KEPLER'S LAWS

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1. Introduction

Astronomy has always been a source of scientific inspiration. Probably the most significant revolution in science was spurred by astronomy. Since antiquity, civilizations have wondered about the laws governing the motion of celestial bodies. The theory in force during the middle-ages was due to Ptolemy. In this model, the sun, the moon, and all the stars revolved about the earth, and their motion was determined by super-imposing circular motions. Some "stars" (which turned out to be the planets) were erratic, and each required many parameters to describe their motion. The theory gave very accurate predictions, but was rather cumbersome.

In the seventeenth century, Johannes Kepler, using data collected by Tycho Brahe, formulated three simple laws to explain the motion of the planets:

1st Law *Every planet moves on an ellipse with the sun at a focus.*

2nd Law *The radius vector from the sun to the planet sweeps out equal areas in equal times.*

3rd Law *The square of the orbital period of the planet is proportional to the cube of the semi-major axis of the orbit.*

Here, only two parameters are required for each planet: the eccentricity of the ellipse, and its semi-major axis. The motion is then determined from these three laws. For example, the second law gives the relative angular velocity of the planet at each point on its orbit. Note for instance that when the planet is close to the sun it is moving faster than when it is further away. The third law is then used to determine its absolute velocity.

In 1687, in his masterpiece *Philosophiae Naturalis Principia Mathematica*, Isaac Newton formulated the laws of dynamics and gravity, and derived Kepler's three laws from these using the tools of the newly developed Calculus. The most remarkable feature, is that the principles he formulated are universal, hence apply to any body moving under the influence of gravity. They are used to describe the motion of asteroids, comets, satellites, in addition to the planets.

In these notes, we will formulate *Newton's Law of Motion*, *Newton's Law of Gravity*, and then derive Kepler's three laws.

2. Newton's Law of Motion

In Newtonian mechanics, the motion of a particle, such as a planet, is described by a parametric curve in 3-space:

$$\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t). \end{cases} \tag{1}$$

It is helpful instead to write (1) as a vector-valued function of t:

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}.$$

The position of the particle at time t is given by the terminal point of the vector $\mathbf{r}(t)$. The velocity of such a particle is:

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt},$$

where differentiation of the vector $\mathbf{r}(t)$ is defined by:

$$\frac{d\mathbf{r}}{dt} = \frac{df}{dt}\,\mathbf{i} + \frac{dg}{dt}\,\mathbf{j} + \frac{dh}{dt}\,\mathbf{k},$$

The acceleration of the particle is then:

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}.$$

We note here for future reference that:

$$\frac{d}{dt}\big(\mathbf{u}\cdot\mathbf{w}\big) = \frac{d\mathbf{u}}{dt}\cdot\mathbf{w} + \mathbf{u}\cdot\frac{d\mathbf{w}}{dt},$$

$$\frac{d}{dt}\big(\mathbf{u}\times\mathbf{w}\big) = \frac{d\mathbf{u}}{dt}\times\mathbf{w} + \mathbf{u}\times\frac{d\mathbf{w}}{dt},$$

for any functions $\mathbf{u}(t)$ and $\mathbf{w}(t)$. This is a direct consequence of Leibniz rule for the differentiation of a product.

Now, *Newton's Law of Motion* states that the acceleration of a particle is proportional to the force exerted on the particle, the constant of proportionality being defined as the *mass* of the particle. This is written:

$$\mathbf{F} = m\mathbf{a}.$$

3. Newton's Law of Gravity

This law states that the gravitational force that a body of mass M exerts on a body of mass m is proportional to the product of their masses, inversely proportional to the square of the distance between the two bodies, and is directed so that it points from the first body to the second. If the first body's position is given by $\mathbf{r}_1$, and the second body's position is given by $\mathbf{r}_2$, then the force is:

$$\mathbf{F} = \frac{GMm\big(\mathbf{r}_2 - \mathbf{r}_1\big)}{\left|\mathbf{r}_2 - \mathbf{r}_1\right|^3}.$$

We wish to consider the case where there are only two bodies. We make the assumption that $m \ll M$, i.e. m is much smaller than M. This is satisfied if, say, the first body is the sun and the second body is a planet. In that case, it is also possible to assume that the position of the first body is fixed at the origin of our coordinate system. Finally, we choose our system of units so that $GM = 1$, and $m = 1$. The only unknown is then the position of the second body which we take to be $\mathbf{r}(t)$. It is to be found by solving the system of differential equations:

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mathbf{r}}{r^3}, \tag{2}$$

where we have put $r = |\mathbf{r}|$. In terms of the components (x, y, z) of $\mathbf{r}$, this reads:

$$\begin{cases} \ddot{x} &= -(x^2 + y^2 + z^2)^{-3/2}\, x \\ \ddot{y} &= -(x^2 + y^2 + z^2)^{-3/2}\, y \\ \ddot{z} &= -(x^2 + y^2 + z^2)^{-3/2}\, z. \end{cases} \tag{3}$$

where the dot above a letter represents differentiation with respect to t. Of course, the solution will depend on the initial position $\mathbf{r}(0)$, and the initial velocity $\dot{\mathbf{r}}(0)$.

4. Conserved Quantities

For any vector-valued function $\mathbf{r}(t)$, define its *energy*:

$$E(t) = \frac{1}{2}\,|\dot{\mathbf{r}}|^2 - \frac{1}{r}.$$

Theorem 1. Conservation of Energy
Let $\mathbf{r}(t)$ be a solution of (2). Then

$$\frac{dE}{dt} = 0.$$

Proof. Note that $r^2 = \mathbf{r} \cdot \mathbf{r}$, so that

$$2r\dot{r} = 2\mathbf{r} \cdot \dot{\mathbf{r}},$$

and it follows that

$$\dot{r} = \frac{\mathbf{r} \cdot \dot{\mathbf{r}}}{r}. \tag{4}$$

Now, $|\dot{\mathbf{r}}|^2 = \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$, so that using (2), we obtain

$$\frac{d}{dt}\,|\dot{\mathbf{r}}|^2 = 2\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = -2\frac{\dot{\mathbf{r}} \cdot \mathbf{r}}{r^3} \tag{5}$$

Therefore, combining (4) with (5), we conclude:

$$\dot{E} = \frac{1}{2}\frac{d}{dt}\,|\dot{\mathbf{r}}|^2 - \frac{d}{dt}\left(\frac{1}{r}\right) = -\frac{\dot{\mathbf{r}} \cdot \mathbf{r}}{r^3} + \frac{1}{r^2}\dot{r} = -\frac{\dot{\mathbf{r}} \cdot \mathbf{r}}{r^3} + \frac{\dot{\mathbf{r}} \cdot \mathbf{r}}{r^3} = 0.$$

□

It follows that if $\mathbf{r}(t)$ is a solution of (2), then $E(t) = E(0) =$ constant. We will use this fact later.

For any vector-valued function $\mathbf{r}(t)$ define its *angular momentum*:

$$\mathbf{L}(t) = \mathbf{r} \times \dot{\mathbf{r}}.$$

Theorem 2. Conservation of Angular Momentum
Let $\mathbf{r}(t)$ *be a solution of* (2). *Then*

$$\frac{d\mathbf{L}}{dt} = 0.$$

Proof. Recall that $\mathbf{w} \times \mathbf{w} = 0$ for any vector $\mathbf{w}$ in 3-space. Thus,

$$\dot{\mathbf{L}} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times \left(-\frac{\mathbf{r}}{r^3}\right) = -\frac{1}{r^3}\,\mathbf{r} \times \mathbf{r} = 0.$$

□

Thus, if $\mathbf{r}(t)$ is a solution of (2), then $\mathbf{L}(t) = \mathbf{L}(0) = \text{constant}$.

5. Integration of the Equations of Motion

Throughout this section $\mathbf{r}(t)$ will denote a solution of (2). From Theorems 1 and 2, we know that there is a constant E and a constant vector $\mathbf{L}$ such that

$$\frac{1}{2}\left|\dot{\mathbf{r}}\right|^2 - \frac{1}{r} = E \tag{6}$$

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{L}. \tag{7}$$

Now, if $\mathbf{L} = \mathbf{0}$, then it is not difficult to check that $\mathbf{r}(t)$ will lie in a fixed line through the origin. In this case, there is always a collision at some time $t = t_0$, i.e. $\mathbf{r}(t_0) = \mathbf{0}$. This case is left as an exercise. From now on, we assume that $\mathbf{L} \neq \mathbf{0}$. Note that, since $\mathbf{r}(t)$ is always perpendicular to the constant vector $\mathbf{L} \neq \mathbf{0}$, we obtain that $\mathbf{r}(t)$ lies in a fixed plane through the origin. We may assume that this is the xy-plane, and we now introduce polar coordinates (r, θ) in this plane, centered at the origin. Note that $r = |\mathbf{r}|$ is the same as before. There are now only two unknowns $r(t)$, and $\theta(t)$.

From the transformation

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

we obtain

$$\begin{cases} \dot{x} = \dot{r}\cos\theta - \dot{\theta}\, r\sin\theta \\ \dot{y} = \dot{r}\sin\theta + \dot{\theta}\, r\cos\theta. \end{cases}$$

It follows that

$$\begin{aligned} |\dot{\mathbf{r}}| &= \dot{x}^2 + \dot{y}^2 \\ &= \dot{r}^2\cos^2\theta + \dot{\theta}^2 r^2 \sin^2\theta - 2\dot{r}\dot{\theta}\cos\theta\sin\theta + \dot{r}^2\sin^2\theta + \dot{\theta}^2 r^2\cos^2\theta \\ &\qquad + 2\dot{r}\dot{\theta}\sin\theta\cos\theta \\ &= \dot{r}^2(\cos^2\theta + \sin^2\theta) + r^2\dot{\theta}^2(\sin^2\theta + \cos^2\theta) \\ &= \dot{r}^2 + r^2\dot{\theta}^2. \end{aligned}$$

and,

$$\begin{aligned}
\mathbf{r} \times \dot{\mathbf{r}} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dot{x} & \dot{y} & 0 \\ x & y & 0 \end{vmatrix} \\
&= (\dot{x}y - x\dot{y})\,\mathbf{k} \\
&= \big(r\sin\theta(\dot{r}\cos\theta - \dot{\theta}r\sin\theta) - r\cos\theta(\dot{r}\sin\theta + \dot{\theta}r\cos\theta)\big) \\
&= -r^2\dot{\theta}\,\mathbf{k}
\end{aligned}$$

Thus, if we define $L = |\mathbf{L}|$, then we have $L = r^2\dot{\theta}$, and equations (6)–(7) can be written

$$\frac{1}{2}\big(\dot{r}^2 + r^2\dot{\theta}^2\big) - \frac{1}{r} = E \tag{8}$$

$$r^2\dot{\theta} = L. \tag{9}$$

From now on, when we say a solution, we mean a pair of functions $\big(r(t), \theta(t)\big)$ which satisfy the equations (8)–(9). From equation (9), we see that $r(t) \neq 0$, and also

$$\dot{\theta} = \frac{L}{r^2}.$$

Substituting into (8), we obtain

$$\frac{1}{2}\dot{r}^2 + \frac{L^2}{2r^2} - \frac{1}{r} = E. \tag{10}$$

We first analyze equation (10) qualitatively. The treatment, although rather sketchy, can be made entirely rigorous. Define

$$V(r) = \frac{L^2}{2r^2} - \frac{1}{r}, \tag{11}$$

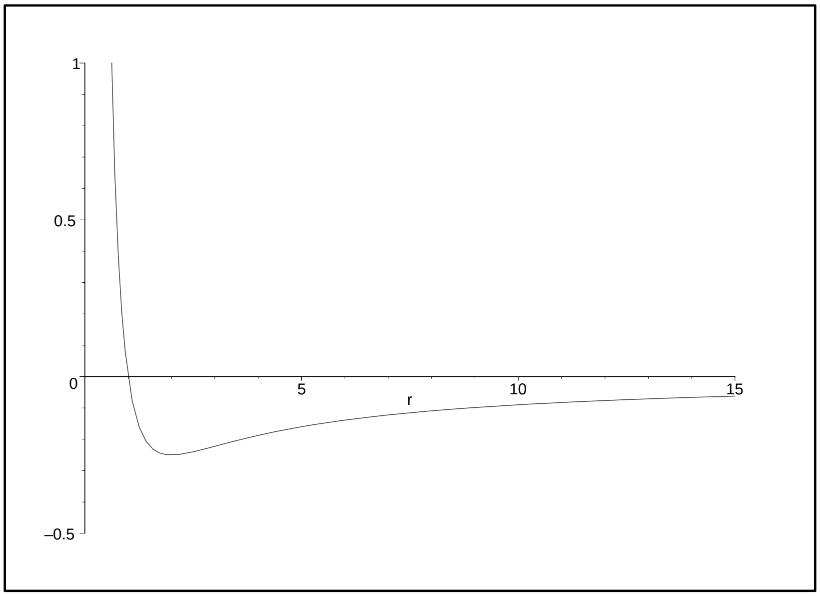
so that

$$E = \frac{1}{2}\dot{r}^2 + V(r).$$

This is known as 1-dimensional motion in the *effective potential* $V(r)$. Since the *kinetic energy* $K = (1/2)\dot{r}^2 \geq 0$, we have

$$V(r) \leq E,$$

for all t. Now, plot $V(r)$ against r, see the plot below.



There are four cases to distinguish:

(a) If the energy $E = V_{\min} = -1/(2L^2)$, then $r = L^2$, and the orbit is a circle.
(b) If the energy $V_{\min} < E < 0$, then there are numbers $0 < r_{\min} < r_{\max}$ such that $r_{\min} < r < r_{\max}$, i.e. r oscillates between $r_{\min}$ and $r_{\max}$.
(c) If the energy $E = 0$, then $r_{\min} < r < \infty$, where $r_{\min} = L^2/2$. Note that $\lim_{r\to\infty} K = 0$, hence also $\lim_{r\to\infty} \dot{r} = 0$, i.e. the *asymptotic velocity* is zero.
(c) If the energy $E > 0$, then there is a number $0 < r_{\min}$ such that $r_{\min} < r < \infty$. However, now we have $\lim_{r\to\infty} K = E > 0$, hence $\lim_{r\to\infty} \dot{r} = \sqrt{2E}$, i.e. the asymptotic velocity is positive.

Note that this analysis depends only on the qualitative features of the potential V. For instance, we obtain the same qualitative behavior if we replace the term $1/r$ by a term $1/r^p$ for any $0 < p < 2$. A good way to think of this problem is to imagine a friction-less bead constrained to roll along the curve $V = V(r)$.

6. Kepler's Laws

In this last section, we prove Kepler's laws.

Theorem 3. Kepler's First Law
Let $\big(r(t), \theta(t)\big)$ be a solution. Then the orbit is a conic section.

Proof. From (8)–(9), we obtain:

$$\dot{r}^2 = 2E + 2/r - L^2/r^2$$
$$\dot{\theta}^2 = L^2/r^4.$$

Thus, using the chain rule, $dr/d\theta = \dot{r}/\dot{\theta}$, we have:

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{L^2}(2Er^4 + 2r^3 - L^2r^2). \tag{12}$$

Differentiating this equation with respect to θ, we get:

$$2\frac{dr}{d\theta}\frac{d^2r}{d\theta^2} = \frac{1}{L^2}(8Er^3 + 6r^2 - 2L^2r)\frac{dr}{d\theta}.$$

Hence

$$\frac{d^2r}{d\theta^2} = \frac{1}{L^2}(4Er^3 + 3r^2 - L^2r). \tag{13}$$

Now, let

$$u = \frac{1}{r},$$

then

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta},$$

and

$$\frac{d^2u}{d\theta^2} = -\frac{1}{r^2}\frac{d^2r}{d\theta^2} + \frac{2}{r^3}\left(\frac{dr}{d\theta}\right)^2.$$

Substituting $(dr/d\theta)^2$ from (12), and $d^2r/d\theta^2$ from (13), we arrive at:

$$\begin{aligned}\frac{d^2u}{d\theta^2} &= -\frac{1}{r^2L^2}(4Er^3 + 3r^3 - L^2r) + \frac{2}{r^3L^2}(2Er^4 + 2r^2 - L^2r^2)\\ &= \frac{1}{L^2} - \frac{1}{r}\\ &= \frac{1}{L^2} - u.\end{aligned}$$

This equation:

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{L^2},$$

is relatively easy to solve. All its solutions are of the form:

$$u = \frac{1}{L^2} + k\cos(\theta - \theta_0),$$

where k and θ_0 are constants. Writing $e = kL^2$, we can write

$$r = \frac{L^2}{1 + e\cos(\theta - \theta_0)}. \tag{14}$$

This is the equation of a conic section in polar coordinates. □

Equation (14) is the equation of a circle if $|e| = 0$, an ellipse if $0 < |e| < 1$, a parabola if $|e| = 1$, and a hyperbola if $1 < |e| < \infty$. An orbit is called *elliptical* if it is a circle or an ellipse, *parabolic* if it is a parabola, and *hyperbolic* if it is a hyperbola; compare with the four cases in section 5.

Theorem 4. KEPLER'S SECOND LAW
Let $\big(r(t), \theta(t)\big)$ be a solution. Then the radius vector $\mathbf{r}$ *sweeps out equal areas in equal time.*

Proof. This follows from the conservation of angular momentum:

$$\frac{d\theta}{dt} = \frac{L}{r^2}.$$

Note that the area swept by the radius vector is given by:

$$A(\theta) = \frac{1}{2}\int_0^\theta r^2 d\theta.$$

Thus,

$$\frac{dA}{d\theta} = \frac{1}{2}r^2,$$

and from the chain rule

$$\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = \frac{1}{2}r^2\frac{L}{r^2} = \frac{1}{2}L.$$

□

Let $\big(r(t), \theta(t)\big)$ be a solution. Let T be the time it takes for the particle to complete one full orbit. We call T the *period*.

Theorem 5. KEPLER'S THIRD LAW
Let $\big(r(t), \theta(t)\big)$ be a solution with an elliptical orbit. Then

$$\frac{T^2}{a^3} = 4\pi^2,$$

where T is the period, and a the semi-major axis of the ellipse.

Proof. From the conservation of angular momentum, we have:

$$\frac{d\theta}{dt} = \frac{L}{r^2}.$$

Thus, $d\theta/dt > 0$, and

$$\frac{dt}{d\theta} = \frac{r^2}{L}.$$

It follows, by substitution, that

$$T = \int_0^T dt = \int_0^{2\pi} \frac{dt}{d\theta}\,d\theta = \frac{1}{L}\int_0^{2\pi} r^2 d\theta.$$

We substitute (14) into this integral to obtain:

$$T = L^3\int_0^{2\pi} \frac{d\theta}{(1 + e\cos\theta)^2} = \frac{2\pi L^3}{(1 - e^2)^{3/2}}.$$

We have taken $\theta_0 = 0$, which clearly does not affect the calculation. We also omitted the computation of the integral, which is rather technical. Now, we have:

$$\begin{aligned} r_{\max} + r_{\min} &= 2a \\ r_{\max} - r_{\min} &= 2c, \end{aligned}$$

where a is the semi-major axis of the ellipse, while c is the distance of the focus from the center. Thus, the eccentricity e of the ellipse is

$$e = \frac{c}{a} = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}}.$$

It follows that

$$\begin{aligned} 1 + e &= \frac{2r_{\max}}{r_{\max} + r_{\min}} \\ 1 - e &= \frac{2r_{\min}}{r_{\max} + r_{\min}}, \end{aligned}$$

hence

$$1 - e^2 = (1+e)(1-e) = \frac{4r_{\max}r_{\min}}{r_{\max} + r_{\min}} = \frac{r_{\min}r_{\max}}{a^2}.$$

On the other hand,

$$\begin{aligned} \frac{L^2}{r_{\max}} &= 1 - e \\ \frac{L^2}{r_{\min}} &= 1 + e, \end{aligned}$$

hence

$$L^2 \left(\frac{1}{r_{\max}} + \frac{1}{r_{\min}} \right) = 2.$$

It follows that

$$L^2 = \frac{2r_{\min}r_{\max}}{r_{\max} + r_{\min}} = \frac{r_{\min}r_{\max}}{a}.$$

It follows that $L^2/(1-e^2) = a$, and therefore

$$T^2 = 4\pi^2 a^3.$$

□

7. Exercises

In all the problems, you may assume that the units are chosen as in the text, i.e. so that $GM = m = 1$. When necessary, the standard units can be recovered by multiplying the final result by the appropriate constant.

In the first set of problems, we deal with the case of radial orbits, i.e. those orbits for which $L = 0$.

Exercise 1. Let $\big(r(t), \theta(t)\big)$ be a solution of the equations of motion (8)–(9) with zero angular momentum $L = 0$. Show that the angular coordinate $\theta(t)$ is constant, and the radial coordinate $r(t)$ satisfies:

$$\frac{d^2 r}{dt^2} = -\frac{1}{r^2}. \tag{15}$$

We call solutions of (15) *radial orbits*.

Exercise 2. Let $r(t)$ be a radial orbit. Show that there is a time t_0 such that $r(t_0) = 0$. We say that a *collision* occurs at that time. Hint: t_0 is not necessarily positive.

Exercise 3. Consider the radial orbits $r(t)$ such that $r(0) = r_0 > 0$. Determine the *escape velocity*, i.e. the minimal 'initial' velocity $\dot{r}(0) > 0$ such an orbit must have so that $\lim_{t\to\infty} r = \infty$. Hint: Use the conservation of energy.

In the next set of exercises, we deal with *circular orbits*, i.e. those orbits for which $r(t) = r_0 > 0$ is constant. For such orbits, it follows from the conservation of angular momentum that $\dot{\theta}$ is constant: $\dot{\theta} = L/r_0^2$.

Exercise 4. Let $\big(r_0, \theta(t)\big)$ be a solution with a circular orbit.

(i) Find its period T in terms of r_0.
(ii) Find its angular momentum L in terms of r_0.
(iii) Find its speed $v = |\dot{\mathbf{r}}|$ in terms of r_0.
(iv) Find its energy E in terms of r_0.

Using this last exercise, it is possible to determine how much energy is required to extract the particle from the system, i.e. if the particle was given that much energy in the form of kinetic energy, it would scatter off in a parabolic or in a hyperbolic orbit. This should be used in the next problem.

BONUS

Exercise 5. A satellite is in a circular orbit at radius r_0 from the center. At time $t = 0$, it is to be accelerated, tangential to its orbit, to such a velocity that it will escape the system, i.e. $\lim_{t\to\infty} r(t) = \infty$. What is the minimum speed at which it should exit its orbit, so this can be achieved?