y). \end{aligned}$$

The above example shows that it is sometimes possible to group terms differently and still be able to factor a given expression. However, in some cases only one specific grouping will lead to a successful factoring.

Example 3. Factor $3x^2 + 7x - 6xy - 14y$.

Solution:

$$\begin{aligned} 3x^2 + 7x - 6xy - 14y &= (3x^2 + 7x) + (-6xy - 14y) \\ &= x(3x + 7) + (-2y)(3x + 7) \\ &= x(3x + 7) - 2y(3x + 7) \\ &= (3x + 7)(x - 2y) \end{aligned}$$

Example 4. Factor $x^3 - 2x^2 + 4x - 8$.

Solution:

$$\begin{aligned} x^3 - 2x^2 + 4x - 8 &= (x^3 - 2x^2) + (4x - 8) \\ &= x^2(x - 2) + 4(x - 2) \\ &= (x - 2)(x^2 + 4). \end{aligned}$$

Remark: In factoring a polynomial, always check for a common factor no matter how many terms the expression has. If there is a common factor, factor it out and this will make the remaining polynomial easier to factor.

Exercise 3.5

Factor each of the following completely.

1. $4xy^2 - 20x^3y$
2. $4x^3y^2 + 6x^2y^3$
3 $x^2 - 16y^2$
4. $3x^2 + 7x + 2$
5. $27z^3 - 8$
6. $16x^2 - 9y^2$
7. $3a^2 - 8a + 5$
8. $2x^2(z + 3y) - 8xy(z + 3y)$
9. $(a + 2b)^2 - 4c^2$
10. $9(x - y)^2 - 4(a + b)^2$
11. $x^4 - 81y^4$
12. $y^3 + 125z^3$
13. $8 - 27x^3$
14. $x^2 + 5x + 6$
15. $3y^2 - y - 10$
16. $6y^2 + 7y - 20$
17. $4x^2 + 5xy - 21y^2$
18. $2y^2 - 2y - 10$
19. $ax - ay + bx - by$
20. $x^3 + 3x^2 - 4x + 12$
21. $4x^3 - 12x^2 - x + 3$
22. $a^2 - 4a + 4 - b^2$

3.6 Operations on Rational Expressions

A *rational expression* or algebraic fraction is an algebraic expression of the form $\dfrac{P}{Q}$, where P and Q are polynomials and $Q \neq 0$. The polynomials P and Q are called the *numerator* and the *denominator*, respectively. Of course, any polynomial P is a rational expression with denominator Q = 1.

The fraction $\dfrac{P}{Q}$ is read as "P over Q".

Illustration 1. $\dfrac{5}{x}$, $\dfrac{2x-5}{x-1}$, $\dfrac{x^2+1}{x^2-3x+4}$, and $\dfrac{2}{3}$ are rational expressions.

Whenever a rational expression is written it will be understood that the values of the variables that make the denominator zero are excluded.

A rational expression is said to be in *lowest terms* if the numerator and denominator have no common factor other 1 and –1.

Illustration 1. a) The fractions $\dfrac{2}{3}$, $\dfrac{7}{5}$, and $\dfrac{3}{4}$ are in lowest terms.

b) The fraction $\dfrac{4xy}{8x^2z}$ is not in lowest terms because the numerator and the denominator have common factors 4 and x.

Simplification of Rational Expression

The procedure of simplifying a fraction is called *reducing the fraction to lowest terms*. It is done by cancellation of all factors common to both numerator and denominator.

Example 1. Reduce the following to lowest terms.

a) $\dfrac{4x^2y}{2xy}$

b) $\dfrac{7a^5b^4}{21a^2b^5}$

c) $\dfrac{a^2x-a^2y}{ax^2-ay^2}$

d) $\dfrac{x^2+3x}{x^2+x-2}$

e) $\dfrac{18x^2+9xy-2y^2}{9x^2-4y^2}$

Solution:

a) $\dfrac{4x^2y}{2xy}=\dfrac{2x(2xy)}{(2xy)}=2x$

b) $\dfrac{7a^5b^4}{21a^2b^5} = \dfrac{a^3(7a^2b^4)}{3b(7a^2b^4)} = \dfrac{a^3}{3b}$

c) $\dfrac{a^2x - a^2y}{ax^2 - ay^2} = \dfrac{a^2(x-y)}{a(x^2-y^2)} = \dfrac{(a)(a)(x-y)}{a(x+y)(x-y)} = \dfrac{a}{x+y}$

d) $\dfrac{x^2+2x}{x^2+x-2} = \dfrac{x(x+2)}{(x-1)(x+2)} = \dfrac{x}{x-1}$

e) $\dfrac{18x^2+9xy-2y^2}{9x^2-4y^2} = \dfrac{(6x-y)(3x+2y)}{(3x-2y)(3x+2y)} = \dfrac{6x-y}{3x-2y}$

Multiplication and Division of Rational Expressions

I. To multiply two or more rational expressions, it is best to factor completely all the numerators and denominators and cancel common factors. After doing so, multiply the remaining numerators and denominators together.

II. To divide rational expression, take the reciprocal of the denominator and proceed to multiplication.

Illustration.

a) $\dfrac{5x}{8} \cdot \dfrac{4}{15x^2} = \dfrac{5x}{2^3} \cdot \dfrac{2^2}{(5x)(3x)} = \dfrac{1}{6x}$

b) $\dfrac{4a}{3b} \cdot \dfrac{3a^2b^2}{10a^4} = \dfrac{2^2a}{3b} \cdot \dfrac{(3b)(a^2b)}{5(2)a^2a^2} = \dfrac{2b}{5a}$

c) $\dfrac{x-4}{2x+4} \cdot \dfrac{4x+8}{x^2-16} = \dfrac{x-4}{2(x+2)} \cdot \dfrac{4(x+2)}{(x+4)(x-4)} = \dfrac{2}{x+4}.$

d) $\dfrac{x^2+2x-3}{x^2+7x+12} \cdot \dfrac{x+1}{x^2+4x-5} = \dfrac{(x+3)(x-1)}{(x+3)(x+4)} \cdot \dfrac{x+1}{(x+5)(x-1)} = \dfrac{x+1}{(x+4)(x+5)}$

e) $\dfrac{4x^3}{9y^2} \div \dfrac{8x^2y^4}{15xy} = \dfrac{4x^3}{9y^2} \cdot \dfrac{15xy}{8x^2y^4} = \dfrac{4x^3}{3(3)y^2} \cdot \dfrac{3(5)xy}{4(2)x^2y^4} = \dfrac{5x^2}{6y^5}$

f) $$\frac{3y^3 - 3y^2}{16y^5 + 8y^4} \div \frac{y^2 + 2y - 3}{4y + 12} = \frac{3y^2(y^2 - 1)}{8y^4(2y+1)} \cdot \frac{4(y+3)}{(y+3)(y-1)}$$
$$= \frac{3y^2(y+1)(y-1)}{8y^4(2y+1)} \cdot \frac{4(y+3)}{(y+3)(y-1)}$$
$$= \frac{3(y+1)}{2y^2(2y+1)}$$

g) $$\frac{x^2 + 4x - 21}{x^2 + 6x - 7} \div \frac{x^2 - 9}{x^2 - x} = \frac{x^2 + 4x - 21}{x^2 + 6x - 7} \cdot \frac{x^2 - x}{x^2 - 9}$$
$$= \frac{(x+7)(x-3)}{(x+7)(x-1)} \cdot \frac{x(x-1)}{(x+3)(x-3)} = \frac{x}{x+3}.$$

Addition and Subtraction of Rational Expressions

To add or subtract two or more fractions with the same denominator take the sum of the numerators and write the result as a single fraction with this sum or difference as the numerator and the common denominator as the denominator. For example,

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \quad \text{and} \quad \frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}.$$

Example 1. Add or subtract the following and reduce the result to lowest terms.

a) $\frac{3}{4x} + \frac{5}{4x}$

b) $\frac{4x}{2x - y} - \frac{2y}{2x - y}$

Solution: a) $\frac{3}{4x} + \frac{5}{4x} = \frac{3+5}{4x} = \frac{8}{4x} = \frac{(4)(2)}{4(x)} = \frac{2}{x}$

b) $\frac{4x}{2x - y} - \frac{2y}{2x - y} = \frac{4x - 2y}{2x - y} = \frac{2(2x - y)}{(2x - y)} = 2$

Two rational expressions are *equivalent* if they are equal when reduced to lowest terms.

Example 2. a) $\frac{15}{18}$ and $\frac{10}{12}$ are equivalent fractions because $\frac{15}{18} = \frac{10}{12} = \frac{5}{6}$.

b) $\dfrac{x}{x+2}$ and $\dfrac{x^2}{x(x+2)}$ are equivalent fractions.

To add or subtract two or more fractions with different denominators, we must replace the fractions with equivalent fractions having the same denominators. It is preferable to use the least common denominator (LCD). The LCD of fractions consists of the product of all unique prime factors in the denominator, each with an exponent equal to the largest exponent with which the factor appears. In other words, the LCD is just the LCM of the denominators of the given fraction.

To add or subtract two or more fractions we may use the following rule:

$$\frac{p}{q} \pm \frac{r}{s} = \frac{p\left(\frac{LCD}{q}\right) \pm r\left(\frac{LCD}{s}\right)}{LCD},$$

where p, q, r, and s represent polynomials.

Illustration 1.

a) The LCD of $\dfrac{7}{18x^2}$ and $\dfrac{5}{8xy^4}$ is $(3^2)(2^3)x^2y^4 = 72x^2y^4$, since $18x^2 = (3^2)(2)x^2$ and $8xy^4 = 2^3xy^4$.

b) The LCD of $\dfrac{2}{x}$ and $\dfrac{x}{x+2}$ is $x(x+2)$.

c) The LCD of $\dfrac{16a}{a-2}$, $\dfrac{a\text{-}2}{(a+2)(a-2)^2}$, and $\dfrac{b+1}{4a+8}$ is $4(a-2)^2(a+2)$.

Illustration 2. Find the sum of $\dfrac{4}{x^2-4x-5}$ and $\dfrac{2}{x^2-1}$.

Solution:

$$\begin{aligned}
\frac{4}{x^2-4x-5} + \frac{2}{x^2-1} &= \frac{4}{(x-5)(x+1)} + \frac{2}{(x+1)(x-1)} \\
&= \frac{4(x-1)+2(x-5)}{(x-5)(x+1)(x-1)} \\
&= \frac{4x-4+2x-10}{(x+5)(x+1)(x-1)} = \frac{6x-14}{(x+5)(x+1)(x-1)} \\
&= \frac{2(3x-7)}{(x+5)(x+1)(x-1)}.
\end{aligned}$$

Example 3. Perform the following addition or subtraction and reduce answer to lowest terms.

a) $\dfrac{3x-1}{5}+\dfrac{4-5x}{6}$ b) $\dfrac{7}{18x^2y}+\dfrac{5}{8xy^4}$ c) $\dfrac{y}{x^2+xy}-\dfrac{x}{xy+y^2}$

Solution:

a) $$\frac{3x-1}{5}+\frac{4-5x}{6}=\frac{(3x-7)(6)+(4-5x)(5)}{30}=\frac{18x-6+20-25x}{30}=\frac{-17x+14}{30}$$

b) $$\frac{7}{18x^2y}+\frac{5}{8xy^4}=\frac{7}{3^2(2)x^2y}+\frac{5}{2^3xy^4}=\frac{7(4y^3)+5(9x)}{3^22^3x^2y^4}=\frac{28y^3+45x}{72x^2y^4}$$

c) $$\frac{y}{x^2+xy}-\frac{x}{xy+y^2}=\frac{y}{x(x+y)}-\frac{x}{y(x+y)}=\frac{y(y)-x(x)}{xy(x+y)}=\frac{y^2-x^2}{xy(x+y)}$$
$$=\frac{(y+x)(y-x)}{xy(x+y)}=\frac{y-x}{xy}$$

Exercise 3.6.

A. Reduce the following to lowest terms.

1. $\dfrac{56}{126}$
2. $\dfrac{a^4x^3y}{a^2xy^3}$
3. $\dfrac{25a^2b^4}{10a^4b^3}$
4. $\dfrac{x^2-1}{x^2-x}$
5. $\dfrac{y^2-6y+6}{y^2+2y-15}$
6. $\dfrac{x^2-16}{x^2-8x+16}$
7. $\dfrac{9x^2-1}{3x^2+5x-2}$
8. $\dfrac{a^2+ab}{3a+2a^3}$
9. $\dfrac{a^2-6a+9}{a^2-9}$
10. $\dfrac{x^2-y^2}{x^2y+xy}$
11. $\dfrac{x^2-7x+10}{2x^2-x-6}$
12. $\dfrac{y^2-3y-4}{y^2-y-12}$
13. $\dfrac{6a^2x-6a^2y}{3ax^2-3ay^2}$
14. $\dfrac{x^2+5x+6}{x^2+3x}$
15. $\dfrac{ax-bx+4a-4b}{x^2-16}$
16. $\dfrac{ax-ay-bx-by}{am-bm-an+bn}$
17. $\dfrac{(4x^2-9y^2)(18x-12)}{(2x-3y)(12x-8)}$
18. $\dfrac{2(x^2-y^2)xy+x^4-y^4}{x^4-y^4}$
19. $\dfrac{4a^2-1}{12a^2+a-4a^3-3}$
20. $\dfrac{(x^2-16)(x^2-4x+16)}{x^3+64}$

B. Perform the indicated operations and reduce answers to lowest terms.

1. $\dfrac{2x+3}{6}-\dfrac{4x-7}{9}$

2. $\dfrac{x+1}{x+2}-\dfrac{x+3}{x}$

3. $\dfrac{x}{x-5}+\dfrac{2}{x+3}$

4. $\dfrac{x}{3}+\dfrac{2x}{5}-\dfrac{3x}{4}$

5. $\dfrac{1}{x}+\dfrac{1}{2x^2}-\dfrac{1}{4x}$

6. $x+y+\dfrac{x^2-xy}{x-y}$

7. $\dfrac{2x+1}{x-y}-\dfrac{3x-1}{x+y}$

8. $\dfrac{3}{a-3}+\dfrac{a^2+2}{a^3-27}$

11. $\dfrac{2x+3}{3x^2+x-2}-\dfrac{3x-4}{2x^2-3x-5}$

12. $\dfrac{2xy}{x^3+y^3}-\dfrac{x}{x^2-xy+y^2}$

13. $\dfrac{2}{x^2-y}-\dfrac{1}{x^2+4x+4}-\dfrac{1}{x^2-4x+4}$

14. $\dfrac{5}{x^2+3x+2}+\dfrac{2}{x^2-1}-\dfrac{1}{x^2+x-2}$

15. $\dfrac{x+1}{x^2-4}+\dfrac{2}{x^2+x-2}-\dfrac{2}{x^2-3x+2}$

16. $\dfrac{2}{x+4}-\dfrac{x-2}{x-3}+\dfrac{3x^2-5x+8}{x^2+x-12}$

17. $\dfrac{2x^2+2xy}{x^2-4y^2}-\dfrac{x+y}{x-2y}-\dfrac{2y}{x+2y}$

18. $\dfrac{x+2}{2x^2-x-3}+\dfrac{x-2}{4x^2-12x+9}+2x-3$

9. $\dfrac{3x-2y}{5x-3}+\dfrac{2x-y}{3-5x}$

10. $\dfrac{2x-1}{4-x}+\dfrac{x+2}{3x-12}$

19. $x+6+\dfrac{5x+1}{12x^2+5x-2}-\dfrac{x}{3x+2}$

20. $\dfrac{2x-1}{2x^2-x-6}+\dfrac{x+3}{6x^2+x-12}-\dfrac{2x-3}{3x^2-10x+8}$

C. Perform the indicated operations and reduce answers to lowest terms

1. $\dfrac{15a^2}{16b^3}\cdot\dfrac{8b}{3a^4}$

2. $\dfrac{9a^3b^5}{24ab^4}\cdot\dfrac{8ab}{27a^2b^3}$

3. $\dfrac{3x-15}{x+3}\div\dfrac{12x+18}{4x+12}$

4. $\dfrac{15x^3y^2z}{2abc^3}\div\dfrac{10x^2y^3z^2}{6a^2bc^2}$

5. $\dfrac{x-y}{x+3y}\cdot\dfrac{x^2-9y^2}{x^2-y^2}$

6. $\dfrac{x^2-y^2}{xy-2y^2}\cdot\dfrac{2x^2-4xy}{x^2-2xy+y^2}$

11. $\dfrac{a^2-4b^2}{4b-2a}\div\dfrac{2a^2+ab-6b^2}{6a-9b}$

12. $\left[\dfrac{3a}{a-3}-\dfrac{3a+2}{a^2-6a+9}\right]\cdot\left[\dfrac{a+2}{a+3}-\dfrac{a}{a^2+6a+9}\right]$

13. $\dfrac{y+1}{x-2}\cdot\dfrac{x^2+2x}{6}\div\dfrac{xy^2-x}{3y-3}$

14. $\dfrac{6x-9}{x^2-25}\cdot\dfrac{x^2-3x-10}{12-4x}$

15. $\dfrac{4y^2-3yz-z^2}{16y^2z^2-z^4}\cdot\dfrac{z^2+4yz}{y-z}$

16. $\dfrac{x^2+10x+24}{x^2+5x+4}\cdot\dfrac{x^2-2x-3}{x^2+9x+18}$

7. $\dfrac{ab+ac}{ab-ac}\cdot\dfrac{b}{b+c}\div\dfrac{b}{b-c}$

8. $\dfrac{x^2+9x+14}{x^2+4x-21}\div\dfrac{x^2-3x-10}{x^2+2x-35}$

9. $\dfrac{x^4-y^4}{(x-y)^2}\cdot\dfrac{y^2}{x^2+y^2}\cdot\dfrac{x-y}{xy+y^2}$

10. $\dfrac{2x^2-3x}{6x^2-21x+18}\div\dfrac{4x^2+27x-7}{2x^2-7x+6}$

17. $\dfrac{y^2-2y-15}{y^2-9}\div\dfrac{12-4y}{y^2-6y+9}$

18. $(a^2-b^2)\div\left[\dfrac{a^2+ab}{b^2+ab}\div\dfrac{a^2-ab}{b^2-ab}\right]$

19. $\left[\dfrac{x^3+4x^2-5x}{x^2-2x+1}\div\dfrac{x^2+x-2}{x^4+8x}\right]\cdot\dfrac{x-4}{x^2-2x+4}$

20. $\dfrac{x+6}{2x^2+13x-7}\div\dfrac{3x-2}{4x^2-8x+3}\cdot\dfrac{2x^2+3x-9}{6x^2-7x+2}$

3.7 Complex or Compound Fractions

A *complex* or *compound fraction* is a fraction which contains fractions in its numerator and/or denominator.

Illustration 1. The following are compound fractions:

$$\frac{\frac{1}{2}}{\frac{3}{2}-1}\ ;\quad \frac{\frac{1}{x}+\frac{3}{y}}{\frac{2}{xy}}\ ;\quad \frac{\frac{1}{a}+\frac{2}{b}}{\frac{2}{a}-\frac{a}{3b}}$$

Simplifying Compound Fractions

Step 1. Express the numerator and denominator to single fractions and then use the following rule for division of fraction:

$$\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$$

Step 2. Simplify and reduce to lowest terms.

Illustration 2. Simplify the following compound fractions.

a) $\dfrac{\frac{1}{x}-\frac{1}{y}}{\frac{4}{x}+\frac{2}{y}}$

b) $\dfrac{9y^2-4x^2}{\frac{y-x}{x-2y}-1}$

Solution:

a) $$\frac{\frac{1}{x}-\frac{1}{y}}{\frac{4}{x}+\frac{2}{y}} = \frac{\frac{y - x}{xy}}{\frac{4y+2x}{xy}} = \frac{y-x}{x+y}.\frac{x+y}{4y+2x} = \frac{y-x}{2(x+2y)}.$$

b) $$\frac{9y^2-4x^2}{\frac{y-x}{x-2y}-1} = \frac{9y^2-4x^2}{\frac{y-x-(x-2y)}{x-2y}} = \frac{9y^2-4x^2}{1}\cdot\frac{x-2y}{-2x+3y}$$

$$= \frac{(3y+2x)(3y-2x)}{1}\cdot\frac{x-2y}{3y-2x}$$

$$= (3x+2x)(x-2y)$$

Exercise 3.7

Simplify the following compound fractions.

1. $\dfrac{\frac{2}{3}+\frac{1}{2}}{\frac{3}{4}-\frac{2}{3}}$

2. $\dfrac{\frac{3}{x}+\frac{5}{y}}{\frac{2}{x}-\frac{3}{y}}$

3. $\dfrac{\frac{4a}{5}-\frac{2b}{3}}{\frac{3a}{5}-\frac{3b}{4}}$

4. $\dfrac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{y^2}-\frac{2}{xy}+\frac{1}{x^2}}$

7. $\dfrac{\frac{y^2}{x^2}-1}{\frac{y^2}{x^2}-\frac{2y}{x}+1}$

8. $\dfrac{\frac{1}{2}-\frac{4}{x^3}}{\frac{1}{x^2}+\frac{1}{4}+\frac{1}{2x}}$

9. $\dfrac{\frac{y}{x}+\frac{x^2}{y^2}}{\frac{x}{y^2}-\frac{1}{y}+\frac{1}{x}}$

10. $\dfrac{\frac{1+y}{1-y}-\frac{1-y}{1+y}}{\frac{1}{1+y}-\frac{1}{1-y}}$

5. $$\frac{\frac{1}{z}+\frac{1}{w}}{\frac{1}{z^2}-\frac{1}{w^2}}$$

6. $$\frac{x^2-\frac{1}{x}}{x+1+\frac{1}{x}}$$

11. $$\frac{x}{1-\frac{1}{1+\frac{1}{x}}}$$