Quantum Field Theory without Infinities

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In relativistic quantum field theories, the traditional renormalization approach leads to Hamiltonians with infinite counterterms. The infinities cancel when this Hamiltonian is used to calculate the renormalized finite S-matrix and related observable properties in a good agreement with experiment. However, time evolution of state vectors and observables cannot be studied without a well-defined finite Hamiltonian. Based on the "clothed particle" approach (O. W. Greenberg and S. S. Schweber, 1958, *Nuovo Cimento* **8**, 378), we reformulate the theory in such a way that ultraviolet infinities appear neither in the S-matrix nor in the Hamiltonian. In this formulation the Hamiltonian is finite and allows us to calculate the time evolution of wave functions, the S-matrix, and other properties by a straightforward application of quantum mechanical rules without renormalization. A rigorous approach to the bound states in quantum field theory is also discussed using the hydrogen atom as an example.

1. INTRODUCTION

Early quantum field theory (QFT) suffered from ultraviolet infinities in higher order contributions to the S-matrix. The breakthrough came in the late 1940s when it was understood how to calculate the S-matrix finite to all orders by renormalizing masses of particles and coupling constants. Since then renormalized theories, especially quantum electrodynamics (QED), achieved an unprecedented accuracy in calculations of the S-matrix and related observable quantities (scattering cross-sections, magnetic moments of particles, energies of bound states, etc.). However, the renormalization was achieved by the introduction of infinite counterterms in the Hamiltonian, and this approach just shifted the problem of infinities from one place (the S-operator) to another (the Hamiltonian H and time evolution operator $\exp(-iHt)$).

Although the S-operator calculated in QFT is sufficient to describe all current experiments, and time-dependent effects are not easily observable at the subatomic level, so that one can avoid a direct use of H and $\exp(-iHt)$, the infinities in H are not acceptable from the point of view of logical consistency and physical unity of the theory. Indeed, presently we cannot use the same (infinite) Hamiltonian that allegedly describes subatomic events to describe also particle interactions in low energy classical physics where time evolution is easily observed. Moreover, future experiments may be able to register radiative corrections to time-dependent processes in the subatomic world (see section 5.2). Thus theoretical foundations of QFT remain unsatisfactory.

In this paper we suggest a rigorous way to remove ultraviolet infinities from QFT Hamiltonians and time evolution operators to all orders in perturbation theory. We will show that given a renormalizable relativistic quantum field theory with arbitrary particle content and with minimal assumptions about the form of interaction, a new *finite* Hamiltonian H^r can be found, while the S-operator and all related quantities remain unchanged (finite).

To obtain H^r we use the *clothed particle* representation first proposed by Greenberg and Schweber [1]. They and other authors derived and analyzed clothed particle pictures for a number of model QFT theories such as scalar-field [2], Lee [3], and Ruijgrok-Van Hove model [4]. A regular way to construct the clothed particle Hamiltonian H^r as a perturbation series was independently suggested by Tani [5], Faddeev [6], and Sato [7]. Shirokov and co-workers [8] further developed these ideas and, in particular, demonstrated how the ultraviolet divergences can be removed from H^r up to the 4th order. However the ability of this approach to remedy QFT divergences in all orders has not been established.

In section 2 we recall some well-known facts from scattering and quantum field theories with the emphasis on the relationships between the Hamiltonian, time evolution and scattering operators. We also introduce the terminology and notation to be used throughout the paper and illustrate them using QED as an example in section 3. For further details the reader may refer to excellent book [9]. In section 4, for a general renormalizable QFT, we prove the existence of a clothed particle Hamiltonian H^r which is finite to all orders in perturbation theory. The bound state problem and prospects for the experimental verification of our approach are qualitatively discussed in section 5 using the hydrogen atom as an example.

We use the system of units in which $\hbar = 1$ and $c = 1$.

2. HAMILTONIAN AND SCATTERING OPERATORS

2.1. S-operator

In quantum theory, time development of observables in the interval $[t', t]$ is described by the evolution operator $U(t \leftarrow t') = \exp(-iH(t-t'))$, where $H = H_0 + V$, H_0 is a free Hamiltonian, and V is an interaction. However, in high energy scattering experiments the interaction acts only during a short time interval of collision $[-\eta, \eta]$, so that inequalities $t' \ll -\eta < 0 < \eta \ll t$ hold. Before and after the collision the exact evolution is well approximated by the free evolutions $U_0(-\eta \leftarrow t')$ and $U_0(t \leftarrow \eta)$, respectively, where $U_0(t \leftarrow t') = \exp(-iH_0(t-t'))$. Therefore, a simplified description is possible in which the evolution is free at all times except sudden change at $t = 0$ described by a unitary operator S:

$$\begin{aligned} U(t \leftarrow t') &= U(t \leftarrow \eta)U(\eta \leftarrow -\eta)U(-\eta \leftarrow t') \\ &\approx U_0(t \leftarrow \eta)U(\eta \leftarrow -\eta)U_0(-\eta \leftarrow t') \\ &\approx U_0(t \leftarrow 0)SU_0(0 \leftarrow t'). \end{aligned} \tag{1}$$

The S-operator may be interpreted as the difference between exact and free evolutions integrated over the collision time interval $[-\eta, \eta]$ (for resonance scattering η is the lifetime of the metastable state). The usual definition of S follows formally from Eq. (1)

$$S = \lim_{t \to +\infty} W(t, -\infty), \tag{2}$$

where

$$W(t, -\infty) = \lim_{t' \to -\infty} U_0(0 \leftarrow t)U(t \leftarrow t')U_0(t' \leftarrow 0). \tag{3}$$

Approximation (1) is well suited for description of scattering experiments in which particles are observed in the asymptotic region. It is clear, however, that only averaged properties of the exact time evolution in the interval $[-\eta, \eta]$ (such as time delay or the lifetime of the metastable state) can be obtained from the S-operator. A great deal of information about the interaction V is lost. Indeed, there are infinitely many Hamiltonians which yield exactly the same S-operator and are therefore equivalent to each other as far as scattering is concerned [10, 11].

2.2. Perturbation theory

The operator $W(t, -\infty)$ satisfies equation

$$\frac{d}{dt}W(t,-\infty) = -iV(t)W(t,-\infty), \tag{4}$$

where

$$V(t) = e^{iH_0 t}Ve^{-iH_0 t}e^{\epsilon t}, \tag{5}$$

and factor $e^{\epsilon t}$ ($\epsilon \to +0$) describes the "adiabatic switching" of the interaction $V(t)$. Operators with time dependence as in Eq.(5) will be called *regular* and will play an important role in our discussion below.

The solution of Eq. (4) with the initial condition $W(-\infty,-\infty) = 1$ can be written in a number of equivalent forms. The most familiar form is the covariant Feynman–Dyson perturbation expansion. However, for our purposes we found more useful an equivalent solution suggested by W. Magnus [12]

$$W(t,-\infty) = \exp[-i\int_{-\infty}^{t} F(t')dt'],$$

where

$$F(t) = V(t) + \frac{i}{2}\int_{-\infty}^{t}[V(t'),V(t)]dt' + \ldots$$

We will be using convenient symbols for time integrals $\underline{Y(t)} = \int_{-\infty}^{t} Y(t')dt'$ and $\underbrace{Y} = \int_{-\infty}^{+\infty} Y(t')dt'$ and often omit the time arguments of operators. Then Hermitian operator F is represented as a series of multiple commutators with time integrals

$$\begin{aligned} F &= V + \frac{i}{2}[\underline{V},V] - \frac{1}{6}[\underline{V},[\underline{V},V]] - \frac{1}{6}[[\underline{\underline{V},V}],V] - \frac{i}{12}[\underline{V},[[\underline{\underline{V},V}],V]] \\ &- \frac{i}{12}[[\underline{V},[\underline{V},V]],\underline{\underline{V}}] - \frac{i}{12}[[\underline{\underline{V},V}],[V,V]] + \ldots \end{aligned} \tag{6}$$

and the S-operator is manifestly unitary: $S = e^{-iG}$, where

$$G = \underbrace{F}. \tag{7}$$

In any physically satisfactory theory the S-operator must be (I) relativistically invariant, (II) cluster separable, and (III) finite. We also assume that (IV) elementary particles of the theory are stable and have well-defined masses $\mu \geq 0$ and one-particle energies $\omega_{\mathbf{p}} = \sqrt{\mu^2 + \mathbf{p}^2}$. The latter condition excludes from consideration a number of important models, such as the electroweak theory, in which there are unstable elementary particles, e.g., $W^{\pm}$, Z^0, muons, etc.

2.3. Relativistic invariance and cluster separability

Condition (I) is a reflection of a more general principle which states that a full description of any system of interacting particles is given by ten operators (linear momentum $\mathbf{P}$, angular momentum $\mathbf{J}$, boost $\mathbf{K}$, and Hamiltonian H) acting in the Hilbert space of states. These operators obey commutation relations of the Poincaré algebra and generally differ from their free counterparts ($\mathbf{P_0}$, $\mathbf{J_0}$, $\mathbf{K_0}$, and H_0). However, it has been shown in [13] that experimentally observed time dilation in moving systems, e.g., slowing-down of the decay of relativistic particles, implies that dynamics has *instant form* [14], i.e., only the Hamiltonian $H = H_0 + V$ and boost $\mathbf{K} = \mathbf{K_0} + \mathbf{W}$ are modified by the interaction so that $\mathbf{P} = \mathbf{P_0}$, $\mathbf{J} = \mathbf{J_0}$, and V is rotationally and translationally invariant

$$[V, \mathbf{J}_0] = [V, \mathbf{P}_0] = 0. \tag{8}$$

With certain conditions of smoothness applied to V and $\mathbf{W}$ [9], the above theory yields a relativistically invariant S-operator, i.e., $[S, H_0] = [S, \mathbf{P_0}] = [S, \mathbf{J_0}] = [S, \mathbf{K_0}] = 0$.

Condition (II) requires that the total scattering amplitude for spatially separated events is given by the product of individual amplitudes [9]. The easiest way to satisfy both relativistic invariance and cluster separability is to build interaction terms V and $\mathbf{W}$ from quantum fields which are linear combinations of creation $\alpha^+(\mathbf{q}, \sigma)$ and destruction $\alpha^-(\mathbf{q}, \sigma)$ operators [$\mathbf{q}$ is momentum and σ is spin (or polarization) component] satisfying the usual commutation/anticommutation relations

$$\alpha^-(\mathbf{q}, \sigma)\alpha^+(\mathbf{q}', \sigma') \pm \alpha^+(\mathbf{q}', \sigma')\alpha^-(\mathbf{q}, \sigma) = \delta(\mathbf{q} - \mathbf{q}')\delta_{\sigma\sigma'}. \tag{9}$$

Then interaction $V(t)$ as well as any regular operator can be written as a sum of normally ordered terms ($q \equiv (\mathbf{q}, \sigma)$)

$$
\begin{aligned}
V(t) &= \sum_{N=0}^{\infty}\sum_{M=0}^{\infty} O_{NM}(t), \\
O_{NM}(t) &= \int dq'_1 \dots dq'_N dq_1 \dots dq_M C_{NM}(q'_1 \dots q'_N q_1 \dots q_M; t) \times \\
&\quad \alpha^+(q'_1) \dots \alpha^+(q'_N)\alpha^-(q_1) \dots \alpha^-(q_M). \qquad (10)
\end{aligned}
$$

The pair of integers (N, M) will be referred to as the *type* of the operator O_{NM}. All known interactions conserve "number of fermions minus number of antifermions". Therefore without loss of generality we can take O_{NM} to be *bosonic*, i.e., having even number of fermion operators $N_f + M_f$. Eqs. (8) and time dependence (5) imply that the numerical factor in (10) has the form

$$
C_{NM}(t) = D_{NM}(q'_1 \dots q'_N q_1 \dots q_M)\delta(\sum_{i=1}^{N} \mathbf{q}'_i - \sum_{j=1}^{M} \mathbf{q}_j) e^{i(E_{NM} - i\epsilon)t},
$$

where the *energy function*

$$
E_{NM}(\mathbf{q}'_1 \dots \mathbf{q}'_N \mathbf{q}_1 \dots \mathbf{q}_M) \equiv \sum_{i=1}^{N} \sqrt{\mu_i^2 + \mathbf{q}'^2_i} - \sum_{j=1}^{M} \sqrt{\mu_j^2 + \mathbf{q}_j^2}
$$

is the difference of energies of particles created and destroyed by O_{NM}, and the *coefficient function* D_{NM} is rotationally invariant. The hypersurface of solutions

$$
E_{NM}(\mathbf{q}'_1 \dots \mathbf{q}'_N \mathbf{q}_1 \dots \mathbf{q}_M) = 0 \qquad (11)
$$

(if exists) is called the *energy shell* of the operator O_{NM}. It was shown in [9] that in order to satisfy condition (II) D_{NM} must not contain such singularities as momentum delta functions [9]. Operators with coefficient functions having the latter property will be called *connected*. Operators whose coefficient functions are bounded will be called *bounded*. Obviously, bounded operators are connected.

2.4. Multiple commutators and diagram technique

Let us prove that *multiple commutator of connected bosonic operators is connected*. First consider a single commutator of two connected bosonic operators of the form (10)

$$O^{(1)}O^{(2)} - O^{(2)}O^{(1)}. \tag{12}$$

To convert the commutator to the normal form we need to move $N^{(2)}$ creation operators in the factor $O^{(2)}$ to the left from $M^{(1)}$ destruction operators in $O^{(1)}$ in the first term and move $N^{(1)}$ creation operators in the factor $O^{(1)}$ to the left from $M^{(2)}$ destruction operators in $O^{(2)}$ in the second term. According to (9) each permutation of particle operators requires two actions. First, if two exchanged operators refer to the same particle we must add a new expression with these operators substituted by a delta function (pairing). This new expression should be also normally ordered which may result in new terms with more pairings, and so on. Second, we must change the sign of the original expression when two fermion operators change places. When the normal ordering is completed we obtain two terms $(O^{(1)}O^{(2)})_{np}$ and $(O^{(2)}O^{(1)})_{np}$ which differ from $O^{(1)}O^{(2)}$ and $O^{(2)}O^{(1)}$ only by the order of particle operators (no pairings) and a number of terms with pairings. Let us show that $(O^{(1)}O^{(2)})_{np} = (O^{(2)}O^{(1)})_{np}$. To obtain $(O^{(1)}O^{(2)})_{np}$ from the original product $O^{(1)}O^{(2)}$ requires $M_f^{(1)}N_f^{(2)}$ permutations of fermion operators. Similarly, obtaining $(O^{(2)}O^{(1)})_{np}$ from $O^{(2)}O^{(1)}$ requires $M_f^{(2)}N_f^{(1)}$ permutations of fermion operators. Additional $N_f^{(2)}N_f^{(1)}$ permutations of fermion creation operators and $M_f^{(2)}M_f^{(1)}$ permutations of fermion destruction operators in $(O^{(2)}O^{(1)})_{np}$ makes it equal to $(O^{(1)}O^{(2)})_{np}$ because the total number of permutations of fermion operators $(N_f^{(1} + M_f^{(1)})(N_f^{(2)} + M_f^{(2)})$ is even since both $N_f^{(1)} + M_f^{(1)}$ and $N_f^{(2)} + M_f^{(2)}$ are even. Therefore terms without pairings cancel in the commutator (12).

Let us now show that all remaining terms with pairings are connected. For each such term with I pairings there are $\mathcal{N} = N^{(1)} + N^{(2)} + M^{(1)} + M^{(2)}$ independent integration variables ($E = \mathcal{N} - 2I$ of them are used as arguments in particle operators) and $I + 2$ momentum delta functions in the integrand. We can integrate out $I + 1$ delta functions. This leaves $I - 1$ integrals by *loop* momenta which can be absorbed into definition of the coefficient function and one delta function which expresses conservation of the total momentum as required by the standard form of a connected operator (10). Repeating the same arguments by induction we see that multiple commutators are also connected.

The terms "connected" and "loop" have their origin in the diagram technique [9]. Factors O_{NM} can be represented by vertices with N outgoing and M incoming lines. Then each term $O^{(1)}O^{(2)}\dots O^{(V)}$ in the multiple commutator is represented by a connected (every two vertices can be joined by a sequence of internal lines corresponding to pairings) diagram.

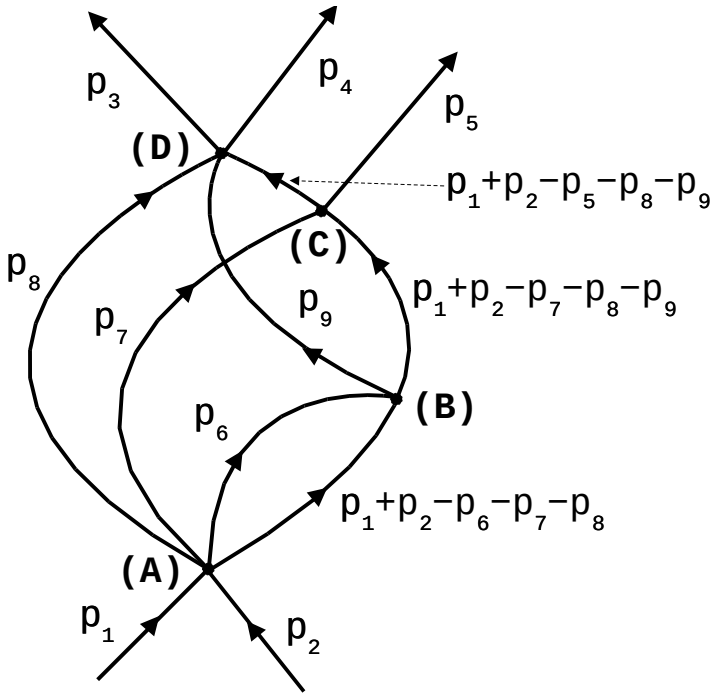

FIG. 1. A diagram representing one term in the product $O^{(D)}O^{(C)}O^{(B)}O^{(A)}$.

Consider for example a term in the product $O = O^{(D)}O^{(C)}O^{(B)}O^{(A)}$ whose diagram is shown in Fig. 1. This diagram has $V = 4$ vertices, $E = 5$ external lines, $I = 7$ internal lines, and $L = I-V+1 = 4$ independent loops. Note that all internal lines are oriented upwards because the couple of paired operators is always in the order $\alpha^-\alpha^+$. There are nine independent integration momenta: 5 momenta of external lines ($\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$, and $\mathbf{p}_5$) and 4 loop momenta ($\mathbf{p}_6$, $\mathbf{p}_7$, $\mathbf{p}_8$, and $\mathbf{p}_9$). It is always possible to chose integration variables in such a way that each loop momentum is present only in the internal lines forming the corresponding loop. For example, momentum $\mathbf{p}_9$ is confined to the loop BDCB. With this choice O acquires the general form (10) with the coefficient function involving four loop integrals

$$
\begin{aligned}
&D_{3,2}(p_3,p_4,p_5;p_1,p_2) = \\
&\int dp_6 dp_7 dp_8 dp_9 D_A(p_6,p_7,p_8,p_1+p_2-p_6-p_7-p_8;p_1,p_2) \times \\
&D_B(p_9,p_1+p_2-p_7-p_8-p_9;p_6,p_1+p_2-p_6-p_7-p_8) \times \\
&D_C(p_5,p_1+p_2-p_5-p_8-p_9;p_7,p_1+p_2-p_7-p_8-p_9) \times \\
&D_D(p_3,p_4;p_8,p_9,p_1+p_2-p_5-p_8-p_9).
\end{aligned}
$$

Consider for example the integral by the loop momentum $\mathbf{p}_9$ as $\mathbf{p}_9 \to \infty$ and all other momenta fixed. Taking into account that at large values of momentum $\omega_{\mathbf{p}} \approx |\mathbf{p}|$ and that energy function E_A does not depend on $\mathbf{p}_9$ we obtain

$$
\begin{aligned}
E_A &\to const, \\
E_B &= \omega_{\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_7-\mathbf{p}_8-\mathbf{p}_9} + \omega_{\mathbf{p}_9} - \omega_{\mathbf{p}_6} - \omega_{\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_6-\mathbf{p}_7-\mathbf{p}_8} \\
&\approx 2|\mathbf{p}_9| \to \infty, \\
E_C &= \omega_{\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_5-\mathbf{p}_8-\mathbf{p}_9} + \omega_{\mathbf{p}_5} - \omega_{\mathbf{p}_7} - \omega_{\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_7-\mathbf{p}_8-\mathbf{p}_9} \\
&\to const, \\
E_D &= \omega_{\mathbf{p}_3} + \omega_{\mathbf{p}_4} - \omega_{\mathbf{p}_8} - \omega_{\mathbf{p}_9} - \omega_{\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_5-\mathbf{p}_8-\mathbf{p}_9} \approx -2|\mathbf{p}_9| \to \infty.
\end{aligned}
$$

In the general case, as in the example above, any loop has a bottom vertex (B), a top vertex (D), and possibly a number of intermediate vertices (C). As the loop momentum goes to infinity, energy functions of the top and bottom vertices tend to infinity, i.e., move away from the energy shell. If corresponding coefficient functions (D_B and D_D in our case) decay suffi-

ciently rapidly in this limit, then the loop integral converges. The ultraviolet divergences in usual QFT theories can occur when coefficient functions do not fall off rapidly outside the energy shell.

2.5. Types of operators in the Fock space

Let us denote

$$O_{NM}(t) \circ \zeta \equiv \int dq'_1 \dots dq'_N dq_1 \dots dq_M C_{NM}(q'_1 \dots q'_N q_1 \dots q_M; t) \times$$
$$\zeta(q'_1 \dots q'_N q_1 \dots q_M) \alpha^+(q'_1) \dots \alpha^+(q'_N) \alpha^-(q_1) \dots \alpha^-(q_M)$$

the operator that differs from a regular O_{NM} (10) only by a numerical factor $\zeta(q'_1 \dots q'_N q_1 \dots q_M)$ in the integrand. Then the following equalities are valid

$$\frac{d}{dt} O_{NM}(t) = O_{NM}(t) \circ i E_{NM},$$
$$\underline{O_{NM}(t)} = O_{NM}(t) \circ \frac{-i}{E_{NM} + i\epsilon}, \tag{13}$$
$$\underbrace{O_{NM}} = O_{NM}(t) \circ 2\pi\delta(E_{NM}). \tag{14}$$

Operator (14) is time-independent, moreover it is non-zero only on the energy shell. We divide operators (10) into four classes (renorm, unphysical, decay, and physical) that have different behavior with respect to normal products, commutators, time derivatives and time integrals (see Table 1), so that any regular operator O has a unique decomposition $O = O^{ren} + O^{unp} + O^{dec} + O^{ph}$.

Renorm operators have either type (0,0) (a numerical constant) or type (1,1) in which case both created and destroyed particle belong to the same species.

Unphysical operators are defined to have types $(1, N)$, $(N, 1)$, $(0, N)$, or $(N, 0)$, with $N \geq 1$. In addition, we require that either energy shell does not exist, i.e., Eq. (11) has no solutions [which is obviously true for $(0, N)$ and $(N, 0)$ operators], or the operator vanishes on the energy shell. Then equation

$$\underbrace{O^{unp}} = 0 \tag{15}$$

holds for any regular operator O.

Decay operators have types $(1,N)$ or $(N,1)$ with $N \geq 1$ [operators (1,1) belong to the decay class if they destroy and create different particle species, otherwise they are renorm] and have a non-empty energy shell where they do not vanish. In the S-matrix approach, particle decay is described by terms G^{dec}, therefore condition (IV) and Eq. (7) imply

$$\underbrace{F^{dec}} = 0. \tag{16}$$

We will also assume that $V^{dec} = 0$ in theories considered here.

Physical operators have at least two creation operators and at least two destruction operators [type (N,M) with $N \geq 2$ and $M \geq 2$] so that the energy shell always exists. A physical operator O_{NM} whose coefficient function $D_{NM}(q'_1 \dots q'_N q_1 \dots q_M)$ falls off rapidly, at least exponentially, when arguments move away from the energy shell will be called *localized*. An example of such a function is $D_{NM} = e^{-E^2_{NM}}$. Any physical operator O^{ph} yields zero when acting on the vacuum or one-particle states.

$$O^{ph}|0\rangle = O^{ph}\alpha^+(q)|0\rangle = 0. \tag{17}$$

Conversely, any Hermitian operator satisfying Eq. (17) is physical [1].

TABLE 1.

Operations with regular operators in the Fock space.

Type of operator O [a]	Product or commutator of O with						
	P	U	D	R	$\frac{dO}{dt}$	$\underline{O}$	$\underbrace{O}$
P	P	P+U+D	P+U+D	P	P	P	P
U	P+U+D	P+U+D+R	P+U+D+R	U	U	U	0
D	P+U+D	P+U+D+R	P+U+D+R	D	D	D	D
R	P	U	D	R	0	NR	∞

[a] Notation: P=physical, U=unphysical, D=decay, R=renorm, NR=non-regular

2.6. Finiteness of the scattering operators

According to condition (III), the time limits in (2) and (3) must exist and be finite. In particular, this implies that for finite t in the limit $t_1 \to -\infty$

$$e^{iH_0 t}W(0,t_1)e^{-iH_0 t} \;=\; e^{iH_0 t}e^{iHt_1}e^{-iH_0 t_1}e^{-iH_0 t}$$

$$
\begin{aligned}
&= e^{iH_0 t} e^{-iHt} e^{+iH(t_1+t)} e^{-iH_0(t_1+t)} \\
&= e^{iH_0 t} e^{-iH(t-t_1)} e^{-iH_0 t_1} \\
&= W(t, t_1).
\end{aligned}
$$

Therefore $W(t, -\infty)$ is regular and finite. A product of any number of regular operators is regular. The time derivative of a regular operator is regular and does not have a renorm part (see Table 1). Thus operator $F(t) = i\frac{d}{dt}[logW(t, -\infty)]$ is regular, finite, and

$$F^{ren} = 0. \tag{18}$$

Eq. (18) together with (15) and (16) implies that $G = \underbrace{F}$ is purely physical $G = G^{ph}$. Any power of a Hermitian physical operator has property (17), is Hermitian, and hence physical. Therefore in the decomposition

$$S = e^{iG} = 1 + iG - \frac{1}{2}G^2 + \ldots$$

the first term is renorm [type(0,0)] and all other terms are physical: $S^{ren} = 1$, $S^{unp} + S^{dec} = 0$, and due to (17) there is no scattering in the vacuum and one-particle states, which is exactly the condition used in standard renormalization theory to ensure that poles and residues of free one-particle propagators are not affected by the renormalized interaction [9].

3. AN EXAMPLE: QUANTUM ELECTRODYNAMICS

As an example of a theory satisfying all conditions (I) - (IV) we consider QED with five kinds of particles and corresponding particle operators: electrons ($a^{\pm}$) and positrons ($b^{\pm}$) with mass m; protons ($d^{\pm}$) and antiprotons ($f^{\pm}$) with mass M; and massless photons ($c^{\pm}$).

Using standard QFT approach [9] we obtain the Hamiltonian with counterterms

$$
\begin{aligned}
H(t) &= H_0 + V(t) \\
&= H_0 + V_1(t) + V_2^{Coul}(t) + R_2 + U_3(t) + \ldots
\end{aligned} \tag{19}
$$

[here and in what follows we denote the power of the coupling constant e (the perturbation order of an operator) by a subscript]. The free Hamiltonian is

$$
\begin{aligned}
H_0 &= \int d\mathbf{p}\omega_{\mathbf{p}} \sum_{\sigma=\pm 1/2} [a^+(\mathbf{p},\sigma)a^-(\mathbf{p},\sigma) + b^+(\mathbf{p},\sigma)b^-(\mathbf{p},\sigma)] \\
&+ \int d\mathbf{p}\Omega_{\mathbf{p}} \sum_{\sigma=\pm 1/2} [d^+(\mathbf{p},\sigma)d^-(\mathbf{p},\sigma) + f^+(\mathbf{p},\sigma)f^-(\mathbf{p},\sigma)] \\
&+ \int_{k\neq 0} d\mathbf{k}|\mathbf{k}| \sum_{\tau=\pm 1} c^+(\mathbf{k},\tau)c^-(\mathbf{k},\tau),
\end{aligned}
$$

where $\omega_{\mathbf{p}} = \sqrt{m^2+\mathbf{p}^2}$ and $\Omega_{\mathbf{p}} = \sqrt{M^2+\mathbf{p}^2}$. The interaction V is constructed from quantum fields

$$
\begin{aligned}
a_\mu(\mathbf{x},t) &= (2\pi)^{-3/2} \int_{k\neq 0} \frac{d\mathbf{k}}{\sqrt{2|\mathbf{k}|}} \sum_\tau [e^{i\mathbf{k}\mathbf{x}-i|\mathbf{k}|t}\epsilon_\mu(\mathbf{k},\tau)c^-(\mathbf{k},\tau) \\
&+ e^{-i\mathbf{k}\mathbf{x}+i|\mathbf{k}|t}\epsilon^*_\mu(\mathbf{k},\tau)c^+(\mathbf{k},\tau)], \\
\psi(\mathbf{x},t) &= (2\pi)^{-3/2} \int d\mathbf{p} \sum_\sigma [e^{i\mathbf{p}\mathbf{x}-i\omega_p t}u(\mathbf{p},\sigma)a^-(\mathbf{p},\sigma) \\
&+ e^{-i\mathbf{p}\mathbf{x}+i\omega_p t}v(\mathbf{p},\sigma)b^+(\mathbf{p},\sigma)], \\
\xi(\mathbf{x},t) &= (2\pi)^{-3/2} \int d\mathbf{p} \sum_\sigma [e^{i\mathbf{p}\mathbf{x}-i\Omega_p t}U(\mathbf{p},\sigma)d^-(\mathbf{p},\sigma) \\
&+ e^{-i\mathbf{p}\mathbf{x}+i\Omega_p t}V(\mathbf{p},\sigma)f^+(\mathbf{p},\sigma)].
\end{aligned}
$$

To simplify the notation we assume summation over repeating indices and introduce operators

$$
\begin{aligned}
A^-_\alpha(\mathbf{p},\sigma) &= u_\alpha(\mathbf{p},\sigma)a^-(\mathbf{p},\sigma), \\
A^+_\alpha(\mathbf{p},\sigma) &= \overline{u}_\alpha(\mathbf{p},\sigma)a^+(\mathbf{p},\sigma), \\
D^-_\alpha(\mathbf{p},\sigma) &= U_\alpha(\mathbf{p},\sigma)d^-(\mathbf{p},\sigma), \\
D^+_\alpha(\mathbf{p},\sigma) &= \overline{U}_\alpha(\mathbf{p},\sigma)d^+(\mathbf{p},\sigma), \\
C^-_{\alpha\beta}(\mathbf{k},\tau) &= \epsilon_\mu(\mathbf{k},\tau)\gamma^\mu_{\alpha\beta}c^-(\mathbf{k},\tau), \\
C^+_{\alpha\beta}(\mathbf{k},\tau) &= \epsilon^*_\mu(\mathbf{k},\tau)\gamma^\mu_{\alpha\beta}c^+(\mathbf{k},\tau).
\end{aligned}
$$

Then writing the total current as a sum of the electron and proton currents $j^\mu = -ie\overline{\psi}\gamma^\mu\psi + ie\overline{\xi}\gamma^\mu\xi$, we obtain the first-order interaction at $t=0$

$$V_1(0) = -\int d\mathbf{x} j^\mu(\mathbf{x},0) a_\mu(\mathbf{x},0) = ie(2\pi)^{-3/2} \int_{k\neq 0} \frac{d\mathbf{p}d\mathbf{k}}{\sqrt{2|\mathbf{k}|}} \sum_{\sigma\sigma'\tau}$$
$$(A^+_\alpha(\mathbf{p}+\mathbf{k},\sigma)A^-_\beta(\mathbf{p},\sigma')C^-_{\alpha\beta}(\mathbf{k},\tau) + A^+_\alpha(\mathbf{p}-\mathbf{k},\sigma)C^+_{\alpha\beta}(\mathbf{k},\tau)A^-_\beta(\mathbf{p},\sigma')$$
$$-D^+_\alpha(\mathbf{p}+\mathbf{k},\sigma)D^-_\beta(\mathbf{p},\sigma')C^-_{\alpha\beta}(\mathbf{k},\tau) - D^+_\alpha(\mathbf{p}-\mathbf{k},\sigma)C^+_{\alpha\beta}(\mathbf{k},\tau)D^-_\beta(\mathbf{p},\sigma')$$
$$+\ldots) \qquad (20)$$

which is unphysical. For brevity, twelve terms involving operators of antiparticles are denoted by dots in (20). The third term on the right hand side of (19)

$$V_2^{Coul}(0) = \int d\mathbf{x}d\mathbf{y} \frac{j_0(\mathbf{x},0)j_0(\mathbf{y},0)}{8\pi|\mathbf{x}-\mathbf{y}|} \qquad (21)$$

is needed to compensate the non-covariance of the photon propagator [9]. When expressed through particle operators, this direct fermion-fermion interaction has physical, unphysical, and renorm parts. R_2 in (19) includes renorm electron and proton self-energy (a^+a^-, b^+b^-, d^+d^-, and f^+f^-) and vacuum polarization (c^+c^-) counterterms. U_3 are unphysical charge renormalization counterterms having the same operator structure as V_1. Dots in (19) denote higher order counterterms (having the same operator structure as R_2 or U_3) needed to ensure that S is finite and consistent with non-relativistic scattering at low energies. These two conditions appear to be sufficient to unambiguously determine infinite counterterms R_i and U_i in all orders and to obtain the S-operator without ultraviolet divergences.

The next step in obtaining the exact S-operator agreeing with experiment – removal of infrared infinities [9, 15] – is beyond the scope of this work. The easiest way to avoid infrared divergences in our calculations is to assign a fictitious small mass to all zero-mass particles, i.e., photons. This, in particular, removes the $\mathbf{k} = 0$ singularities from interaction operators (20).

The relativistic invariance and cluster separability of QED were discussed in detail in [9, 16]. There are no decay terms in the interaction operator V, and all decay terms in G are forbidden by the charge, energy, and angular momentum conservation laws. It was shown in [17] that operators describing the decay of a photon into odd number of photons [e.g., $c^+(\mathbf{k}_1,\tau_1)c^+(\mathbf{k}_2,\tau_2)c^+(\mathbf{k}_3,\tau_3)c^-(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3,\tau_4)$] are zero on the energy shell (when momenta $\mathbf{k}_i$ are collinear). Therefore they are unphysical and do not contribute to G. Thus QED satisfies all conditions (I) - (IV).

4. CONSTRUCTION OF THE CLOTHED PARTICLE HAMILTONIAN

4.1. Formulation of the problem

There are two difficulties with the current formulation of QFT presented above. First, coefficient functions in the interaction Hamiltonian V usually do not tend to zero sufficiently rapidly as their arguments go to infinity. Therefore, as discussed in section 2.4, loop integrals may be divergent. Second, interaction V usually contains unphysical terms: $V^{unp} \neq 0$. The reconciliation of this property with the absence of renorm terms in F [see eq. (18) which is a necessary requirement in any theory with a finite S-matrix] is a non-trivial task: Unphysical terms in V give rise to renorm terms in F due to commutators like $[\underline{V^{unp}}, V^{unp}]^{ren}$. To ensure the absence of such terms we need to assume that $V^{ren} \neq 0$ and that interaction V has such a balance between unphysical and renorm parts that all renorm terms in F cancel. Such cancellation is achieved in current renormalization theory by introducing (usually infinite) unphysical and renorm counterterms in V. The presence of V^{unp} and V^{ren} leads to a number of spurious effects: Unphysical terms in V create clouds of virtual particles and particle-antiparticle pairs out of vacuum and around each physical particle. Renorm terms in V modify (often by an infinite amount) particle masses and energies.

These difficulties indicate that interaction Hamiltonians in QFT are not correct and should be modified. For example, renorm terms in F can be avoided if interaction Hamiltonian is made physical (such Hamiltonians are usually referred to as clothed particle Hamiltonians [1]). Indeed, for physical interactions, property (18) follows automatically because commutators and time integrals of physical operators in (6) are physical (see Table 1). Moreover, a physical interaction acts only when two or more particles are present [see Eq. (17)], so that there are no clouds of virtual particles and self-interaction effects [1] .

Of course, modification of interactions cannot be done arbitrarily. Since S-operator computed using interaction V is very accurate, we demand that any modification of the QFT Hamiltonian should preserve the S-operator. For example, Hamiltonians H and

$$H^r = e^{i\Phi} H e^{-i\Phi}, \tag{22}$$

related by a unitary transformation $e^{i\Phi}$, yield the same scattering as long as conditions

$$\lim_{t\to\pm\infty} e^{iH_0 t} \Phi e^{-iH_0 t} = 0 \tag{23}$$

are satisfied. Indeed, in the limit $t \to +\infty, t' \to -\infty$ we obtain [10]

$$\begin{aligned} S^r &= e^{iH_0 t} e^{-iH^r(t-t')} e^{-iH_0 t'} \\ &= e^{iH_0 t}(e^{i\Phi} e^{-iH(t-t')} e^{-i\Phi}) e^{-iH_0 t'} \\ &= (e^{iH_0 t} e^{i\Phi} e^{-iH_0 t}) e^{iH_0 t} e^{-iH(t-t')} e^{-iH_0 t'} (e^{iH_0 t'} e^{-i\Phi} e^{-iH_0 t}) \\ &= e^{iH_0 t} e^{-iH(t-t')} e^{-iH_0 t'} \\ &= S. \end{aligned}$$

Although Hamiltonians H and H^r are scattering equivalent they are not physically equivalent: they result in different time evolutions.

Let us now state the goal of this paper in a more formal way. We will consider a general renormalizable quantum field theory with instant form interaction V having a non-zero unphysical part and all infinite counterterms required to ensure that S-operator satisfies conditions (I)-(IV). In particular this implies that

$$F^{dec} = 0, \tag{24}$$
$$F^{ren} = 0. \tag{25}$$

We will assume that all operators can be written as expansions in powers of the coupling constant and introduce a simplifying technical condition (which is valid, for example, in QED)

$$V_1^{ph} = V_1^{ren} = V_1^{dec} = 0. \tag{26}$$

We will be looking for the clothed particle Hamiltonian H^r in the form (22)

$$\begin{aligned} H^r &= H_0 + V^r \\ &= e^{i\Phi}(H_0 + V)e^{-i\Phi} \\ &= (H_0 + V) + i[\Phi, (H_0 + V)] - \frac{1}{2!}[\Phi, [\Phi, (H_0 + V)]] + \dots \end{aligned} \tag{27}$$

Our goal is to prove that there exists a *clothing transformation* $e^{i\Phi}$ such that

(A) V^r is Hermitian;
(B) V^r is physical: (B1) $(V^r)^{unp} = 0$, (B2) $(V^r)^{dec} = 0$, and (B3) $(V^r)^{ren} = 0$;
(C) V^r is finite: (C1) V^r is finite on the energy shell and (C2) V^r rapidly falls off outside the energy shell (is localized);
(D) V^r is connected;
(E) S-matrices calculated with V and V^r are identical;
(F) The transformed theory belongs to the relativistic instant form dynamics, in particular, (F1) $[V^r, \mathbf{P}_0] = [V^r, \mathbf{J}_0] = 0$.

4.2. Clothing transformation in a regulated theory

We begin with introducing a regularization, e.g., a cutoff at large integration momenta so that all loop integrals converge and all operators are bounded. (Singularities at $|\mathbf{k}| = 0$ like in (20) can be removed by assigning a small mass to photons as discussed in section 3.) We assume that regulated interaction V remains Hermitian, connected, and Eqs. (8) hold, however the regularization may destroy relativistic invariance. First we will prove that in this regulated theory a bounded operator Φ in (27) can be chosen so that conditions (A), (B1), (C), (D), and (F1) are satisfied in all orders. Multiplying (27) by $e^{iH_0 t}$ from the left and $e^{-iH_0 t}$ from the right we enforce the time dependence (5) characteristic for regular operators. Using property $[\Phi, H_0] = id\Phi/dt$ and collecting together terms of equal order we obtain an infinite set of equations

$$V_i^r = -d\Phi_i/dt + Z_i + V_i, \tag{28}$$

where Z_i is a sum of multiple commutators involving Φ_j, $d\Phi_j/dt$ and V_j from lower orders ($1 \le j < i$), for example,

$$Z_2 = i[\Phi_1, V_1] - \frac{i}{2!}[\Phi_1, d\Phi_1/dt]. \tag{29}$$

We solve equations (28) (i.e., select operators Φ_i so that properties (A), (B1), (C), (D), and (F1) are satisfied) using mathematical induction. In the first order $Z_1 = 0$, and we can choose $\Phi_1^{ph} = \Phi_1^{dec} = \Phi_1^{ren} = 0$ and

$$\Phi_1^{unp} = \underline{V_1}. \tag{30}$$

Then $V_1^r = 0$ so that conditions (A), (B1), (C), (D), (F1) are trivially satisfied. V_1 is unphysical (26) and therefore either does not have energy

shell in which case $E_{V_1} \neq 0$ or vanishes simultaneously with its energy function E_{V_1} on the energy shell. In both cases $\Phi_1 = V_1 \circ (-iE_{V_1}^{-1})$ is bounded.

Assume that in all orders j $(j < i)$ we have constructed Φ_j, so that (A), (B1), (C), (D), and (F1) are satisfied, and Φ_j and Z_j are bounded. Then, to satisfy these conditions in order i we can choose $\Phi_i^{dec} = \Phi_i^{ren} = 0$ and

$$\Phi_i^{unp} = \underline{Z_i^{unp}} + \underline{V_i^{unp}}, \tag{31}$$

$$\Phi_i^{ph} = \underline{(Z_i^{ph} + V_i^{ph}) \circ (1 - \zeta_i)} + \Sigma_i, \tag{32}$$

where Σ_i is any localized bounded physical Hermitian operator commuting with $\mathbf{J}_0$ and $\mathbf{P}_0$, and ζ_i is any real localized bounded rotationally invariant function such that $\zeta_i = 1$ on the energy shell. Substituting (31) to (28) we see that (B1) is satisfied.

If A and B are Hermitian and commute with $\mathbf{P}_0$ and $\mathbf{J}_0$ then $i[A, B]$, $\underline{A}$, and $\frac{dA}{dt} \equiv -i[A, H_0]$ are Hermitian and commute with $\mathbf{P}_0$ and $\mathbf{J}_0$ as well. Therefore operators F, Φ, and V^r expressed through multiple commutators, time integrals and time derivatives of V are Hermitian, translationally and rotationally invariant, and, as shown in section 2.4, connected [conditions (A), (D), and (F1)]. Z_i is bounded due to regularization, and time integrations in (31) and (32) do not introduce singularities because the integrands either do not have energy shell or vanish on the energy shell. Therefore V_i^r is bounded [condition (C)]. Moreover Φ_i and $\Phi = \sum_i \Phi_i$ are bounded and we can apply the Riemann-Lebesgue lemma to show that Eq. (23) holds and that condition (E) is also satisfied [10]. Now we are in a position to prove (B2) and (B3). Indeed, if $(V^r)^{dec} \neq 0$ then $F^{dec} \neq 0$ in contradiction to (24). If $(V^r)^{ren} \neq 0$ then $F^{ren} \neq 0$ in contradiction to (25).

4.3. The limit of removed regularization

We assume that removing regularization does not affect the validity of (A), (B), (C2), (D), (E), and (F1). However, we still need to prove (C1) and (F). Inserting V^r instead of V in (6) and (7) we obtain equality

$$\begin{aligned} G &= \underbrace{F_2} + \underbrace{F_3} + \underbrace{F_4} + \dots \\ &= \underbrace{V_2^r} + \underbrace{V_3^r} + \underbrace{V_4^r} + \frac{i}{2}\underbrace{[V_2^r, V_2^r]} \dots \end{aligned}$$

which is valid even in the non-regulated theory. Comparing these two forms order by order and using Eqs. (15), (24), and (25) we obtain a set

of conditions for V_i^r on the energy shell where F_i^{ph} are finite and known [they can be computed from V using Eq. (6)]

$$\underbrace{V_2^r} = \underbrace{F_2^{ph}}, \tag{33}$$

$$\underbrace{V_3^r} = \underbrace{F_3^{ph}}, \tag{34}$$

$$\underbrace{V_i^r} = \underbrace{F_i^{ph}} + \underbrace{Q_i}, \ i \geq 4. \tag{35}$$

Q_i denotes a sum of multiple commutators of V_j^r from lower orders ($2 \leq j \leq i-2$) with time integrations, for example $Q_4 = \frac{i}{2}[V_2^r, V_2^r]$.

V_2^r and V_3^r are finite on the energy shell due to (33), (34), and finiteness of F. Let us now use mathematical induction and assume that for all j, such that $j < i$, operators V_j^r satisfy (C1). We wish to prove that the same property holds for Q_i, and therefore for V_i^r. Let us first consider operators $\tilde{Q}_i$ which differ from Q_i only by the absence of time integrations. As discussed in section 2.4, $\tilde{Q}_i$ can be represented by a connected diagram and the convergence of loop integrals is determined by the asymptotic behavior of vertex coefficient functions. All coefficient functions in V_j^r are localized [condition (C2)], i.e., they fall off rapidly outside the energy shell. Therefore all loop integrals are convergent. This conclusion is only strengthened when we take into account time integrals in Q_i, because they introduce inverse energy factors (13) which make convergence properties even better.

Applying the obtained clothing transformation $e^{i\Phi}$ to all generators of the Poincaré group and using (F1) we obtain

$$\{\mathbf{P}_0, \mathbf{J}_0, \mathbf{K}^r, H^r\} = e^{i\Phi}\{\mathbf{P}_0, \mathbf{J}_0, \mathbf{K}, H\}e^{-i\Phi},$$

so that transformed operators satisfy condition (F) [8]. This concludes the proof of existence of a physical finite relativistically invariant clothed particle Hamiltonian H^r in a theory whose S-operator satisfies conditions (I) - (IV).

4.4. General properties of H^r

It should be noted that the above construction does not allow to obtain full information about V^r. Functions ζ_i and σ_i (the coefficient function of Σ_i) can be chosen rather arbitrarily. They just need to be localized, bounded, rotationally invariant, and $\zeta_i = 1$ on the energy shell. This uncertainty reflects the one-to-many correspondence between the S-operator and Hamiltonians [10, 11]. It means that there is a class of finite physical interactions $\{V^r\}$ all of which can be used for S-matrix calculations without

encountering divergent integrals. However, there should exist the unique correct interaction $V^r \in \{V^r\}$ (and thus unique correct functions ζ_i and σ_i) such that $H^r = H_0 + V^r$ is the exact time evolution generator. As we are not aware of any theoretical conditions allowing to determine the off-shell behavior of ζ_i and σ_i, these functions need to be fitted to experimental measurements. Such experiments are bound to be rather challenging because they must go beyond usual information contained in the S-operator (scattering cross-sections, energies and lifetimes of bound states, etc.) and should be capable of measuring radiative corrections to the wave functions and time evolution of observables (more on this at the end of the next section).

However, there are some properties of H^r which are independent on the choice of ζ_i and σ_i: Since Hamiltonians H and H^r are related by a unitary transformation and yield the same S-matrices, they have the same energies (eigenvalues) and widths of bound states (the corresponding eigenfunctions are, of course, different). Due to (33) - (35), the choice of ζ_i and σ_i does not affect V_2^r and V_3^r [and some higher order terms for which Q_i on the right hand side of (35) is zero] on the energy shell. These terms can be explicitly calculated near the energy shell (see next section). Note that the on-shell Hamiltonian is exactly what we need for applications, because in all processes (excluding very short virtual events) the total energy is approximately conserved at all times.

5. CLOTHED PARTICLE HAMILTONIAN IN QED

5.1. Electron-proton interaction

The operator structure and physical meaning of some of the terms in the perturbation series representing the clothed particle interaction V^r in QED are reported in Table 2. Bold numbers in the third column indicate perturbation orders in which the operators can be unambiguously obtained near the energy shell independent on the choice of functions ζ_i and σ_i.

Consider for example the electron-proton interaction term at $t = 0$

$$V^r[d^+a^+d^-a^-] = \int_{\mathbf{k}\neq 0} d\mathbf{p}d\mathbf{q}d\mathbf{k} \sum_{\sigma\sigma'\sigma''\sigma'''} v^{\alpha\beta\gamma\delta}(\mathbf{pqk}, \sigma\sigma'\sigma''\sigma''') \times$$
$$D_\alpha^+(\mathbf{p}-\mathbf{k},\sigma)A_\gamma^+(\mathbf{q}+\mathbf{k},\sigma')D_\beta^-(\mathbf{p},\sigma'')A_\delta^-(\mathbf{q},\sigma''').$$

From (20), (21), (28) - (30), and (32) the 2nd order approximation for the coefficient function is

TABLE 2.

Examples of terms in the clothed particle interaction V^r in QED.

Operator	Physical meaning	Perturbation Orders
	Elastic potentials	
$a^+a^+a^-a^-$	electron-electron	$\mathbf{2}, 4, 6, \ldots$
$d^+a^+d^-a^-$	electron-proton	$\mathbf{2}, 4, 6, \ldots$
$a^+c^+a^-c^-$	electron-photon	$\mathbf{2}, 4, 6, \ldots$
$d^+c^+d^-c^-$	proton-photon	$\mathbf{2}, 4, 6, \ldots$
$c^+c^+c^-c^-$	photon-photon	$4, 6, \ldots$
$a^+a^+a^+a^-a^-a^-$	3-electron	$4, 6, \ldots$
	Inelastic potentials	
$a^+b^+c^-c^-$	$e^- - e^+$ pair creation	$\mathbf{2}, 4, 6, \ldots$
$c^+c^+a^-b^-$	$e^- - e^+$ annihilation	$\mathbf{2}, 4, 6, \ldots$
$d^+a^+c^+d^-a^-$	$p^+ - e^-$ bremsstrahlung	$\mathbf{3}, 5, \ldots$

$$v_2^{\alpha\beta\gamma\delta}(\mathbf{pqk}, \sigma\sigma'\sigma''\sigma''') = -i\sigma_2^{\alpha\beta\gamma\delta}(\mathbf{pqk}, \sigma\sigma'\sigma''\sigma''')E(\mathbf{p},\mathbf{q},\mathbf{k})$$
$$+\frac{e^2}{4(2\pi)^3}\zeta_2(\mathbf{pqk}, \sigma\sigma'\sigma''\sigma''')[T(\mathbf{p},\mathbf{q},\mathbf{k})(\delta_{ln} - \frac{k_l k_n}{\mathbf{k}^2})\frac{\gamma^l_{\alpha\beta}\gamma^n_{\gamma\delta}}{|\mathbf{k}|} - \frac{4\gamma^0_{\alpha\beta}\gamma^0_{\gamma\delta}}{\mathbf{k}^2}],$$

where indices l and n run over values x, y, and z,

$$E(\mathbf{p},\mathbf{q},\mathbf{k}) = \Omega_{\mathbf{p}-\mathbf{k}} + \omega_{\mathbf{q}+\mathbf{k}} - \Omega_{\mathbf{p}} - \omega_{\mathbf{q}},$$
$$T(\mathbf{p},\mathbf{q},\mathbf{k}) = \frac{1}{\omega_{\mathbf{q}+\mathbf{k}} - \omega_{\mathbf{q}} + |\mathbf{k}|} - \frac{1}{\omega_{\mathbf{q}+\mathbf{k}} - \omega_{\mathbf{q}} - |\mathbf{k}|}$$
$$+\frac{1}{\Omega_{\mathbf{p}-\mathbf{k}} - \Omega_{\mathbf{p}} + |\mathbf{k}|} - \frac{1}{\Omega_{\mathbf{p}-\mathbf{k}} - \Omega_{\mathbf{p}} - |\mathbf{k}|}.$$

Near the energy shell ($\zeta_2 \approx 1$, $E \approx 0$)

$$v_2 \approx -\frac{e^2}{(2\pi)^3}\frac{\gamma^0_{\alpha\beta}\gamma^0_{\gamma\delta}}{\mathbf{k}^2} \tag{36}$$
$$+\frac{e^2}{2(2\pi)^3}(\frac{1}{\omega_{\mathbf{q}+\mathbf{k}} - \omega_{\mathbf{q}} + |\mathbf{k}|} - \frac{1}{\omega_{\mathbf{q}+\mathbf{k}} - \omega_{\mathbf{q}} - |\mathbf{k}|})(\delta_{ln} - \frac{k_l k_n}{\mathbf{k}^2})\frac{\gamma^l_{\alpha\beta}\gamma^n_{\gamma\delta}}{|\mathbf{k}|}$$

does not depend on ζ_2 and σ_2.

5.2. The hydrogen atom

The total electron-proton interaction operator $V^r[d^+a^+d^-a^-]$ leaves invariant the "one electron + one proton" subspace $\mathcal{H}_{pe}$ of the Fock space. Therefore we may try to find the bound states of the hydrogen atom as eigenstates of the mass operator

$$\begin{aligned} M|\Psi_\varepsilon> &\equiv \sqrt{(H^r)^2-\mathbf{P}_0^2}|\Psi_\varepsilon> \\ &\approx \sqrt{(H_0+V^r[d^+a^+d^-a^-])^2-\mathbf{P}_0^2}|\Psi_\varepsilon>=\varepsilon|\Psi_\varepsilon> \end{aligned}$$

with $\varepsilon < m+M$. This equation assumes more familiar form of the stationary Schrödinger equation in the center of mass frame ($\mathbf{P}_0 = 0$)

$$(H_0+V^r[d^+a^+d^-a^-])|\Psi_\varepsilon>=\varepsilon|\Psi_\varepsilon>. \tag{37}$$

Using the "plane wave" decomposition

$$|\Psi_\varepsilon\rangle = \int d\mathbf{p}d\mathbf{q}\sum_{\sigma\sigma'}\Psi_\varepsilon(\mathbf{p},\sigma;\mathbf{q},\sigma')d^+(\mathbf{p},\sigma)a^+(\mathbf{q},\sigma')|0\rangle$$

valid in $\mathcal{H}_{pe}$, Eq. (37) can be transformed into an integral equation with respect to the wave function $\Psi_\varepsilon(\mathbf{p},\sigma;\mathbf{q},\sigma')$

$$\begin{aligned} &\varepsilon\Psi_\varepsilon(\mathbf{p},\sigma;\mathbf{q},\sigma') = (\omega_{\mathbf{p}}+\Omega_{\mathbf{q}})\Psi_\varepsilon(\mathbf{p},\sigma;\mathbf{q},\sigma') \\ &+ \int_{\mathbf{k}\neq 0}d\mathbf{k}\sum_{\sigma''\sigma'''}Y(\mathbf{p},\mathbf{q},\mathbf{k},\sigma\sigma'\sigma''\sigma''')\Psi_\varepsilon(\mathbf{p}+\mathbf{k},\sigma'';\mathbf{q}-\mathbf{k},\sigma'''), \end{aligned} \tag{38}$$

where

$$\begin{aligned} Y(\mathbf{p},\mathbf{q},\mathbf{k},\sigma\sigma'\sigma''\sigma''') \equiv v^{\alpha\beta\gamma\delta}(\mathbf{p}+\mathbf{k},\mathbf{q}-\mathbf{k},\mathbf{k},\sigma\sigma'\sigma''\sigma''')\times \\ \overline{U}_\alpha(\mathbf{p},\sigma)\overline{u}_\gamma(\mathbf{q},\sigma')U_\beta(\mathbf{p}+\mathbf{k},\sigma'')u_\delta(\mathbf{q}-\mathbf{k},\sigma'''). \end{aligned} \tag{39}$$

It would be interesting to demonstrate that in the second-order approximation (36) equation (38) yields the fine structure of the hydrogen spectrum,

i.e., provides an alternative to the familiar Dirac equation. Note that unlike Dirac spinors, the wave functions $\Psi_\varepsilon(\mathbf{p},\sigma;\mathbf{q},\sigma')$ have a direct interpretation as probability amplitudes in the two-particle momentum-spin space. In the non-relativistic limit ($|\mathbf{k}|, |\mathbf{p}|, |\mathbf{q}| \ll m$) the following approximations are valid [9]:

$$\overline{u}_\alpha(\mathbf{p}_1,\sigma_1)\gamma^n_{\alpha\beta}u_\beta(\mathbf{p}_2,\sigma_2) \approx \overline{U}_\alpha(\mathbf{p}_1,\sigma_1)\gamma^n_{\alpha\beta}U_\beta(\mathbf{p}_2,\sigma_2) \approx 0,$$

where $n = x, y$, or z,

$$\overline{u}_\alpha(\mathbf{p}_1,\sigma_1)\gamma^0_{\alpha\beta}u_\beta(\mathbf{p}_2,\sigma_2) \approx \overline{U}_\alpha(\mathbf{p}_1,\sigma_1)\gamma^0_{\alpha\beta}U_\beta(\mathbf{p}_2,\sigma_2) \approx \delta_{\sigma_1\sigma_2},$$

$\omega_\mathbf{p} \approx m+\mathbf{p}^2/(2m)$, and $\Omega_\mathbf{p} \approx M+\mathbf{p}^2/(2M)$. Substituting these approximations and (36) to (39) and (38) and performing the Fourier transformation we see that (38) reduces to the usual Schrödinger equation describing energy levels and stationary state functions of the hydrogen atom as expected

$$\begin{aligned}&(-\frac{1}{2m}\nabla^2_\mathbf{r} - \frac{1}{2M}\nabla^2_\mathbf{R} - \frac{e^2}{4\pi|\mathbf{r}-\mathbf{R}|})\Psi_\varepsilon(\mathbf{r},\sigma;\mathbf{R},\sigma')\\ =\ &(\varepsilon - m - M)\Psi_\varepsilon(\mathbf{r},\sigma;\mathbf{R},\sigma').\end{aligned}$$

Adding 3rd and higher order interactions to the 2nd order approximation in (38) would affect its solutions in three major ways: broadening and shift of energy levels and modifications of eigenfunctions Ψ_ε. The largest contribution to the broadening is provided by the 3rd order operators $d^+a^+d^-a^-c^-$ and $d^+a^+c^+d^-a^-$ which describe absorption and spontaneous emission of a photon by the atom. These terms in V^r, rather than "zero-point vibrations of the electromagnetic field", are responsible for the instability of excited atomic levels.

The Lamb shifts of energy levels are due to $V^r_4[d^+a^+d^-a^-]$ and higher order terms in $V^r[d^+a^+d^-a^-]$. Unlike $V^r_2[d^+a^+d^-a^-]$, these terms do not have the $|\mathbf{k}|^{-2}$ singularity at $|\mathbf{k}| = 0$ which is responsible for the long-range $-e^2/(4\pi|\mathbf{r}-\mathbf{R}|)$ behavior of the potential in the position space; this would be inconsistent with the low-energy classical limit of the Coulomb scattering. Thus the Coulomb asymptotics of the potential is insensitive to the choice of ζ_i and σ_i in all orders. However, this choice does affect the short range behavior of the potential and the shape of the wave functions

Ψ_ε. For example, according to (35), the 4th order operator $V_4^r[d^+a^+d^-a^-]$ depends on second order functions ζ_2 and σ_2 through commutators like

$$[\underline{V_2^r[d^+a^+d^-a^-]}, V_2^r[d^+a^+d^-a^-]].$$

The radiative corrections to the wave functions do not affect scattering experiments, but they are reflected in the dynamics of electron wave packets formed as superpositions of states $|\Psi_\varepsilon\rangle$. Observations of such dynamics may reveal important information about the Hamiltonian H^r and allow, in principle, to fit ζ_i and σ_i to experiments. However time-resolved studies of the wave packets [18] are associated with serious difficulties: The low-lying atomic levels (where the short-range effect of higher order interactions is most pronounced) have large energy separations, and the dynamical time scale is in the femtosecond range. Observing small radiative corrections with such a resolution is a challenging experimental task.

6. SUMMARY

In relativistic quantum field theories the non-renormalized interactions can be used only in the lowest non-vanishing perturbation order: in higher orders the calculated S-matrix is infinite. The remedy suggested by the renormalization theory is to add counterterms to the interaction Hamiltonian V. Unfortunately, the counterterms are infinite in all physically relevant field theories, and the time evolution cannot be properly described.

In this paper we demonstrate that important properties of relativistic invariance and cluster separability can be satisfied in a broad class of theories that are related by certain unitary transformations to the traditional field theories. Using a unitary clothing transformation $e^{i\Phi}$ it is possible to find in this class a finite relativistically invariant clothed particle Hamiltonian $H^r = e^{i\Phi}(H_0 + V)e^{-i\Phi}$ which provides a full dynamical description of time-dependent processes. H^r remains finite in the limit of removed regularization because infinities present in Φ exactly cancel infinities in V. Since transformation $e^{i\Phi}$ has an additional property that it conserves the S-operator, the theory constructed from the Hamiltonian H^r is consistent with all experimentally verified predictions of the old theory, but unlike the old theory it is completely free from ultraviolet divergences. Any physical information, including time evolution of wave functions and observables, can be calculated from H^r by a straightforward application of quantum mechanical rules without renormalization corrections. In addition, there are no self-interactions and clouds of virtual particles in the vacuum and one-particle states. A close collaboration between theory and experiment is needed in order to find exact expressions for H^r in high perturbation orders and outside the energy shell.

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